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# A NOTE ON THE RATES OF ASYMPTOTIC NORMALITY OF LINEAR PERMUTATION STATISTICS

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**Abstract**

In his pervious paper[3], Prof. Chen Xiru proposed a question: can the rate of asymptotic normality of linear permutation statistics be arbitrarily slow when one of $\{A_n\}$ and $\{B_n\}$ satisfies the condition $W$ and the other satisfies the condition $N$? This note shows that it can. Moreover, a wrong statement about this problem is corrected.

## § 1. Introduction

Let $A_n=\{a_{n1}, \cdots, a_{nN_n}\}$ and $B_n=\{b_{n1}, \cdots, b_{nN_n}\}$, $n=1, 2, \cdots$, be two sequences of real vectors, and $\{R_{n1}, \cdots, R_{nN_n}\}$ be a random vector with a uniform distribution over the set of all permutations of $(1, 2, \cdots, N_n)$, where $\lim\limits_{n\to\infty} N_n=\infty$。 Call

$$L_n=\sum_{i=1}^{N_n} b_{ni}\, a_{nR_{ni}} \tag{1}$$

the linear permutation statistics generated by $A_n$ and $B_n$. Define

$$\bar{a}_n=\frac{1}{N_n}\sum_{i=1}^{N_n} a_{ni}$$

$$\mu_r(A_n)=\frac{1}{N_n}\sum_{i=1}^{N_n}(a_{ni}-\bar{a}_n)^r,\ n=1, 2, \cdots, \text{ and } r=2, 3, \cdots,$$

and similarly for $\bar{b}_n$, $\mu_r(B_n)$. We have

$$\lambda_n=EL_n=N_n\bar{a}_n\bar{b}_n,\quad \sigma_n^2=\operatorname{Var} L_n=\frac{N_n^2}{N_n-1}\mu_2(A_n)\mu_2(B_n).$$

We say that $\{A_n\}$ satisfies the condition $W$ or $N$, if

$$\sup_n \mu_r(A_n)/[\mu_2(A_n)]^{r/2}\leqslant M<\infty, \text{ for each integer } r\geqslant 3 \tag{2}$$

or

$$\lim_{n\to\infty} N_n^{-\frac{r}{2}+1}\mu_r(A_n)/[\mu_2(A_n)]^{r/2}=0, \text{ for each } r\geqslant 3 \tag{3}$$

respectively, where $M$ is iudependent of $r$. Likewise for $\{B_n\}$.

The well-known theorem of Wald-Wolfowitz-Noether [1, 2] guarantees the

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asymptotic normality of $(L_n-\lambda_n)/\sigma_n$ under the condition that one of $\{A_n\}$ and $\{B_n\}$ satisfies the condition $W$ while the other satisfies the condition $N$. Recently, Chen [3] considered the rate of this convergence and obtained some results. Among the results in Chen's paper, there is an attempt to establish the fact that the above-mentioned convergence rate can be arbitrarily slow when $\{A_n\}$ and $\{B_n\}$ both satisfy the condition N. But in reality, his result does not establish this fact for obvious reason. One aim of this note is to establish the correct result (Theorem 2 of the present paper. Note the difference between our Theorem 2 and Theorem 4 of [3]). Chen also proposed the question: does this result remain true in the case that one of $\{A_n\}$ and $\{B_n\}$ satisfies the condition $W$ and the other satisfies the condition $N$? The second aim of this note is to answer affirmatively this question (Theorem 2, too).

## § 2. The Main Results and Proofs

Choose a sequence of even positive integers $\{N_n\}$ with $\lim\limits_{n\to\infty} N_n=\infty$, and a sequence of strictly decreasing positive numbers $\{q_n\}$, such that

$$\lim_{n\to\infty} q_n^2 N_n^\alpha=\infty \tag{4}$$

and

$$\lim_{n\to\infty} q_n=0, \tag{5}$$

where $0<\alpha<1/6$ is a constant. Without losing generality we assume that $q_n<\frac{1}{2}$ for all $n\geqslant 1$. Write $p_n=\frac{1}{2}-q_n$. Define

$$A_n=\begin{cases} a_{ni}=1/D_n & \text{for } i=1,\ 2,\ \cdots,\ [p_nN_n] \\ a_{ni}=-1/D_n & \text{for } i=[p_nN_n]+1,\ \cdots,\ 2[p_nN_n] \\ a_{ni}=0 & \text{for } i=2[p_nN_n]+1,\ \cdots,\ N_n-2 \\ a_{nN_n-1}=q_n^{\frac{1}{2}}N_n^{\frac{1}{2}}/D_n=-a_{nN_n}, & \end{cases} \tag{6}$$

