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GENERALIZED LERAY FORMULA ON POSITIVE COMPLEX LAGRANGE-GRASSMANN MANIFOLDS

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Abstract

A full proof of a matrix lemma stated in [1] is given, and the notions concerning cannonical argument and signature of a triple of the Lagrange planes in a complex phase space is formulated. Then a formula is established, which generalizes that one of J. Leray's in real phase space case. Finally, some applications of the formula are given.

§ 0. Introduction

In this paper we shall discuss in full the topic touched briefly in § 3 of [1]. In § 1, we give the full proof of a matrix lemma, i. e. Lemma 1.6, which generalizes a lemma used by J. J. Duistermaat in [3], and wasannounced in § 3 of [1]. In § 2, we proceed to formulate the notions concerning cannonical argument and signature of a triple of Lagrange planes in a complex phase space, among which one is real in, essence, the other negative semi-definite and the third positive semi-definite. In Theorem 2.16 of § 2, we establish a fomula which generalizes that one of J. Leray in real phase space case. In spite of its elementary nature the proof is fairly long and a bit intricated, as we did not expect before.

The main potential application of the results outlined above we have conceived of is using them to develop «Analyse Lagrangienne» for complex phase case, parallel to what J. Leray did in [2] for the real phase case. It seems to be quite possible, and we expect to work it out in another occasion. In this paper we only give other two comparatively minor applications in § 3 and § 4 respectively. In § 3, we show that Hörmander's cross index suitablely generalized is still the backbone in the notion of almost analytic Maslov line bundles, used in [4], as well as in the real phase case. In § 4, we give an invariant formulation for a one dimensional Čech co-cycle with coefficients in the sheaf of germs of real valued continuous funcrion on a positive complex Lagrange-Grassmbnn manifold, which was used in several Soviet literatures such as [7] and [8] without complete rigor, as it appeared to us.

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§ 1. A fundamental lemma

Definition 1.1. *Let P be an $n\times n$ complex symmetric matrix, $\lambda_1\lambda_2\cdots\lambda_n$ be the eigenvalues of it. If* $\operatorname{Im}P\geqslant 0$, *then* $\operatorname{Im}\lambda_j\geqslant 0$ (or *if* $\operatorname{Im}P\leqslant 0$, *then* $\operatorname{Im}\lambda_j\leqslant 0$), $j=1, 2, \cdots, n$. *We define the canonical argument of the matrix P by setting*

$$k_+-\arg P=\sum_{j=1}^{n}\arg\lambda_j\left(\text{or } k_--\arg P=\sum_{j=1}^{n}\arg\lambda_j\right), \tag{1.1}$$

where

$$0\leqslant\arg\lambda_j\leqslant\pi\,(\text{or}-\pi\leqslant\arg\lambda_j\leqslant 0),\, j=1,\, 2,\, \cdots,\, n,$$
$$\arg\lambda_j=0 \quad \textit{for}\ \lambda_j=0.$$

And we define the signature of the matrix P by setting

$$\operatorname{sgn}P=\begin{cases} n-\dim(\ker P)-\dfrac{2}{\pi}\,k_--\arg P & \textit{when}\ \operatorname{Im}P\leqslant 0,\\ \\ n-\dim(\ker P)-\dfrac{2}{\pi}\,k_+-\arg P & \textit{when}\ \operatorname{Im}P\geqslant 0.\end{cases} \tag{1.2}$$

(For a real symmetric matrix P, this definition agrees with the common one, whether $\operatorname{Im}P$ is considered to be$\geqslant 0$ or$\leqslant 0$).

Remark 1.2. This definition is introduced by the suggestion of the method of stationary phase. Following the conventional usage, by $GL(n, C)$ we denote the set of all $n\times n$ complex nondegenerate matrices. Set

$$G_+(n)=\{P\in GL(n, C);\ P={}^tP,\ \operatorname{Im}P\geqslant 0\},$$
$$G_-(n)=\{P\in GL(n, C);\ P={}^tP,\ \operatorname{Im}P\leqslant 0\}.$$

Obviously, both $G_+(n)$ and $G_-(n)$ are closed contractible subsets of $GL(n, C)$.
It is easy to see that $k_+-\arg P$(or $k_--\arg P$) is just tnat single-valued branch of the argument of $\det P$ which vanishes for $P=I$ and is a continuous function of P in $G_+(n)$ $(G_-(n))$.

It is not difficult to verify that if P is a complex symmetric $n\times n$ matrix, with $\operatorname{Im}P\geqslant 0$(or $\operatorname{Im}P\leqslant 0$), then

$$k_+-\arg P=k_--\arg(-P)+[n-\dim(\ker P)]\pi$$
$$(\text{or } k_--\arg P=k_+-\arg(-P)-[n-\dim(\ker P)]\pi). \tag{1.3}$$

If, in addition, P is nondegenerate, then $\operatorname{Im}P^{-1}\leqslant 0$(or $\operatorname{Im}P^{-1}\geqslant 0$), and

$$k_+-\arg P=-k_--\arg P^{-1}(\text{or } k_--\arg P=-k_+-\arg P^{-1}). \tag{1.4}$$

Proposition 1.3. *Suppose that P is a complex symmetric matrix, and that* $\operatorname{Im}P\geqslant 0$(or $\operatorname{Im}P\leqslant 0$), *then a necessary and sufficient condition that $x+iy\in\ker P$ is that x, $y\in\ker(\operatorname{Re}P)\cap\ker(\operatorname{Im}P)$.*

Proof Let $X=\operatorname{Re}p$. $Y=\operatorname{Im}P$. Assume $x+iy\in\ker P$, i. e. $(X+iY)(x+iy)=0$, then

$$Xx=Yy,\ Yx=-Xy. \tag{1.5}$$

Hence

$$ {}^t y Y y + {}^t x Y x = 0. $$

Since $Y \geqslant 0$ (or $Y \leqslant 0$), we have ${}^t y Y y = 0$, ${}^t x Y x = 0$, thus $Yy = 0$, $Yx = 0$. And by (1.5) we obtain $Xx = 0$, $Xy = 0$. Therefore $x, y \in \ker X \cap \ker Y$. This completes the proof of the necessary part of the proposition. The other part of it is obvious.

Corollary 1.4 *Let P be a non-zero matrix and satisfy the conditions of Proposition 1.3. Then*

(i) *there exists such a real orthogonal matrix Q that*

$$ {}^t Q P Q = \left(\begin{array}{c|c} P_0 & 0 \\ \hline 0 & 0 \end{array} \right), \quad \det P_0 \neq 0; \tag{1.6} $$

(ii) $k_+ - \arg P$ (or $k_- - \arg P$) *is just that branch of the argument of the determinant of the restriction of P on the orthogonal complement of* $\ker P$ (i.e. *on the range of P*), *as that mentioned in Remark* 1.2.

Proof Suppose that P is a $n \times n$ complex matrix as stateed in the corollary, $\dim \ker P = n - r$. We choose a real orthogonal matrix

$$ Q = (u_1, \cdots, u_n), $$

where $u_{r+1}, \cdots, u_n \in \ker (\operatorname{Re} P) \cap \ker (\operatorname{Im} P)$. Then it can be easily verified that this Q is just the one we need.

Proposition 1.5. *Let matrix P satisfy the condition in Definition* 1.1, *and matrix T be real nondegenerate, $S = {}^t T P T$. Then*

$$ k_+ - \arg S = k_+ - \arg P \ (\text{or } k_- - \arg S = k_- - \arg P). \tag{1.7} $$

Proof Let us first consider the case when $\det P \neq 0$. Since T can be decomposed into a product of an orthogonal matrix and a positive definite real symmetric matrix, by Definitition 1.1 we may assume with no loss of generality that T is positive definite. Then we only need to consider the following homotopy

$$ S_\theta = T_\theta P T_\theta, \quad 0 \leqslant \theta \leqslant 1, \quad T_\theta = (1-\theta) T + \theta I, $$

to complete the proof. In fact, since $\det S = \det P \times (\det T)^2$, there existts an integer-valued function $k(\theta)$ such that

$$ k_\pm - \arg S_\theta = k_\pm - \arg P + 2\pi k(\theta), \quad 0 \leqslant \theta \leqslant 1. \tag{1.8} $$

By Remark 1.2 $k(\theta)$ must be a continuous function of θ, and therefore, is actually a constant. But $k(1) = 0$, hence $k(\theta) \equiv 0$.

