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GLOBAL SOLVABILITY IN THE WHOLE SPACE FOR A CLASS OF FIRST ORDER QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract

For reducible quasilinear hyperbolic systems and for general quasilinear hyperbolic systems of diagonal form in two independent variables, some sufficient conditions satisfied by the system itself are given to guarantee the existence of nontrivial global smooth solutions in the whole (t, x) space.

§ 1. Introduction and Principal Results

It is well known that, for the Cauchy problem for first order quasilinear hyperbolic systems, in general the solution may occur singularities in a finite time, even if the initial data are smooth. However, some additional conditions on the system and on the initial data have been obtained to guarantee the existence of global smooth solutions on $t \geqslant 0$ or to guarantee the development of singularities in a finite time (cf. [1, 2] and their references). In this paper we consider instead of the Cauchy problem the global solvability in the whole (t, x) space for first order quasilinear hyperbolic systems. That is, we are going to determine the conditions satisfied by the system itself in order that the system admits global smooth solutions in the whole (t, x) space (GSSWS).

In the case of a single quasilinear equation, a necessary and sufficient condition has been given in [3, 4] for the existence of GSSWS. In particular, the equation

$$\frac{\partial U}{\partial t}+\lambda(U)\frac{\partial U}{\partial x}=0 \tag{1.1}$$

possesses nontrivial (i. e. $\not\equiv$ constant) bounded GSSWS iff there exists an interval $[U_1, U_2]$ such that

$$\lambda'(U)\equiv 0, \qquad \forall\, U\in[U_1, U_2]. \tag{1.2}$$

That is, equation (1.1) must be essentially linear on the domain under consideration.

This paper will deal with the same kind of problems for first order quasilinear

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hyperbolic systems in two variables t and x. For a reducible hyperbolic system, introducing the Riemann invariants r and s as new unknown functions, we can reduce the system to the following diagonal form

$$\begin{cases} \frac{\partial r}{\partial t}+\lambda(r, s)\frac{\partial r}{\partial x}=0, \\ \frac{\partial s}{\partial t}+\mu(r, s)\frac{\partial s}{\partial x}=0, \end{cases} \tag{1.3}$$

where we suppose that on the domain under consideration λ and μ are smooth and system (1.3) is strictly hyperbolic:

$$\lambda(r, s)>\mu(r, s). \tag{1.4}$$

We restrict ourselves here to nontrivial GSSWS, here "nontrivial" means that any unknown function is not equal to a constant. otherwise, the conclusion will be trivial (when the unknown function are all equal to constants) or we can reduce the original problem to a simpler problem in which equations and unknown functions have decreased in number (when a part of unknown functions are equal to constants).

Using the results in [1] (Theorems 2 and 3), it is easy to get the following two Theorems.

Theorem 1. *If one characteristic value (say, λ) of system (1.3) is genuinely nonlinear in the sense of P. D. Lax:*

$$\frac{\partial \lambda}{\partial r}\neq 0 \tag{1.5}$$

on the domain under consideration, then for any bounded GSSWS of system (1.3) (if it exists) it always holds that

$$r\equiv\text{const.} \tag{1.6}$$

This is a trivial case in our discussion.

Theorem 2. *If the characteristic values λ and μ are of the following special form*

$$\begin{cases} \lambda(r, s)=\lambda_1(r)+\lambda_2(s), \\ \mu(r, s)=\mu_1(r)+\mu_2(s), \end{cases} \tag{1.7}$$

then system (1.3) admits nontrivial bounded GSSWS iff there exist intervals $[r_1, r_2]$ and $[s_1, s_2]$ such that

$$\begin{cases} \lambda_1'(r)\equiv 0, \ \forall r\in[r_1, r_2], \\ \mu_2'(s)\equiv 0, \ \forall s\in[s_1, s_2]. \end{cases} \tag{1.8}$$

The main aim of this paper is to prove the following

Theorem 3. *Suppose that there exist intervals $[r_1, r_2]$ and $[s_1, s_2]$ such that*

$$\lambda(\bar{r}, \bar{s})>\mu(\bar{\bar{r}}, \bar{\bar{s}}), \ \forall \bar{r}, \bar{\bar{r}}\in[r_1, r_2], \ \bar{s}, \bar{\bar{s}}\in[s_1, s_2]. \tag{1.9}$$