$$B_n=\begin{cases} b_{ni}=N_n^{-\frac{1}{2}} & \text{for } i=1,\ 2,\ \cdots,\ \frac{1}{2}N_n \\ b_{ni}=-N_n^{-\frac{1}{2}} & \text{for } i=\frac{1}{2}N_n+1,\ \cdots,\ N_n, \end{cases} \tag{7}$$

where

$$D_n^2=N_n^{-1}(2[p_nN_n]+2q_nN_n)=1-2\theta_n/N_n,\ 0\leqslant\theta_n\leqslant 1. \tag{8}$$

It is not difficult to verify that $\{A_n\}$ satisfies the condition $N$ with $\bar{a}_n=0$ and $\mu_2(A_n)=1$ while $\{B_n\}$ satisfies the condition $W$ with $\bar{b}_n=0$, $\mu_2(B_n)=1/N_n$.

Denote by $L_n$ the linear permutation statistic generated by $A_n$ and $B_n$ and denote by $F_n$ the distribution of $L_n/\sqrt{\operatorname{Var} L_n}$.

**Theorem 1.** *Under the above notations we have*

(i) $F_n \xrightarrow{c} \Phi$, *as* $n\to\infty$,

(ii) $\|F_n-\Phi\| \geqslant c\, q_n^2$, *for n large enough, where c is a positive constant independent of n.*

*Proof* Assertion (i) follows from Noether's Theorem. For the proof of (ii), set

$$A_n^* = \begin{cases} a_{ni}^* = a_{ni} \text{ for } i=1, 2, \cdots, N_n-2, \\ a_{ni}^* = 0 \quad \text{for } i=N_n-1, N_n, \end{cases} \tag{9}$$

and $L_n^* = \sum_{i=1}^{N_n} b_{ni} a_{nR_{ni}}^*$. It is easy to verify that $\{A_n^*\}$ satisfies the condition $W$. Denote by $F_n^*$ the distribution of $L_n^*$. By Theorem 1 in [3]

$$\begin{aligned} \sup_x |F_n^*(x) - \Phi(x/\sqrt{2p_n})| &\leqslant \sup_x |P(L_n^*/\sqrt{\operatorname{Var} L_n^*} \leqslant x/\sqrt{\operatorname{Var} L_n^*}) - \Phi(x/\sqrt{\operatorname{Var} L_n^*})| \\ &\quad + \sup_x |\Phi(x/\sqrt{\operatorname{Var} L_n^*}) - \Phi(x/\sqrt{2p_n})| \\ &= o(N_n^{-\alpha}) + \sup_x |\Phi(x/\sqrt{\operatorname{Var} L_n^*}) - \Phi(x/\sqrt{2p_n})|. \end{aligned} \tag{10}$$

Since $\operatorname{Var} L_n^* = \frac{1}{N_n-1}(2[p_nN_n]/D_n^2) = 2p_n(1+O(1/N_n))$, the second term on the right hand side of (10) is equal to $O(1/N_n)$. Hence

$$\sup_x |F_n^*(x) - \Phi(x/\sqrt{2p_n})| = o(N_n^{-\alpha}). \tag{11}$$

Lot $Q_n = L_n - L_n^*$. By the definition of $A_n$, $A_n^*$ and $B_n$, we see that $Q_n$ assumes only three possible values: 0 and $\pm\ 2q_n^{\frac{1}{2}}/D_n$. Thus we have

$$\begin{aligned} P(L_n\leqslant x) &= P(L_n\leqslant x,\ Q_n=0) + P(L_n\leqslant x,\ Q_n=-2q_n^{\frac{1}{2}}/D_n) + P(L_n\leqslant x,\ Q_n=2q_n^{\frac{1}{2}}/D_n) \\ &= P(L_n^*\leqslant x,\ Q_n=0) + P(L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n,\ Q_n=-2q_n^{\frac{1}{2}}/D_n) \\ &\quad + P(L_n^*\leqslant x-2q_n^{\frac{1}{2}}/D_n,\ Q_n=2q_n^{\frac{1}{2}}/D_n) = I_1+I_2-I_3, \end{aligned} \tag{12}$$

where

$$\begin{cases} I_1 = P(L_n^*\leqslant x), \\ I_2 = P(x<L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n,\ Q_n=-2q_n^{\frac{1}{2}}/D_n) \\ I_3 = P(x-2q_n^{\frac{1}{2}}/D_n<L_n^*\leqslant x,\ Q_n=2q_n^{\frac{1}{2}}/D_n). \end{cases} \tag{13}$$