Now assume that $\dim (\ker P) = n - r$, $0 < r < n$; thus, by Corollary 1.4 (and notice that $\dim (\ker S) = \dim (\ker P)$) and Definition 1.1, we may suppose that

$$ S = \left(\begin{array}{c|c} S_0 & 0 \\ \hline 0 & O_{n-r} \end{array} \right), \quad P = \left(\begin{array}{c|c} P_0 & 0 \\ \hline 0 & O_{n-r} \end{array} \right), $$

$$ \det S_0 \neq 0, \quad \det P_0 \neq 0. $$

Therefore, multiplying both sides of the equality

$$\left(\begin{array}{c|c} S_0 & 0 \\ \hline 0 & O_{n-r} \end{array}\right) = {}^tT \left(\begin{array}{c|c} P_0 & 0 \\ \hline 0 & O_{n-r} \end{array}\right) T \text{ (possiblly with a new } T\text{)}$$

by $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & O_{n-r} \end{array}\right)$ both from the left and from the right simultaneously, we obtain

$$\left(\begin{array}{c|c} S_0 & 0 \\ \hline 0 & O_{n-r} \end{array}\right) = \left(\begin{array}{c|c} {}^tT_0P_0T_0 & 0 \\ \hline 0 & O_{n-r} \end{array}\right),$$

where T_0 is the block at the upper left corner in the block expression of T similar to those of S and P. Then it follows that

$$k_\pm-\arg S = k_\pm-\arg S_0 = k_\pm-\arg P_0 = k_\pm-\arg P. \tag{1.9}$$

Lemma 1.6. *Let P be an invertible $n\times n$ complex symmetric matrix with* $\operatorname{Im} P\leqslant 0$. *Suppose that for P and P^{-1} we have the following block partitioned expressions, respectively,*

$$P=\left(\begin{array}{c|c} A & B \\ \hline {}^tB & C \end{array}\right), \quad P^{-1}=\left(\begin{array}{c|c} R & S \\ \hline {}^tS & T \end{array}\right). \tag{1.10}$$

Then we have

$$\begin{aligned} k_--\arg P &= k_--\arg A - k_+-\arg T - \pi\dim(\ker A) \\ &= k_--\arg C - k_+-\arg R - \pi\dim(\ker C), \end{aligned} \tag{1.11}$$

$$\begin{aligned} k_+-\arg P^{-1} &= k_+-\arg T - k_--\arg A + \pi\dim(\ker A) \\ &= k_+-\arg R - k_--\arg C + \pi\dim(\ker C). \end{aligned} \tag{1.12}$$

Remark 1.7. When P is real, this is just the lemma proved by Duistermaat (cf. [3] Lemma 4.1.2.). The result in general case is due to Wang Rouhwai (cf. [1]).

To prove Lemma 1.6, we need to prove an auxilliary proposition beforehand.

Proposition 1.8. *If the conditions of Lemma* 1.6 *are satisfied and in addition A is nondegenerate, then*

$$k_--\arg P = k_--\arg A + k_--\arg(C-{}^tBA^{-1}B) = k_--\arg A - k_+-\arg T. \tag{1.13}$$

Proof Consider the homotopy

$$P_\theta=\theta P - i(1-\theta)I_n,\ A_\theta=\theta A - i(1-\theta)I_r,\ C_\theta=\theta C - i(1-\theta)I_{n-r}\ (0\leqslant\theta\leqslant 1),$$

where we suppose that A is an $r\times r$ matrix, and that C is a $(n-r)\times(n-r)$ marix. It is easy to see that P_θ, $A_\theta (0\leqslant\theta\leqslant 1)$ are nondegenerate and

$$P_\theta=\left(\begin{array}{c|c} A_\theta & \theta B \\ \hline \theta\,{}^tB & C_\theta \end{array}\right) \quad (0\leqslant\theta\leqslant 1).$$

From the equality

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline -\theta^t BA_\theta^{-1} & I_{n-r} \end{array}\right)\left(\begin{array}{c|c} A_\theta & \theta B \\ \hline \theta^t B & C_\theta \end{array}\right)\left(\begin{array}{c|c} I_r & -\theta A_\theta^{-1}B \\ \hline 0 & I_{n-r} \end{array}\right)$$

$$=\left(\begin{array}{c|c} A_\theta & 0 \\ \hline 0 & C_\theta-\theta^{2t}BA_\theta^{-1}B \end{array}\right) \quad (0\leqslant\theta\leqslant 1), \tag{1.14}$$

we conclude that $C_\theta-\theta^{2\,t}BA_\theta^{-1}B$ is nondegenerate and

$$P_\theta^{-1}=\left(\begin{array}{c|c} \times\times & * \\ \hline {}^t* & (C_\theta-\theta^{2t}BA^{-1}B)^{-1} \end{array}\right) \quad (0\leqslant\theta\leqslant 1).$$

Therefore, $\operatorname{Im}(C_\theta-\theta^{2t}BA_\theta^{-1}B)\leqslant 0$ and $(C-{}^tBA^{-1}B)^{-1}=T$. From (1.14), we obtain

$$\det P_\theta=\det A_\theta \det(C_\theta-\theta^{2t}BA_\theta^{-1}B) \quad (0\leqslant\theta\leqslant 1).$$

Moreover, from Remark 1.2 it follows that

$$k_- -\arg P_\theta=k_- -\arg A_\theta+k_- -\arg(C_\theta-\theta^{2t}BA_\theta^{-1}B)+2\pi m_\theta \quad (0\leqslant\theta\leqslant 1), \tag{1.15}$$

wher m_θ is an integer-valued function on the interval $0\leqslant\theta\leqslant 1$. Because we know all the terms in (1.15) are continuous functions on the interval $0\leqslant\theta\leqslant 1$ except $2\pi m_\theta$, m_θ must also be so. In view of $m_0=0$, we have $m_\theta\equiv 0\,(0\leqslant\theta\leqslant 1)$. Set $\theta=1$ in the equality (1.15), then (1.13) follows.

Proof af Lemma 1.6 From (1.3) and (1.4) oue may see that we only need to prove the first equality in (1.11). To do this, assume that A is an $r\times r$ matrix. Note that if $\det A\neq 0$, then Proposition 1.8 says that the fact we want to prove is true. Therefore, only the case when $\det A=0$ needs to be discussed.

We first consider the extreme case when both A and C are zero-matrices. In this case, the facts that P is nondegenerate and $\operatorname{Im}P\leqslant 0$ imply that B is a real nondegenerate marix, and

$$P^{-1}=\left(\begin{array}{c|c} 0 & {}^tB^{-1} \\ \hline B^{-1} & 0 \end{array}\right).$$

Hence, if we notice that the characteristic polynomial of P is

$$\det\left(\begin{array}{c|c} \lambda I & -B \\ \hline -{}^tB & \lambda I \end{array}\right)=\det\left(\begin{array}{c|c} \lambda I & 0 \\ \hline -{}^tB & \lambda I-\lambda^{-1}\,{}^tBB \end{array}\right)=\det(\lambda^2-{}^tBB),$$

and thus that the eigenvalues of P are $\pm\mu_j(j=1, 2, \cdots, r)$, where $\mu_j^2(j=1, 2, \cdots, r)$ are the eigenvalues of the positive definite matrix ${}^tBB$, so we can see that (1.11) holds indeed in this case.

Now suppose that one of A and C, for instance, A is a non-zero matrix. In this case, by Corollary 1.4, there exists a real orthogonal matrix Q such that

$$^tQAQ=\left(\begin{array}{c|c} A_0 & 0 \\ \hline 0 & 0 \end{array}\right),$$

where A_0 is an $s\times s$ matrix, $0<s<r$, $\det A_0\neq 0$. Hence

$$\left(\begin{array}{c|c} ^tQ & 0 \\ \hline 0 & I_{n-r} \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline ^tB & C \end{array}\right)\left(\begin{array}{c|c} Q & 0 \\ \hline 0 & I_{n-r} \end{array}\right)=\left(\begin{array}{c|c|c} A_0 & 0 & B_1 \\ 0 & 0 & B_2 \\ \hline ^tB_1 & ^tB_2 & C \end{array}\right),$$

where $\left(\frac{B_1}{B_2}\right)={}^tQB$. Thus, by Proposition 1.5 and 1.8, we have

$$k_--\arg P=k_--\arg A_0+k_--\arg\left(\begin{array}{c|c} O_{r-s} & B_2 \\ \hline ^tB_2 & C-{}^tB_1A_0^{-1}B_1 \end{array}\right)$$

$$=k_--\arg A+k_--\arg\left(\begin{array}{c|c} O_{r-s} & B_2 \\ \hline ^tB_2 & C-{}^tB_1A_0^{-1}B_1 \end{array}\right).$$

Moreover, if notice that

$$\left(\begin{array}{c|c} O_{r-s} & B_2 \\ \hline ^tB_2 & C-{}^tB_1A_0^{-1}B_1 \end{array}\right)^{-1}=\left(\begin{array}{c|c} \times\times & * \\ \hline ^t* & T \end{array}\right),$$

then it can be seen that by means of induction on the dimension n, the proof of this lemma can be reduced to the simple case when

$$P=\begin{pmatrix} 0 & b \\ b & c \end{pmatrix} \quad (b,\ c \text{ are complex numbers}). \tag{1.16}$$