Suppose further that there exist r_0, s_0 such that

$$r_0\in[r_1, r_2], \ s_0\in[s_1, s_2] \tag{1.10}$$

and

$$\begin{cases} \frac{\partial \lambda}{\partial r}(r, s_0) \equiv 0, \ \forall r \in [r_1, r_2], \\ \frac{\partial \mu}{\partial s}(r_0, s) \equiv 0, \ \forall s \in [s_1, s_2]. \end{cases} \tag{1.11}$$

Then system (1.3) *admits nontrivial bounded GSSWS.*

It is well known (cf. [1]) that if system (1.3) is linearly degenerate in the sense of P. D. Lax:

$$\frac{\partial \lambda}{\partial r} \equiv 0, \quad \frac{\partial \mu}{\partial s} \equiv 0 \tag{1.12}$$

on the domain under consideration, then for arbitrary initial data with bounded C^1 norms, system (1.3) always possesses a unique GSSWS. Theorem 3 shows that in order to get the existence of nontrivial GSSWS, system (1.3) is not necessary to be linearly degenerate. We can also find a concrete example in [5] to explain this fact. Besides, from Theorem 2 it follows easily that condition (1.11) in Theorem 3 is necessary under assumption (1.7), but it is still a question if condition (1.11) is also necessary for the general case.

Applying Theorems 1 and 3 to the system of the nonlinear vibrating string

$$\begin{cases} \frac{\partial r}{\partial t} + k(r-s)\frac{\partial r}{\partial x} = 0, \\ \frac{\partial s}{\partial t} - k(r-s)\frac{\partial s}{\partial x} = 0, \end{cases} \quad (k>0) \tag{1.13}$$

we get immediately

Theorem 4. (i) *If there exists an interval* $[\alpha_1, \alpha_2]$ *such that*

$$k'(\alpha) \equiv 0, \ \forall \alpha \in [\alpha_1, \alpha_2], \tag{1.14}$$

then system (1.13) *admits nontrivial GSSWS*;

(ii) *If* $k' \neq 0$, *then all GSSWS of system* (1.13) *are trivial*: $r \equiv$ const. *and* $s \equiv$ const.

In Sec. 2 we shall at first state a generalization of Theorem 3 to general quasilinear systems of diagonal form with n equations, then give the proof of this general result. In Sec.3 we shall discuss briefly the case of non-strictly hyperbolic systems.

§ 2. Global Solvability in the Whole Space for Quasilinear Systems of Diagonal Form

In this section we shall consider GSSWS for the following quasilinear system of diagonal form

$$\frac{\partial U_i}{\partial t} + \lambda_i(U)\frac{\partial U_i}{\partial x} = 0 \quad (i=1, \cdots, n), \tag{2.1}$$

in which $U = (U_1, \cdots, U_n)$.

Theorem 5. *Suppose that there exists* $U^{(1)}$ *and* $U^{(2)}$ *such that*

$$\lambda_i(U) > \lambda_j(V), \ \forall U_k, V_k \in [U_k^{(1)}, U_k^{(2)}], \ (k=1, \cdots, n) \tag{2.2}$$

for $i, j=1, \cdots, n$ *and* $i<j$. *Suppose further that there exists* $U^{(0)}$ *such that*

$$U_i^{(0)} \in [U_i^{(1)}, U_i^{(2)}] \quad (i=1, \cdots, n) \tag{2.3}$$

and

$$\frac{\partial \lambda_i}{\partial U_i}(U_1^{(0)}, \cdots, U_{i-1}^{(0)}, U_i, U_{i+1}^{(0)}, \cdots, U_n^{(0)}) \equiv 0, \ \forall U_i \in [U_i^{(1)}, U_i^{(2)}] \tag{2.4}$$

for $i=1, \cdots, n$. *Then system* (2.1) *admits nontrivial bounded GSSWS.*

Obviously, Theorem 5 is a direct generalization of Theorem 3.

Proof It is sufficient to prove that under the assumption of Theorem 5 we can suitably choose a bounded initial data $\varphi(x) = (\varphi_1(x), \cdots, \varphi_n(x))$ such that the Cauchy problem

$$\begin{cases} (2.1), \\ t=0: U=\varphi(x) \end{cases} \tag{2.5}$$

admits a nontrivial global smooth solution on both $t \geqslant 0$ and $t \leqslant 0$.