Applying Eq. (11), we have

$$I_1 = \Phi(x/\sqrt{2p_n}) + o(N_n^{-2}). \tag{14}$$

Furthermore

$$I_2 = P(x<L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n \mid Q_n=-2q_n^{\frac{1}{2}}/D_n)\, P(Q_n=-2q_n^{\frac{1}{2}}/D_n). \tag{15}$$

It is easy to compute that

$$P(Q_n=-2q_n^{\frac{1}{2}}/D_n) = \left(\frac{1}{2}N_n\right)^2 \Big/ N_n(N_n-1) = \frac{1}{4}+O(1/N_n). \tag{16}$$

On the other hand, we can split the event $(Q_n=-2q_n^{\frac{1}{2}}/D_n)$ into $\left(\frac{1}{2}N_n\right)^2$ mutual-disjoint events $E_{ij} = (R_{nN_{n-1}}=i,\ R_{nN_n}=j)$, $i=\frac{1}{2}N_n+1, \cdots, N_n$; $j=1, \cdots, \frac{1}{2}N_n$.

It is easy to see that

$$P(x<L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n|E_{ij})=P(x\leqslant L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n|E_{N_n,1})$$

for $i=\frac{1}{2}N_n+1, \cdots, N_n$; $j=1, \cdots, \frac{1}{2}N_n$. Hence

$$\begin{aligned}P(x<L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n|Q_n=-2q_n^{\frac{1}{2}}/D_n)\\ =\sum_{i=\frac{1}{2}N_n+1}^{N_n}\sum_{j=1}^{\frac{1}{2}N_n}P(x<L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n|E_{ij})P(E_{ij})/P(Q_n=-2q_n^{\frac{1}{2}}/D_n)\\ =P(x<L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n|E_{N_n,1}).\end{aligned} \tag{17}$$

Set

$$\tilde{A}_n=\{\tilde{a}_{ni}=a_{ni}, \quad \text{for } i=1, 2, \cdots, N_n-2\},$$

$$\tilde{B}_n=\begin{cases}\tilde{b}_{ni}=N_n^{-\frac{1}{2}}, & \text{for } i=1, 2, \cdots, \frac{1}{2}N_n-1\\ \tilde{b}_{ni}=-N_n^{-\frac{1}{2}}, & \text{for } i=\frac{1}{2}N_n, \cdots, N_n-2,\}\end{cases}$$

and let $\tilde{L}_n=\sum_{i=1}^{N_n-2}\tilde{b}_{ni}\tilde{a}_{n\tilde{R}_{ni}}$, where $(\tilde{R}_{n1}, \cdots, \tilde{R}_{nN_n-2})$ is a random vector, each of the $(N_n-2)!$ permutations of $(1, 2, \cdots, N_n-2)$ being with the equal probability $1/(N_n-2)!$. Similarly, as in the derivation of Eq. (11), we can prove that

$$P(x<\tilde{L}_n\leqslant x+2q_n^{\frac{1}{2}}/D_n)=\Phi((x+2q_n^{\frac{1}{2}})/\sqrt{2p_n})-\Phi(x/\sqrt{2p_n})+o(N_n^{-\alpha}). \tag{18}$$

It is easy to see that

$$P(x<\tilde{L}_n\leqslant x+2q_n^{\frac{1}{2}}/D_n)=P(x<L_n^*\leqslant x+2q_n^{\frac{1}{2}}/D_n|E_{N_n,1}).$$

From this relation and Eq. (15), (16), (17), (18), we oatain

$$I_2=\frac{1}{4}(\Phi((x+2q_n^{\frac{1}{2}})/\sqrt{2p_n})-\Phi(x/\sqrt{2p_n}))+o(N_n^{-\alpha}). \tag{19}$$

In a similar way we can derive that

$$I_3=\frac{1}{4}(\Phi(x/\sqrt{2p_n})-\Phi((x-2q_n^{\frac{1}{2}})/\sqrt{2p_n}))+o(N_n^{-\alpha}). \tag{20}$$