By the nondegeneracy of P, we have $b\neq 0$, and by $\operatorname{Im}P\leqslant 0$, b must be a real number. Besides, obviously

$$P^{-1}=\begin{pmatrix} -b^2c & b^{-1} \\ b^{-1} & 0 \end{pmatrix}.$$

Set

$$P_\theta=\begin{pmatrix} 0 & b \\ b & \theta c \end{pmatrix} \quad (0\leqslant\theta\leqslant 1),$$

then $\det P_\theta=-b^2$, so

$$k_--\arg P_\theta=-\pi+2\pi m_\theta \quad (0\leqslant\theta\leqslant 1), \tag{1.18}$$

where m_θ is an integer-valued function on $0\leqslant\theta\leqslant 1$. By a similar argument as the one in Proposition 1.8, it follows that $m_\theta=m_0(0\leqslant\theta\leqslant 1)$. For $\theta=0$

$$P_0=\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix},$$

which has eigenvalues b and $-b$. So

$$k_- - \arg P_0 = -\pi.$$

Hence $m_0=0$, and $m_\theta \equiv 0\,(0\leqslant\theta\leqslant 1)$. Substituting this into (1.18), and taking $\theta=1$, we obtain

$$k_- - \arg P = -\pi.$$

Therefore, for the case when P and P^{-1} are expressed as in (1.16) and (1.17) respectively, the lemma is true, and this also completes the proof of the lemma.

§ 2. The Leray formula on the positive complex Lagrange-Grassmann manifold

Let (M, σ) be a real symplectic vector space of dimension $2n$, with symplectic bilinear form σ, (M^c, σ) be the complexifieation of (M, σ). We denote by $\Lambda(M^c)$ the set of Lagrangean subspaces of M^c, and write

$\mathscr{L}_R = \{L\in\Lambda(M^c);\ L$ is real, in essence, namely, $L=\bar{L}\}$,

$\mathscr{L}_+ = \{L\in\Lambda(M^c);\ L$ is positive semi-definite, namely, $\operatorname{Im}\sigma(u, \bar{u})\geqslant 0\ \forall u\in L\}$,

$\mathscr{L}_- = \{L\in\Lambda(M^c);\ L$ is negative semi-definite, namely, $\operatorname{Im}\sigma(u, \bar{u})\leqslant 0\ \forall u\in L\}$.

Definition 2.1. *Let $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, $A\cap B=\{0\}$, $B\cap C=\{0\}$. Then there exists a unique linear mapping $T: A\to B$, having C as its graph. We define*

$$Q(a, a') = \sigma(Ta, a'),\ \forall a, a'\in A. \tag{2.1}$$

It is not hard to check that $Q(a, a')$ is a symmetric bilinear form on A. By $k_+ - \arg(A, B, C)$ and $\operatorname{sgn}(A, B, C)$ we denote, respectively, the canonical argument and the signature of the matrix associated with Q relative to any real basis of A.

Remark 2.2. From $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, we have

$$0\leqslant \operatorname{Im}\sigma(a+Ta, \overline{a+Ta}) = \operatorname{Im}[\sigma(a, \bar{a})+\sigma(Ta, \bar{a})+\sigma(a, \overline{Ta})+\sigma(Ta, \overline{Ta})]$$
$$\leqslant 2\operatorname{Im}Q(a, \bar{a}),\ \forall a\in A.$$

Therefore $k_+ - \arg(A, B, C)$ in the above definition is full of meaning. Again by Proposition 1.5, we see that the definitions of $k_+ - \arg(A, B, C)$ and $\operatorname{sgn}(A, B, C)$ are independent of the choice of real basis of A.

By analogy with this definition, for $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, $A\cap C=\{0\}$, $B\cap C=\{0\}$, we may define $k_- - \arg(A, C, B)$ and $\operatorname{sgn}(A, C, B)$.

Proposition 2.3. *If $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, then $B\cap C$ is real in essence.*

Proof We choose a basis of M such that $(M^c, \sigma)\approx \mathbf{C}^n_x\times\mathbf{C}^n_\xi$, and that

$$B:\ x=B_0\xi,$$
$$C:\ x=C_0\xi,$$

where B_0, C_0 are symmetric matrices, $\operatorname{Im}B_0\geqslant 0$, $\operatorname{Im}C_0\leqslant 0$ (both B and C are transversal to $\mathbf{C}^n_x$).

If $(x, \xi)\in B\cap C$, then $(B_0-C_0)\xi=0$. Set

$$B_1=\operatorname{Re}B_0,\ B_2=\operatorname{Im}B_0,\ C_1=\operatorname{Re}C_0,\ C_2=\operatorname{Im}C_0,\ u=\operatorname{Re}\xi,\ v=\operatorname{Im}\xi.$$

By Proposition 1.3

$$(B_1-C_1)u=0,\ (B_1-C_1)v=0,\ (B_2-C_2)u=0,\ (B_2-C_2)v=0.$$

Therefore by $B_2\geqslant 0$ and $C_2\leqslant 0$, it follows immediately that

$${}^tuB_2u=0,\ {}^tvB_2v=0,\ {}^tuC_2u=0,\ {}^tvC_2v=0,\ B_2u=0,\ B_2v=0,\ C_2u=0,\ C_2v=0.$$

Hence

$$\bar{x}=B_0\xi,\ \bar{x}=C_0\xi,$$

namely, $(\bar{x}, \xi)\in B\cap C$; this completes the proof.

With D being any subspace of M^c, we set

$$D^\sigma=\{z\in M^c;\ \sigma(z, y)=0,\ \forall y\in D\}.$$

If D is real in essence and isotropic, namely, $D\subset D^\sigma$, we define

$$\pi^*\sigma(\tilde{y}_1, \tilde{y}_2)=\sigma(y_1, y_2)\forall \tilde{y}_1, \tilde{y}_2\in D^\sigma/D,$$

where y_1, y_2 are representatives of $\tilde{y}_1$, $\tilde{y}_2$ respectively. Obviouly, $(D^\sigma/D, \pi^*\sigma)$ is a complex symplectic vector space.

For any $E\in\Lambda(M^c)$, we define

$$E/D=[E\cap D^\sigma+D]/D=(E\cap D^\sigma)/(D\cap E).$$

It is not hard to check that $E/D\in\Lambda(D^\sigma/D)$ and for $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, we have that A/D, B/D, C/D are real in essence, semi-positive, semi-negative, respectively.

Proposition 2.4. *Let* $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$.

(i) *If* $A\cap B=\{0\}$, $B\cap C=\{0\}$, *then*

$$\operatorname{sng}(A, B, C)=\operatorname{sgn}(A/(A\cap C), B/(A\cap C), C/(A\cap C)). \tag{2.2}$$

(ii) *If* $A\cap C=\{0\}$, $B\cap C=\{0\}$, *then*

$$\operatorname{sgn}(A, C, B)=\operatorname{sgn}(A/(A\cap B), C/(A\cap B), B/(A\cap B)). \tag{2.3}$$

Proof We only prove (ii), because the proof of (i) is analogeous. If $A\cap B=\{0\}$, there is nothing to prove. Therefore assame that $A\cap B\neq\{0\}$. It is not hard to check that $A/(A\cap B)$, $B/(A\cap B)$, $C/(A\cap B)$ are mutually transversal, hence according to Definition 2.1 the right of (2.3) is well defined.

Choose $X\in\mathscr{L}_R$, such that X is transversal to any of A, B and C. Next choose a symplective basis of M, such that $(M^c, \sigma)\approx\mathbf{C}^n_x\times\mathbf{C}^n_\xi$, $X\approx\mathbf{C}^n_x$, $A\approx\mathbf{C}^n_\xi$, and moreover

$$B:\ x=B_0\xi,\ B_0=\left(\begin{array}{c|c} V & 0 \\ \hline 0 & 0 \end{array}\right),\ \det V\neq 0$$

$$C:\ x=C_0\xi,\ \det C_0\neq 0,$$

where B_0, C_0 are $n\times n$ complex symmetric matrices, $\operatorname{Im} B_0\geqslant 0$, $\operatorname{Im} C_0\leqslant 0$. Corresponding to partitioned form of B_0, the standard real symplectic basis chosen above will be denoted as e^I, $e^{\hat{I}}$, f_I, $f_{\hat{I}}$, where

$$I\cup\hat{I}=\{1, 2, \cdots, n\},$$

$$X=\{e^I, e^{\hat{I}}\},\ A=\{f_I, f_{\hat{I}}\}.$$

It is obvious that

$$B=\{f_I+e^IV,\ f_{\hat{I}}\},\ A\cap B=\{f_{\hat{I}}\},$$
$$A+B=\{e^I,\ f_I,\ f_{\hat{I}}\}.$$

If we also assume that

$$C_0^{-1}=\left(\begin{array}{c|c} R & S \\ \hline {}^tS & T \end{array}\right),$$

then

$$C=\{e^I+f_IR+f_{\hat{I}}{}^tS,\ e^{\hat{I}}+f_IS+f_{\hat{I}}T\}.$$

Therefore it is easily seen that

$$A/(A\cap B)\approx\{f_I\},\ B/(A\cap B)\approx\{f_I+e^IV\},$$
$$C/(A\cap B)\approx\{e^{\mp}+f_IR\},\ X/(A\cap B)\approx\{e^I\}.$$

Hence

$$\operatorname{sgn}(A/(A\cap B),\ C(A\cap B),\ B/(A\cap B))=\operatorname{sgn}(R-V^{-1})^{-1}.$$

Observing that

$$\operatorname{sgn}(A,\ C,\ B)=\operatorname{sgn}C_0(B_0-C_0)^{-1}B_0,$$

and

$$C_0(B_0-C_0)^{-1}B_0=(B_0C_1^{-0}-I)^{-1}B_0=\left(\begin{array}{c|c} (R-V^{-1})^{-1} & 0 \\ \hline 0 & 0 \end{array}\right),$$

we conclude that (2.3) is true.

In Definition 2.1, we considered the case only when B is transversal to A and C. For the general case, imitating Leray's idea, we give the following

Definition 2.5. *Let $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$. Set $D=A\cap B+B\cap C+C\cap A$, then A/D, B/D, $C/D\in\Lambda(D^\sigma/D)$ are mutually transversal. We define.*

$$\begin{aligned} k_+-\arg(A,\ B,\ C)&=k_+-\arg(A/D,\ B/D,\ C/D),\\ \operatorname{sgn}(A,\ B,\ C)&=\operatorname{sgn}(A/D,\ B/D,\ C/D),\\ k_--\arg(A,\ C,\ B)&=k_--\arg(A/D,\ C/D,\ B/D),\\ \operatorname{sgn}(A,\ C,\ B)&=\operatorname{sgn}(A/D,\ C/D,\ B/D). \end{aligned} \tag{2.4}$$

Moreover, we also define

$$\operatorname{sgn}(B,\ A,\ C)=-\operatorname{sgn}(A,\ B,\ C),\ \operatorname{sgn}(C,\ B,\ A)=-\operatorname{sgn}(A,\ B,\ C), \tag{2.5}$$
$$\operatorname{sgn}(C,\ A,\ B)=-\operatorname{sgn}(A,\ C,\ B),\ \operatorname{sgn}(B,\ C,\ A)=-\operatorname{sgn}(A,\ C,\ B).$$

Remark 2.6. Certainly, it is possible that $D^\sigma=D$. (for instance, when among A, B and C at least two of them coincide). In this case, $D^\sigma/D=\{0\}$, hence we naturally regard $k_+-\arg(A,\ B,\ C)$, etc, as zero.

Remark 2.7. By Proposition 2.4, it is in agreement with Definition 2.1 to define sgn $(A,\ B,\ C)$ and $\operatorname{sgn}(A,\ C,\ B)$ by (2.4). Now we explain that it is also right to define $\operatorname{sgn}(B,\ A,\ C)$, $\cdots$ by (2.5). This is not very obvious. For instance, denoting by sgn(B, A, C) the sgn(B, A, C) defined by (2.5) for the time being, the problem, when $A, B\in\mathscr{L}_R$, $C\in\mathscr{L}_+$, is whether $\widetilde{\mathrm{sgn}}(B, A, C)$ equals to sgn(B, A, C) defined by (2.4).

Proposition 2.8. *Let* $A, B\in\mathscr{L}_R$, $C\in\mathscr{L}_+$, *then*

$$\widetilde{\mathrm{sgn}}\,(B, A, C)=\mathrm{sgn}(B, A, C). \tag{2.6}$$

Proof By (2.4), the general problem reduces to the case when A, B, C are mutually transversal. We can choose a real symplectic basis of M such that $(M^c, \sigma)\approx\mathbf{C}^n_x\times\mathbf{C}^n_\xi$, $B\approx\mathbf{C}^n_x$, $A\approx\mathbf{C}^n_\xi$, and moreover

$$C:\ x=C_0\xi,\ \det C_0\neq 0.$$

Then by Definition 2.1, we obtain

$$\begin{aligned}\mathrm{sgn}(A, B, C)&=\mathrm{sgn}(-C_0),\\ \mathrm{sgn}(B, A, C)&=\mathrm{sgn}\,C_0^{-1}.\end{aligned} \tag{2.7}$$

But from (1.2), (1.3) and (1.4), we have

$$\mathrm{sgn}(-C_0)=-\mathrm{sgn}\,C_0^{-1},$$

which, combined with (2.5) and (2.7), gives (2.6).

Proposition 2.9. *For* $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, sgn(A, B, C) *is an altermate function.*

Proof By (2.4) and (2.5), it suffices to prove

$$\mathrm{sgn}(A, C, B)=-\mathrm{sgn}(A, B, C) \tag{2.8}$$

when $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$ are mutually transveral.

We choose such $X\in\mathscr{L}_R$ that X is transversal to A, B, C. And we choose a symplectic basis of M, such that $(M^c, \sigma)\approx\mathbf{C}^n_x\times\mathbf{C}^n_\xi$, $X\approx\mathbf{C}^n_x$, $A\approx\mathbf{C}^n_\xi$, and moreover

$$\begin{aligned}B&:\ x=B_0\xi,\ \det B_0\neq 0,\\ C&:\ x=C_0\xi,\ \det C_0\neq 0,\end{aligned}$$

where $\mathrm{Im}\,B_0\geqslant 0$, $\mathrm{Im}\,C_0\leqslant 0$, and C_0-B_0 is nondegenerate.

It is obvious that

$$\begin{aligned}\mathrm{sgn}(A, B, C)&=\mathrm{sgn}\,B_0(C_0-B_0)^{-1}C_0,\\ \mathrm{sgn}(A, C, B)&=\mathrm{sgn}\,C_0(B_0-C_0)^{-1}B_0.\end{aligned}$$

Hence, noticing that

$$B_0(C_0-B_0)^{-1}C_0=-C_0(B_0-C_0)^{-1}B_0,$$

we obtain (2.8).

Proposition 2.10. *For any real in essence linear subspace* D' *of* $D=A\cap B+B\cap C+C\cap A$, *we have*

$$\begin{aligned}k_+-\arg(A, B, C)&=k_+-\arg(A/D', B/D', C/D'),\\ \mathrm{sgn}(A, C, C)&=\mathrm{sgn}(A/D', B/D', C/D').\end{aligned} \tag{2.9}$$

Here we assume $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$.

Proof Set

$$D''=A/D'\cap B/D'+B/D'\cap C/D'+C/D'\cap A/D'.$$

It is not hard to verify step by step that

$$D''\approx D/D',\ D''^{\sigma}\approx D^{\sigma}/D',\ D''^{\sigma}/D''\approx D^{\sigma}/D,\ (A/D')/D''\approx A/D,$$
$$(B/D')/D''\approx B/D,\ (C/D')/D''\approx C/D.$$

And these evidently imply (2.9).

Definition 2.11. *Let $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, $B\cap C=\{0\}$, and define*

$$\hat{Q}(a,\ a')=\sigma(a',\ P_B a),\ \forall a,\ a'\in A, \tag{2.10}$$

where $P_B\, a$ is the projection of a upon B along C (similarly $P_C\, a$ is the projection of a upon C along B). It is not hard to check that $\hat{Q}(a,\ a')$ is a symmetric bilinear form on A. We shall denote by $\hat{k}_+-\arg(A,\ B,\ C)$ and $\widehat{\operatorname{sgn}}(A,\ B,\ C)$, respectively, the canonical argument and the signature of the matrix associated with Q relative to any real basis of A.

Here something similar to Remark 2.2 applies, we do not explain it in detail.

Proposition 2.12. *Let $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, $B\cap C=\{0\}$, then*

$$\widehat{\operatorname{sgn}}(A,\ B,\ C)=\operatorname{sgn}(A,\ B,\ C). \tag{2.11}$$

Proof (i) The case when $A\cap B=\{0\}$.

For every $a\in A$, Ta in Definition 2.1, is just$-P_B\, a$ in Definition 2.11, so that

$$\hat{Q}(a,\ a')=Q(a,\ a'),\ \forall a,\ a'\in A,$$

Hence (2.11) is true.

(ii) The case when $A\cap B\neq\{0\}$.