From (12), (13), (14), (19), (20) we obtain

$$\begin{aligned}P(L_n\leqslant x)&=\Phi(x/\sqrt{2p_n})+\frac{1}{4}(\Phi((x+2q_n^{\frac{1}{2}})/\sqrt{2p_n})-\Phi(x/\sqrt{2p_n}))\\ &\quad-\frac{1}{4}(\Phi(x/\sqrt{2p_n})-\Phi((x-2q_n^{\frac{1}{2}})/\sqrt{2p_n}))+o(N_n^{-\alpha}))\\ &=\Phi(x)+\Delta_1+\Delta_2+o(N_n^{-\alpha}),\end{aligned} \tag{21}$$

where

$$\Delta_1=\Phi(x/\sqrt{2p_n})-\Phi(x),$$

$$\Delta_2=\frac{1}{4}\{\Phi((x+2q_n^{\frac{1}{2}})/\sqrt{2p_n})-2\Phi(x/\sqrt{2p_n})+\Phi((x-2q_n^{\frac{1}{2}})/\sqrt{2p_n})\}.$$

First we observe that

$$\Delta_1=\int_x^{x/\sqrt{2p_n}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt=\int_{-x\left(\frac{1}{\sqrt{2p_n}}-1\right)}^{0}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(t+\frac{x}{\sqrt{2p_n}}\right)^2\right\}dt$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{4p_n}\right\}\int_0^{x\left(\frac{1}{\sqrt{2p_n}}-1\right)}\exp\left\{-\frac{1}{2}t^2+\frac{xt}{\sqrt{2p_n}}\right\}dt$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{4p_n}\right\}\int_0^{x\left(\frac{1}{\sqrt{2p_n}}-1\right)}\left(1+\frac{xt}{\sqrt{2p_n}}+O(t^2)\right)dt$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{4p_n}\right\}\left[x\left(\frac{1}{\sqrt{2p_n}}-1\right)+\frac{x^3}{2\sqrt{2p_n}}\left(\frac{1}{\sqrt{2p_n}}-1\right)^2+O\left(\left(\frac{1}{\sqrt{2p_n}}-1\right)^3\right)\right].$$

Since $1/\sqrt{2p_n}=1+q_n+\frac{3}{2}q_n^2+O(q_n^3)$, we have

$$\Delta_1=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{4p_n}\right\}\left[x\,q_n+\frac{3}{2}x\,q_n^2+\frac{x^3}{2}q_n^2\right]+o(q_n^2). \tag{22}$$

Secondly, we observe that

$$\Delta_2=\frac{1}{4\sqrt{2\pi}}\left\{\int_{x/\sqrt{2p_n}}^{(x+2q_n^{\frac{1}{2}})/\sqrt{2p_n}}-\int_{(x-2q_n^{\frac{1}{2}})/\sqrt{2p_n}}^{x/\sqrt{2p_n}}\right\}e^{-\frac{1}{2}t^2}dt$$

$$=\frac{1}{4\sqrt{2\pi}}\left\{\int_0^{2q_n^{\frac{1}{2}}/\sqrt{2p_n}}\left(\exp\left(-\frac{1}{2}(t+x/\sqrt{2p_n})^2\right)-\exp\left(-\frac{1}{2}(t-x/\sqrt{2p_n})^2\right)dt\right\}\right.$$

$$=\frac{1}{4\sqrt{2\pi}}\exp\left\{-\frac{x^2}{4p_n}\right\}\int_0^{2q_n^{\frac{1}{2}}/\sqrt{2p_n}}\left[-\frac{2xt}{\sqrt{2p_n}}+\frac{xt^3}{\sqrt{2p_n}}-\frac{1}{3}\left(\frac{xt}{\sqrt{2p_n}}\right)^3+O(t^4)\right]dt\Big\}$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{4p_n}\right\}\left\{-x\,q_n(2p_n)^{-\frac{3}{2}}+x\,q_n^2(2p_n)^{-\frac{5}{2}}-\frac{x^3}{3}q_n^2(2p_n)^{-\frac{7}{2}}\right\}+o\,(q_n^{\frac{5}{2}})$$

$$=\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{4p_n}\right\}\left\{-x\,q_n-2x\,q_n^2-\frac{x^3}{3}q_n^2\right\}o+(q_n^2). \tag{23}$$

Combining (21), (22), (23) and noticing the fact that $e^{-\frac{x^2}{4p_n}}=e^{-\frac{x^2}{2}}(1+O(1))$, we get

$$P(L_n\leqslant x)=\Phi(x)-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\left(\frac{1}{2}x-\frac{1}{3}x^3\right)q_n^2+o(q_n^2).$$

Since Var $L_n=\frac{N_n^2}{(N_n-1)}\mu_2(A_n)\mu_2(B_n)=N_N/(N_N-1)=1+O(1/N_n)$, we get

$$F_n(x)=P(L_n\leqslant x\,\mathrm{Var}\,L_n)=\Phi(x)-\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\left(\frac{1}{2}x-\frac{1}{3}x^3\right)q_n^2+o(q_n^2).$$

Inserting $x=1$, we get

$$\|F_n-\Phi\|\geqslant|F(1)-\Phi(1)|\geqslant\frac{1}{6\sqrt{2\pi e}}q_n^2+o(q_n^2).$$

From this and assumption (5), assertion (ii) follows and the proof is concluded.