We follow the idea in the proof of Proposition 2.4 and adopt the notations in it, but here we replace the assertion $\det C_0\neq 0$ by $\det\ (B_0-C_0)\neq 0$, and assume

$$(B_0-C_0)^{-1}=\left(\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline {}^tQ_{12} & Q_{12} \end{array}\right),$$

where the block partitioning is similar to that of B_0. Then it is easy to see that

$$C=\{f_I Q_{11}+f_{\hat{I}}{}^tQ_{12}+e^I(VQ_{11}-I),\ f_I Q_{12}+f_{\hat{I}}Q_{22}+e^I VQ_{12}-e^{\hat{I}}\}.$$

Hence

$$A/(A\cap B)=\{f_I\},\ B/(A\cap B)=\{f_I+e^I V\},$$
$$C/(A\cap B)=\{e^I(VQ_{11}-I)+f_I Q_{11}\},\ X/(A\cap B)=\{e^I\}.$$

Moreover, we can verify that $B/(A\cap B)$ is transversal to $A/(A\cap B)$ and $C/(A\cap B)$. Thus, by Proposition 2.10 we have

$$\operatorname{sgn}(A,\ B,\ C)=\operatorname{sgn}(A/(A\cap B),\ B/(A\cap B),\ C/(A\cap B))=\operatorname{sgn}(V-VQ_{11}V).$$

But it can be seen that

$$\widehat{\operatorname{sgn}}(A,\ B,\ C)=\operatorname{sgn} B_0(C_0-B_0)^{-1}C_0=\operatorname{sgn}[B_0-B_0(B_0-C_0)^{-1}B_0]$$
$$=\operatorname{sgn}\left(\begin{array}{c|c} V-VQ_{11}V & 0 \\ \hline 0 & 0 \end{array}\right).$$

Hence (2.11) is true.

Based upon Proposition 2.12, we will adopt only the notation $\operatorname{sgn}(A, B, C)$ and not $\widehat{\operatorname{sgn}}(A, B, C)$. It is worth while to notice that when $B\cap C=\{0\}$, to compute $\operatorname{sgn}(A, B, C)$ according to Definition 2.11 is often convenient.

Proposition 2.13. *Let $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, $B\cap C=\{0\}$, then the formula*

$$-\operatorname{sgn}(A, B, C)=\operatorname{sgn}(A, X, B)+\operatorname{sgn}(B, X, C)+\operatorname{sgn}(C, X, A) \tag{2.12}$$

is valid for all $X\in\mathscr{L}_R$ transversal to A, B, C.

Proof We choose a symplectic basis of M such that $(M^c, \sigma)\approx\mathbf{C}^n_x\times\mathbf{C}^n_\xi$, $X\approx\mathbf{C}^n_x$, $A\approx\mathbf{C}^n_\xi$, and moreover

B: $x=B_0\xi$,

C: $x=C_0\xi$,

where B_0, C_0 are $n\times n$ complex symmetric matrices, $\operatorname{Im}B_0\geqslant 0$, $\operatorname{Im}C_0\leqslant 0$, $\det(B_0-C_0)\neq 0$.

By Definition 2.5 (or Proposition 2.12), it is not hard to establish that

$$\operatorname{sgn}(A, B, C)=\operatorname{sgn}B_0(C_0-B_0)^{-1}C_0,$$
$$\operatorname{sgn}(A, X, B)=\operatorname{sgn}(-B_0),$$
$$\operatorname{sgn}(B, X, C)=-\operatorname{sgn}(C_0-B_0)^{-1},$$
$$\operatorname{sgn}(C, X, A)=-\operatorname{sgn}(-C_0).$$

Hence, (2.12) is just the equality

$$-\operatorname{sgn}B_0(C_0-B_0)^{-1}C_0=\operatorname{sgn}(-B_0)-\operatorname{sgn}(C_0-B_0)^{-1}-\operatorname{sgn}(-C_0)$$

and by the definition (1.2) this in turn is the same as

$$\begin{aligned}&k_--\arg(B_0-C_0)^{-1}+k_--\arg(-B_0)\\ &=k_--\arg B_0(B_0-C_0)^{-1}C_0-k_+-\arg(-C_0)-\pi\dim(\ker C_0).\end{aligned} \tag{2.13}$$

Here we have used the equality

$$\dim(\ker B_0(B_0-C_0)^{-1}C_0)=\dim(\ker B_0)+\dim(\ker C_0),$$

which follows easily from the reasoning appeared in the last part of the proof of Proposition 2.12.

(i) The case when $A\cap B=\{0\}$.

In this case, $\det B_0\neq 0$; thus using (1.4) we may write (2.13) as

$$\begin{aligned}&k-\arg(B_0-C_0)^{-1}-k_+-\arg(-B_0^{-1})\\ &=k_--\arg B_0(B_0-C_0)^{-1}C_0-k_+-\arg(-C_0)-\pi\dim(\ker C_0).\end{aligned} \tag{2.14}$$

But by Lemma 1.6 and

$$\left(\begin{array}{c|c}-B_0^{-1} & I_n\\ \hline I_n & -C_0\end{array}\right)^{-1}=\left(\begin{array}{c|c}B_0(B_0-C_0)^{-1}C_0 & B_0(B_0-C_0)^{-1}\\ \hline (B_0-C_0)^{-1}B_0 & (B_0-C_0)^{-1}\end{array}\right),$$

we know that (2.14) is valid.

(ii) The case when $A\cap B\neq\{0\}$.

In this case $\det B_0=0$. Certainly, we may assume that B_0 is non-zero (otherwise what to be proved is clear). Thus, by Corollary 1.4, we may assume with no loss of generality that

$$B_0=\left(\begin{array}{c|c} V & 0 \\ \hline 0 & 0 \end{array}\right),\quad V-r\times r \text{ matrix, } \det V\neq 0.$$

Corresponding to this, we write $(B_0-C_0)^{-1}$ in a similar partitioned form

$$(B_0-C_0)^{-1}=\left(\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline {}^tQ_{12} & Q_{22} \end{array}\right).$$

Then

$$B_0(B_0-C_0)^{-1}=\left(\begin{array}{c|c} VQ_{11} & VQ_{12} \\ \hline 0 & 0 \end{array}\right),\quad (B_0-C_0)^{-1}B_0=\left(\begin{array}{c|c} Q_{11}V & 0 \\ \hline {}^tQ_{n}V & 0 \end{array}\right)$$

$$B_0(B_0-C_0)^{-1}C_0=\left(\begin{array}{c|c} VQ_{11}V-V & 0 \\ \hline 0 & 0 \end{array}\right).$$

And it is not hard to check that

$$\left(\begin{array}{c|c|c|c} -V^{-1} & 0 & I_n & 0 \\ \hline 0 & I_{n-r} & 0 & 0 \\ \hline I_r & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$-C_0$}} \\ \cline{1-2} 0 & 0 & & \end{array}\right)^{-1}=\left(\begin{array}{c|c|c|c} VQ_{11}V-V & 0 & VQ_{11} & VQ_{12} \\ \hline 0 & I_{n-r} & 0 & 0 \\ \hline Q_{11}V & 0 & Q_{11} & Q_{12} \\ \hline {}^tQ_{12}V & 0 & {}^tQ_{12} & Q_{22} \end{array}\right)$$

$$=\left(\begin{array}{c|c|c} VQ_{11}V-V & 0 & \multirow{2}{*}{$B_0(B_0-C_0)^{-1}$} \\ \cline{1-2} 0 & I_{n-r} & \\ \hline \multicolumn{2}{c|}{(B_0-C_0)^{-1}B_0} & (B_0-C_0)^{-1} \end{array}\right).$$

Hence, by Lemma 1.6, we get

$$k_- -\arg\left(\begin{array}{c|c} VQ_{11}V-V & 0 \\ \hline 0 & I_{n-r} \end{array}\right)-k_+ -\arg(-C_0)-\pi\dim\left[\ker\left(\begin{array}{c|c} VQ_{11}V-V & 0 \\ \hline 0 & I_{n-r} \end{array}\right)\right]$$

$$=k_- -\arg(B_0-C_0)^{-1}-k_+ -\arg\left(\begin{array}{c|c} -V^{-1} & 0 \\ \hline 0 & I_{n-r} \end{array}\right),$$

namely

$$\begin{aligned} &k_- -\arg B_0(B_0-C_0)^{-1}C_0-k_+ -\arg(-C_0)-\pi\dim[\ker(VQ_{11}V-V)] \\ &=k_- -\arg(B_0-C_0)^{-1}+k_- -\arg(-B_0). \end{aligned} \tag{2.15}$$

In view of

$$\dim(\ker B_0)+\dim(\ker C_0)=\dim[\ker B_0(B_0-C_0)^{-1}C_0]$$
$$=\dim[\ker(VQ_{11}V-V)]+(n-r),$$
$$\dim(\ker B_0)=n-r,$$

we have

$$\dim[\ker(VQ_{11}V-V)]=\dim(\ker C_0).$$

Substituting it into (2.15), we obtain (2.13).

Proposition 2.14. *Let (M^c, σ) be the complexification of a real symplectic vector space of dimension 2. And let $A\in\mathscr{L}_R$, $B\in\mathscr{L}_-$, $C\in\mathscr{L}_+$, then the formula (2.12) is valid for all $X\in\mathscr{L}_R$ transversal to A, B, C.*

This may be verified directly.

Proposition 2.15. *Let A, B, $X\in\mathscr{L}_R$, $C\in\mathscr{L}_+$, then (2.12) is valid.*

Proof It will be enough to give the proof for X transversal to A, B, C. In fact, if this has been done, by choosing such $Y\in\mathscr{L}_R$ that it is transversal to A, B, C and X, then we shall have immediately

$$\begin{aligned}
&\operatorname{sgn}(A, X, B)+\operatorname{sgn}(B, X, C)+\operatorname{sgn}(C, X, A)=-\operatorname{sgn}(A, Y, X)\\
&\quad-\operatorname{sgu}(X, Y, B)-\operatorname{sgn}(B, Y, A)-\operatorname{sgn}(B, Y, X)-\operatorname{sgn}(X, Y, C)\\
&\quad-\operatorname{sgn}(C, Y, B)-\operatorname{sgn}(C, Y, X)-\operatorname{sgn}(X, Y, A)-\operatorname{sgn}(A, Y, C)\\
&=\operatorname{sgn}(A, Y, B)+\operatorname{sgn}(B, Y, C)+\operatorname{sgn}(C, Y, A)\\
&=-\operatorname{sgn}(A, B, C).
\end{aligned}\tag{2.16}$$

We shall make induction on $n=\frac{1}{2}\dim M$ to finish our proof. When $n=1$, by Proposition 2.14, the assertion is valid. Now we assume that when n equals 1, 2, $\cdots$, $m-1$, the assertion is valid, and prove that it will also be true when $n=m$.

Case I $A\cap B\cap C\neq\{0\}$. We take $D'=A\cap B\cap C$, then from the induction hypothesis and Proposition 2.10, we obtain:

the left side of (2.12) $=-\operatorname{sgn}(A/D', B/D', C/D')=\operatorname{sgn}(A/D', X/D', B/D')+\operatorname{sgn}(B/D', X/D', C/D')+\operatorname{sgn}(C/D', X/D', A/D')=$ the right side of (2.12).

Case II $A\cap B\cap C=\{0\}$, $A\cap B\neq\{0\}$, $B\cap C\neq\{0\}$, $C\cap A\neq\{0\}$. Choosing $Y\in\mathscr{L}_R$ such that it contains the real in essence isotopic subspace $D=A\cap B+B\cap C+C\cap A$ of M^c and a real in essence subspace D' of $A\cap B$, then from the induction hypothesis and Proposition 2.10, we have

$$\begin{aligned}
&-\operatorname{sgn}(A, B, C)=-\operatorname{sgn}(A/D', B/D', C/D')\\
&\quad=\operatorname{sgn}(A/D', Y/D', B/D')+\operatorname{sgn}(B/D', Y/D')+\operatorname{sgn}(C/D', Y/D', A/D')\\
&\quad=\operatorname{sgn}(A, Y, B)+\operatorname{sgn}(B, Y, C)+\operatorname{sgn}(C, Y, A).
\end{aligned}$$

Hence using the result of Case I and imitating the proof of (2.16), we obtain (2.12).

Case III $B\cap C=\{0\}$ or $A\cap C=\{0\}$. By Proposition 2.13 we know that (2.12) is valid.

Case IV $A\cap B=\{0\}$, $B\cap C\neq\{0\}$, $C\cap A\neq\{0\}$. We write $D'=A\cap C$ and choose such $Y\in\mathscr{L}_R$ that $Y\supset D'+B\cap C$ and $Y\cap X=\{0\}$. (By Proposition 2.3 and Theorem

3.4.2 in [3], we know this is possible). By the induction hypothesis and Proposition 2.10, we have

$$\begin{aligned}-\mathrm{sgn}(A, B, C) &= -\mathrm{sgn}(A/D', B/D', C/D')\\ &= \mathrm{sgn}(A/D', Y/D', B/D') + \mathrm{sgn}(B/D', Y/D', C/D') + \mathrm{sgn}(C/D', Y/D', A/D')\\ &= \mathrm{sgn}(A, Y, B) + \mathrm{sgn}(B, Y, C) + \mathrm{sgn}(C, Y, A).\end{aligned} \tag{2.17}$$

Because $Y \cap A \cap C = D' \neq \{0\}$, $Y \cap B \cap C = B \cap C \neq \}0\}$ and X is transversal to A, B, C and Y, using the result of Case I, we get

$$\begin{aligned}-\mathrm{sgn}(C, Y, A) &= \mathrm{sgn}(C, X, Y) + \mathrm{sgn}(Y, X, A) + \mathrm{sgn}(A, X, C), -\mathrm{sgn}(B, Y, C)\\ &= \mathrm{sgn}(B, X, Y) + \mathrm{sgn}(Y, X, C) + \mathrm{sgn}(C, X, B).\end{aligned} \tag{2.18}$$

But in view of $A \cap B = \{0\}$ and Proposition 2.13, we have

$$-\mathrm{sgn}(A, Y, B) = \mathrm{sgn}(A, X, Y) + \mathrm{sgn}(Y, X, B) + \mathrm{sgn}(B, X, A). \tag{2.19}$$

Combining (2.17), (2.18) and (2.19), we obtain (2.12).

After these preparations, we may prove the major theorem in this section.

Theorem 2.16. *Let A, $X \in \mathscr{L}_R$, $B \in \mathscr{L}_-$, $C \in \mathscr{L}_+$, then the generalized Leray's formula* (2.12) *is valid.*

Proof We also use induction on $n = \frac{1}{2}\dim M$. Almost repeating word for word the proof of Proposition 2.15, we may complete the proof in Cases I and II. Now we consider the other cases.

Case III $B \cap C = \{0\}$. By Proposition 2.13, we know that (2.12) is valid.

Case V $B \cap C \neq \{0\}$, $A \cap B = \{0\}$ or $A \cap C = \{0\}$. We write $B \cap C = D'$ and choose $Y \in \mathscr{L}_R$ such that $Y \supset D'$. By the induction hypothesis and Proposition 2.10, we then obtain (2.17). By assumption A, $Y \in \mathscr{L}_R$ and Proposition 2.15, we see that the first equality in (2.18) and (2.19) are valid. Again because $Y \cap B \cap C = D' \neq \{0\}$, by the result of Case I, we see that the last equality in (2.18) is also valid. Combining (2.17), (2.18) and (2.19), we obtain (2.12). The proof of the theorem is completed.

§ 3. A remark on the notion of the almost analytic Maslov line bundles

As an application of the results in the previous two sections, we shall now show that the transition functions of the almost analytic Maslov line bundle on a complex positive conic Lagrangean manifold constructed ingeniously in [4] may be expressed invariantly in terms of what we call the generalized Hörmander cross indices. This serves to bring out the parallelism between the notions of Maslov line bundles in real and complex cases more clearly.

As in § 2, let (M, σ) be a real symplectic symplectic vector space of dimension $2n$, and let M^c be its complexification. And we shall use the notations $\Lambda(M^c)$, $\mathscr{L}_+ =$

$\mathscr{L}_+(M^c)$, $\mathscr{L}_-=\mathscr{L}_-(M^c)$, $\mathscr{L}_R=\mathscr{L}_R(M^c)$, etc., without repeating their definitions.

Fix a Lagrangean plane $\hat{F}\in\mathscr{L}_R$, and for each $L\in\Lambda(M^c)$, let

$$\Lambda^+(L)=\{\Lambda\in\mathscr{L}_+;\ \Lambda\cap L=\{0\}\},$$
$$\Lambda^-(L)=\{\Lambda\in\mathscr{L}_-;\ \Lambda\cap L=\{0\}\}.$$

Definition 3.1. *For every pair (L_1, L_2) of $\mathscr{L}_-$ and every $\Lambda\in\Lambda^+(L_1)\cap\Lambda^+(L_2)$, we define*

$$\sigma_{L_1,L_2}(\Lambda)=\frac{1}{2}[\operatorname{sgn}(\widetilde{F}, L_2, \Lambda)-\operatorname{sgn}(\widetilde{F}, L_1, \Lambda)] \tag{3.1}$$

as the generalized Hörmander cross index of the pairs $(\widetilde{F}, \Lambda)$ and (L_1, L_2).