**Theorem 2.** *For an arbitrarily given non-increasing positive-valued function $\varphi(x)$, $\varphi(x)\to0$ as $x\to\infty$, there exist $\{A_n\}$ and $\{B_n\}$ satisfying the conditions $N$ and $W$*

*respectively, such that the distribution $F_n$ of $(L_n-\lambda_n)/\sigma_n$ convergescto $\Phi$, but for $n$ large enough*

$$\|F_n-\Phi\|\geqslant\varphi(N_n).$$

*Proof* Choose arbitrarily a sequence of positive even integers $N_n$ such that $N_n\to\infty$ as $n\to\infty$. Then $\varphi(N_n)$ is a sequence of positive numbers with $\varphi(N_n)\to 0$. Choose a sequence of positive numbers $\{q_n\}$ such that (1) $q_n^2\geqslant 12\sqrt{2\pi e}\,\varphi(N_n)$ for all large $n$, (2) $q_n^2N_n^\alpha\to\infty$ as $n\to\infty$, (3) $q_n\to 0$ as $n\to\infty$.

Then $\{A_n\}$ and $\{B_n\}$, constructed according to Theorem 1, satisfying the conditions $N$ and $W$ respectively, are such that

$$\|F_n-\Phi\|\geqslant\frac{1}{12\sqrt{2\pi e}}\,q_n^2\geqslant\varphi(N_n) \text{ for all large } n,$$

which proves Theorem 2.

In Theorem 1, if we construct $A_n$ and $B_n$ with $q_n=N_n^{-2c}$, where $c\in\left(0,\ \frac{1}{2}\right)$ is a constant, then $\{A_n\}$ satisfies a condition stronger than the condition $N$, i. e.

$$\mu_r(A_n)/[\mu_2(A_n)]^{r/2}=O(N_n^{(\frac{1}{2}-c)r-1}). \tag{24}$$

We shall say that $\{A_n\}$ satisfies condition $N_c$ if (24) holds. In this case we have $\|F_n-\Phi\|\geqslant CN_n^{-4c}$. Therefore we get

**Theorem 3.** *For each $c\in(0,\ 1/24)$, there exist $\{A_n\}$ satisfying the condition $N_c$ and $\{B_n\}$ satisfying the condition $W$, such that*

$$\|F_n-\Phi\|\geqslant CN_n^{-4c},$$

*where $F_n$ is the distribution of $(L_n-\lambda_n)/\sigma_n$ which is the standardized linear permutation statistics generated by $\{A_n\}$ and $\{B_n\}$, while $C$ is a constant independent of $n$.*

**Remark 1.** If the bound $M$ in (2) is dependent of $r$, we say that $\{A_n\}$ satisfies the condition $W^*$. In Theorem 1, choose $N_n$ being $6n$ set

$$B_n=\begin{cases} b_{ni}=1/\sqrt{3n} \text{ if } i=1,\ 2,\ \cdots,\ 2n,\\ b_{ni}=\ -1/2\sqrt{3n} \text{ if } i=2n+1,\ \cdots,\ 6n,\end{cases}$$

and define $\{A_n\}$ as before, then we can prove $\|F_n-\Phi\|\geqslant Cq_n^{3/2}$.

Hence we have

**Theorem 3′** *If $c\in(0,\ 1/18)$ is a constant, then there exist $\{A_n\}$ satisfying the condition $N_c$ and $\{B_n\}$ satisfying the condition $W^*$, such that $\|F_n-\Phi\|\geqslant CN_n^{-3c}$.*

**Remark 2.** Since the estimate (29) in [3] is not true, we cannot know whether the assertion of Theorem 3 in [3] is true. If so, Theorem 3 and Theorem 3′ would be valid also for $c\in(0,\ 1/8)$ and $c\in(0,\ 1/6)$, respectively.

## References

[1] Wald, A. & Wolfowitz, J., *Ann. Math. Stat.*, **15** (1944), 358—372.
[2] Noether, G. E., *Ann. Math. Stat.*, **20** (1949), 455—558.
[3] 陈希孺，应用数学学报, **4**:4 (1981), 342—355.