Proposition 3.2. i) *$\forall L_1, L_2\in\mathscr{L}_-$, $\sigma_{L_1,L_2}(\Lambda)$ is a continuous function of $\Lambda\in\Lambda^+(L_1)\cap\Lambda^+(L_2)$;*

ii) *$\sigma_{L_1,L_2}(\Lambda)$ is continuous with respect to $L_1, L_2\in\Lambda^-(\widetilde{F})$ and $\Lambda\in\Lambda^+(L_1)\cap\Lambda^+(L_2)$.*

Proof i) $\forall\Lambda_0\in\Lambda^+(L_1)\cap\Lambda^+(L_2)$, choose $\widetilde{X}\in\mathscr{L}_R$ such that it intersects $\widetilde{F}$, L_1, L_2 and Λ_0 trans versally. By formula (2.12), we have

$$\sigma_{L_1,L_2}(\Lambda)=\frac{1}{2}[\operatorname{sgn}(\widetilde{F}, L_2, \widetilde{X})-\operatorname{sgn}(\widetilde{F}, L_1, \widetilde{X})+\operatorname{sgn}(\widetilde{X}, L_2, \Lambda)-\operatorname{sgn}(\widetilde{X}, L_1, \Lambda)]. \tag{3.2}$$

But whenever $\Lambda\in\Lambda^+(L_1)\cap\Lambda^+(L_2)$ is close enough to Λ_0, $\widetilde{X}$ will also intersect Λ transversally, so the expression on the right of (3.2) is obviously continuous with respect to $\Lambda\in\Lambda^+(L_1)\cap\Lambda^+(L_2)$ at Λ_0, This completes the proof.

The proof of ii) is similar.

By the way, we point out that, since$\Lambda^+(L)$ and $\Lambda^+(L)\cap\Lambda^+(L')$ are contractible whenever L and $L'\in\mathscr{L}_-$, and the collections $\{\Lambda^+(L),\ L\in\mathscr{L}^-\}$ and $\{\Lambda^+(L);\ L\in\mathscr{L}_R\}$, or even certain finite subcollections of them, all constitute relative open coverings for $\mathscr{L}_+$, the assertion i) of the above proposition has its implication relative to the cohomolgy of $\mathscr{L}_+$. Here $\mathscr{L}^-$, as in [4], denotes the totallity of the strictly negative definite members of $\mathscr{L}_-$.

Now, in order to fix the terminologies and notations, let us repeat the Definition 6.1 of [4].

Definition 3.3. *Let the $\widetilde{F}\in\mathscr{L}_R$ fixed arbitrarily is the complexification of a Lagrangean plane F of M Then for a plane $\Lambda\in\mathscr{L}_+$, we say a basis $e=(e_1, \cdots, e_n)$ for Λ is admissible if there is a basis $f=(f_1, \cdots, f_n)$ in F and a plane $L\in\mathscr{L}^-$ such that e_j is the projection of f_j along L for all j. We write this as*

$$e=E_\Lambda(f, L),$$

and denote by $\mathscr{B}(\Lambda)$ the set of admissible bases, equipped with the product topology from $\Lambda\times\cdots\times\Lambda$ (n-times).

Proposition 3.4. *The unique function S_Λ: $\mathscr{B}(\Lambda)\times\mathscr{B}(\Lambda)\to C\setminus\{0\}$, specified and constructed in Proposition 6.2 of [4], can be expressed as*

$$S_A(e, e') = |e/e'|^{\frac{1}{2}} e^{\frac{\pi i}{2}\sigma_{L,L'}(A)}$$

when $$e = E_A(f, L),\ e' = E_A(f', L'), \tag{3.3}$$

where $$e/e' = (e_1 \wedge \cdots \wedge e_n)/(e'_1 \wedge \cdots \wedge e'_n).$$

Proof Set up symplectic linear coordinates (x, ξ) in M as that in the proof of the Proposition 6.2 in [4], so that F is given by $x=0$, A by $\tilde{x}=A\tilde{\xi}$, L and L' by $\tilde{\xi}=B\tilde{x}$ and $\tilde{\xi}=B'\tilde{x}$, respectively. Then e and e' can be identified with $(I-BA)^{-1}R$, $(I-B'A)R'$, respectively, where R and $R' \in GL(n, R)$; thus

$$e/e' = \det(I-BA)^{-1}R/[\det(I-B'A)R'].$$

Now, notice that.

$$\det(I-BA) = (-1)^n \det\begin{pmatrix} -A & I \\ I & -B \end{pmatrix},$$

$$\begin{pmatrix} -B & I \\ I & -A \end{pmatrix}^{-1} = \begin{pmatrix} A(I-BA)^{-1} & (I-BA)^{-1} \\ (I-BA)^{-1} & (I-BA)^{-1}B \end{pmatrix} \triangleq P;$$

so

$$\det(I-BA)^{-1} = |\det(I-BA)|^{-1}\exp\{in\pi + ik_- - \arg P\}.$$

But, by meas of Lemma 1.6, we have

$$\begin{aligned} k_- - \arg P &= k_- - \arg A(I-BA)^{-1} - k_+ - \arg(-A) - \pi \dim(\ker A) \\ &= k_+ - \arg A(BA-I)^{-1} - k_+ - \arg(-A) - n\pi \\ &= k_+ - \arg[\tilde{F}, L, A) - k_+ - \arg(\tilde{F}, \tilde{X}, A) - n\pi, \end{aligned}$$

where $\tilde{X}$ is the plane $\tilde{\xi}=0$; therefore

$$(\det(I-BA)^{-1})^{1/2} = |\det(I-BA)|^{-1/2} e^{\frac{\pi i}{2}\sigma_{L,L'}(A)}.$$

This and the similar result for $(\det(I-B'A)^{-1})^{1/2}$ yield

$$e/e' = \pm[|e/e'|^{1/2} e^{\frac{\pi i}{2}\sigma_{L,L'}(A)}]^2, \tag{3.4}$$

where the plus sign is valid precisely when R and R', that is f and f', have the same orientation.

We assert that the phase factor $e^{\frac{\pi i}{2}\sigma_{L,L'}(A)}$ is uniquely determined by $e=E_A(f, L)$ and $e'=E_A(f', L')$. First, suppose $e=e'$. Then in view of the homotopy

$$\begin{cases} B_t = (1-t)B + tB',\ \operatorname{Im} B_t < 0 \quad (0 \leqslant t \leqslant 1), \\ R_t = (I-B_tA)(I-B'A)^{-1}R' = (1-t)R + tR' \in GL(n, R^n), \\ \quad (I-BA)^{-1}R = (I-B_tA)^{-1}R_t = (I-B'A)R', \end{cases}$$

we can infer from (3.4) that

$$1 = \pm e^{\frac{\pi i}{2}\sigma_{L(t),L'}(A)},\quad 0 \leqslant t \leqslant 1, \tag{3.5}$$

where $L(t)$ is the plane $\xi = B_t\tilde{x}$ belonging to $\mathscr{L}^-$. But according to Proposition 3.2, the phase factor on the right side of (3.5) is a continuous function of $t \in [0, 1]$, so only the plus sign is correct in (3.5), because this is so when $t=1$. Set $t=0$ in (3.5), we get therefore

$$1=e^{\frac{\pi i}{2}\sigma_{L,L'}(\Lambda)} \text{ if } \exists f, f'\in \mathscr{B}(F) \ s.t. E_\Lambda(f, L)=E_\Lambda(f', L'). \tag{3.6}$$

From this the truth of the assertion follows easily:

$$e^{\frac{\pi i}{2}\sigma_{L,L'}(\Lambda)}=e^{\frac{\pi i}{2}[\sigma_{L^0,L}(\Lambda)+\sigma_{L,L'}(\Lambda)+\sigma_{L',L'^0}(\Lambda)]}=e^{\frac{\pi i}{2}\sigma_{L^0,L'^0}(\Lambda)},$$

if $e=E_\Lambda(f, L)=E_\Lambda(f^0, L^0)$ and $e'=E_\Lambda(f', L')=E_\Lambda(f'^0, L'^0)$,

In short, if we define $S_\Lambda(e, e')$ by (3.3), then it is single valued, and it has all the properties specified in Proposition 6.2 of [4], as can be seen from Proposition 3.2 and (3.4). This concludes the proof.

Since the construction of the transition functions of the almost analytic "Maslov" line bundle on a positive conic Lagrangean manifold in [4] is based solely on the function S_Λ and the notion of admissible coordinates, our Proposition 3.4 already implies what we promised to show at the beginning of this section.

§ 4. A coordinates free description of the Maslov Co-cycle on the positive Lagrange-Grassmann manifold

As before, we denote by (M, σ) a real symplectic vector space of dimension $2n$, and M^c its complexification. We shall also make use of the related notations $\mathscr{L}_\pm=\mathscr{L}_\pm(M^C)$, $\mathscr{L}_R=\mathscr{L}_R(M^C)$ and so forth, as before.

Consider $\{\Lambda^+(L); L\in\mathscr{L}_+(M^C)\}$, where $\Lambda^+(L)=\{\Lambda\in\mathscr{L}_+(M^C); \Lambda\cap L=\{0\}\}$. It is known that there are some finite sub-collections of this, each of which covers $\mathscr{L}_+(M^C)$ completely. By the way, we recall that each $\Lambda^+(L)$ is a relative open and contractible subset of $\mathscr{L}_+(M^C)$.

Fix a plane $\widetilde{F}\in\mathscr{L}_R(M^C)$ arbitrarily, and for every $L_1, L_2\in\mathscr{L}_R(M^C)$ we define

$$m_{L_1,L_2}(\Lambda)=\frac{1}{2}[\operatorname{sgn}(\widetilde{F}, L_2, \Lambda)+\dim(\widetilde{F}\cap L_2)-\operatorname{sgn}(\widetilde{F}, L_1, \Lambda) -\dim(\widetilde{F}\cap L_1)] \text{ for } \Lambda\in\Lambda^+(L_1)\cap\Lambda^+(L_2). \tag{4.1}$$

By Proposition 3.2, this is a real valued continuous function with domain $\Lambda^+(L_1)\cap\Lambda^+(L_2)$, which is also contractible. Thus the collection $\{m_{L_1,L_2}(\Lambda)\}$, or even any one of its sub-collections, for which the corresponding $\{\Lambda^+(L_1), \Lambda^+(L_2)\}$ covers $\mathscr{L}_+(M^C)$, makes up a one dimensional Čech co-cycle on the positive Lagrange-Grassmann manifold $\mathscr{L}_+(M^C)$, which we shall call the Maslov co-cycle. We shall show that this is just the one used implicitly in Maslov's work[5], and [6] and explicitly in [7] and [8].

To do this, choose a symplectic basis $\{e_1, \cdots, e_n; f^1, \cdots, f^n\}$ for (M, σ), so that all $f^j\in\widetilde{F}$, and that we can identify M^C with $T^*(C^n_{\tilde{x}})=C^n_{\tilde{x}}\times C^n_{\tilde{\xi}}$, $\widetilde{F}$ with $C^n_{\tilde{\xi}}$. Then every vector $\nu\in M^C$ will be identified with its corresponding symplectic coordinates $(\tilde{x}, \tilde{\xi})=(\tilde{x}^1, \cdots, \tilde{x}^n; \tilde{\xi}_1, \cdots, \tilde{\xi}_n)$. And for each subset $I\subset\{1, \cdots, n\}$, with its complement denoted by $\bar{I}$, we denote by F_I the coordinate Lagrangean plane $\{\tilde{x}^{\bar{I}}=0, \tilde{\xi}_I=0\}\in \mathscr{L}_R(M^C)$, where $\tilde{x}_{\bar{I}}=(\tilde{x}^{\nu_1}, \cdots, \tilde{x}^{\nu_{n-k}})$, $\xi_I=(\tilde{\xi}_{\mu_1}, \cdots, \tilde{\xi}_{\mu_k})$, whenever $I=(\mu_1, \cdots, \mu_k)$, $\bar{I}=(\nu_1, \cdots, \nu_{n-k})$.

Cosider the collection $\{U_J=\Lambda^+(F_I)\}$ with I ranges over all subsets of $\{1, \cdots, n\}$, including the empty one. It is known that $\{U_I\}$ covers $\mathscr{L}_+(M^C)$.

For every $\Lambda\in U_I\cap U_J$, because each of the $(\tilde{x}^{\bar{I}}, \tilde{\xi}_I)$ and $(\tilde{x}^{\bar{J}}, \tilde{\xi}_J)$ can be taken as linear coördinates on Λ, the following are three symmetric complex matrices with their imaginary parts non-negative definite associated with Λ:

$$P_I(\Lambda)=-\frac{\partial\tilde{x}^{\bar{I}}}{\partial\tilde{\xi}_I}\Big|_\Lambda,\quad P_J(\Lambda)=-\frac{\partial\tilde{x}^{\bar{J}}}{\partial\tilde{\xi}_J},\quad P_{I,J}(\Lambda)=\frac{\partial(\tilde{x}^{I_3}, \tilde{\xi}_{J_3})}{\partial(\tilde{\xi}_{I_2}, \tilde{x}^{I_3})}\Big|_\Lambda,$$

$$(I_1=I\cap J,\ I_2=I\cap\bar{J},\ I_3=\bar{I}\cap J,\ I_4=\bar{I}\cap\bar{J}). \tag{4.2}$$

Computing by the methods developed in § 2, it is not hard to find out

$$\begin{cases} k_+-\arg(\tilde{F}, F_I, \Lambda)=k_+-\arg P_I(\Lambda)\\ k_+-\arg(\tilde{F}, F_J, \Lambda)=k_+-\arg P_J(\Lambda)\\ \operatorname{sgn}(\tilde{F}, F_I, \Lambda)=\operatorname{sgn}P_I(\Lambda),\\ \operatorname{sgn}(\tilde{F}, F_J, \Lambda)=\operatorname{sgn}P_J(\Lambda),\\ \operatorname{sgn}(\tilde{F}, F_I, F_J)=0,\ \operatorname{sgn}(F_J, F_I, \Lambda)=\operatorname{sgn}P_{I,J}(\Lambda). \end{cases} \tag{4.3}$$

And applying the generalized Leray formula (2.12), we get

$$\operatorname{sgn}(\tilde{F}, F_I, \Lambda)=\operatorname{sgn}(\tilde{F}, F_J, \Lambda)+\operatorname{sgn}(F_J, F_I, \Lambda)+\operatorname{sgn}(\tilde{F}, F_I, F_J). \tag{4.4}$$

Now we are ready to state the following result.

Theorem 4.1. i) *We have*

$$m_{F_I,F_J}(\Lambda)=\frac{1}{\pi}k_+-\arg P_{I,J}(\Lambda)-|I_3|,\ \forall\Lambda\in U_I\cap U_J. \tag{4.5}$$

And hence the collection of functions shown on the right side really constitutes a one dimensional Čech Co-cycle with coefficients in the sheaf of germs of real valued continuous functions on $\mathscr{L}_+(M^C)$.

ii) *We also have*

$$\frac{1}{\pi}k_--\arg\frac{\partial\tilde{x}^{\bar{I}}}{\partial\tilde{\xi}_I}\Big|_\Lambda-\frac{1}{\pi}k_--\arg\frac{\partial\tilde{x}^{\bar{J}}}{\partial\tilde{\xi}_J}\Big|_\Lambda=k_+-\arg P_{I,J}(\Lambda)-|I_2|$$

$$=m_{F_I,F_J}(\Lambda)+|I_3|-|I_2|,\ \forall\Lambda\in U_I\cap U_J. \tag{4.6}$$

Thus the collection of functions shown on the left side also constitutes a one dimensional Čech Co-cycle of the same kind as above.

Proof i) From (4.3), (4.4), and Definition (4.1), we see immediately that

$$m_{F_I,F_J}(\Lambda)=\frac{1}{2}[|\bar{J}|-|\bar{I}|-\operatorname{sgn}P_{I,J}(\Lambda)]$$

$$=\frac{1}{2}\Big[|I_2|-|I_3|-|I_2|-|I_3|+\frac{2}{\pi}k_+-\arg P_{I,J}(\Lambda)\Big]$$

$$=\frac{1}{\pi}k_+-\arg P_{I,J}(\Lambda)-|I_3|.$$

Since $P_{I,J}(\Lambda)$ is invertible, when $\Lambda\in U_I\cap U_J$, we have

$$\operatorname{sgn} P_{I,J}(\Lambda)=|I_2|+|I_3|-\frac{2}{\pi}k_+-\arg P_{I,J}(\Lambda).$$

ii) Note

$$\operatorname{sgn}(\widetilde{F},\ F_I,\ \Lambda)=n-\dim\widetilde{F}\cap F_I-\dim\widetilde{F}\cap\Lambda-\frac{2}{\pi}k_+-\arg(\widetilde{F},\ F_I,\ \Lambda),$$

$$\operatorname{sgn} P_I(\Lambda)=|I|-\dim(\ker P_I(\Lambda))-\frac{2}{\pi}k_+-\arg P_I(\Lambda),$$

so in view of (4.3) we have

$$\dim\ker(P_I\Lambda)=|I|+|\bar{I}|-n+\dim\widetilde{F}\cap\Lambda=\dim\widetilde{F}\cap\Lambda,$$

and

$$\operatorname{sgn}(\widetilde{F},\ F_I,\ \Lambda)=-|I|+\dim\widetilde{F}\cap\Lambda-\frac{2}{\pi}k_--\arg(-P_I(\Lambda)).$$

Similarly

$$\operatorname{sgn}(\widetilde{F},\ F_J,\ \Lambda)=-|J|+\dim\widetilde{F}\cap\Lambda-\frac{2}{\pi}k_--\arg(-P_J(\Lambda)),$$

and therefore

$$m_{F_I,F_J}(\Lambda)=\frac{1}{\pi}k_--\arg(-P_I(\Lambda))-\frac{1}{\pi}k_--\arg(-P_J(\Lambda))+|I_2|-|I_3|.$$

Comparing this with (4.5) yields (4.6). The proof is completed.

To end up, we should like to point out that the proof of the second half of the assertion i) given in [7] and [8] seemed incomplete. And it was the motivation of giving an invariant formulation, and hence an accurate proof, that led one of the authors [1] to consider the generalized Duistermaat Lemma, i. e. Lemma 1.6 in §1.

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