For $i=1, \cdots, n$, let $\varphi_i(x)$ be smooth and satisfy

$$\begin{cases} \varphi_i(x) \equiv U_i^{(0)} & \text{for } |x| \geqslant 1; \\ \varphi_i(x) \in [U_i^{(1)}, U_i^{(2)}] & \text{for } |x| < 1 \end{cases} \tag{2.6}$$

and

$$\sup_{x \in \mathbf{R}^1} |\varphi_i'(x)| \neq 0. \tag{2.7}$$

In what follows we shall prove that if

$$\eta = \sup_{\substack{i=1, \cdots, n \\ x \in \mathbf{R}^1}} |\varphi_i'(x)| \tag{2.8}$$

is suitably small, then the Cauchy problem (2.5) with the chosen initial data $\varphi(x)$ gives a nontrivial bounded GSSWS of system (2.1).

According to the well known results on the existence of local smooth solutions to the Cauchy problem for first order quasilinear hyperbolic systems (cf. [6]), in order to get the existence of global smooth solutions on $t \geqslant 0$ for the Cauchy problem (2.5), it is sufficient to prove that for any $T_0 > 0$, if the Cauchy problem (2.5) admits a C^1 solution on the domain

$$R(T) = \{(t, x) \mid 0 \leqslant t \leqslant T, \ |x| < \infty\} \quad (T \leqslant T_0), \tag{2.9}$$

the C^1 norm of the solution on $R(T)$ depends only on T_0.

Since the C^0 norm of the solution is easy to estimate, we need only to pay our attention to the estimation of the C^0 norm of the first derivatives $\frac{\partial U_i}{\partial x}$ $(i=1, \cdots, n)$.

Let $x = x_i(t, \alpha)$ be the i-th characteristic curve passing through the point $(0, \alpha)$

$$\begin{cases} \dfrac{dx_i(t, \alpha)}{dt} = \lambda_i(U'(t, x_i(t, \alpha)), \varphi_i(\alpha), U''(t, x_i(t, \alpha))), \\ x_i(0, \alpha) = \alpha, \end{cases} \tag{2.10}$$

in which

$$U' = (U_1, \cdots, U_{i-1}), \ U'' = (U_{i+1}, \cdots, U_n). \tag{2.11}$$

Since $U_i(t, x_i(t, \alpha)) = \varphi_i(\alpha)$ we have

$$\frac{\partial U_i}{\partial x}(t, x_i(t, \alpha)) = \frac{\varphi_i'(\alpha)}{\frac{\partial x_i}{\partial \alpha}(t, \alpha)}. \tag{2.12}$$

Differentiation (2.10) in α gives

$$\frac{\partial x_i}{\partial \alpha}(t, \alpha) = e^{F(t,\alpha)}\Big(1+\varphi_i'(\alpha)\int_0^t \frac{\partial \lambda_i}{\partial U_i}(U'(\tau, x_i(\tau,\alpha)), \varphi_i(\alpha), U''(\tau, x_i(\tau, \alpha)))e^{-F(\tau,\alpha)}d\tau\Big), \tag{2.13}$$

where

$$F(t, \alpha) = \int_0^t \sum_{j\neq i} \frac{\partial \lambda_i}{\partial U_j}(U'(s, x_i(s, \alpha)), \varphi_i(\alpha), U''(s, x_i(s, \alpha)))\cdot\frac{\partial U_j}{\partial x}(s, x_i(s, \alpha))ds. \tag{2.14}$$

Using condition (2.2) and noticing the definition of $\varphi(x)$, it is easy to see that there exists a constant $T_1>0$ independent of η such that for $t>T_1$,

$$\operatorname{Supp}\frac{\partial U_i}{\partial x}(t, x) \cap \operatorname{Supp}\frac{\partial U_j}{\partial x}(t, x) = \emptyset, \quad i\neq j \tag{2.15}$$

and

$$x_i > x_j, \ \forall\, x_i \in \operatorname{Supp}\frac{\partial U_i}{\partial x}(t, x), \ x_j\in \operatorname{Supp}\frac{\partial U_j}{\partial x}(t, x), \quad i<j. \tag{2.16}$$

For $t\leqslant T_1$ we have the following

Lemma. *If η is suitably small such that*

$$\eta < \frac{1}{2enMT_1}, \tag{2.17}$$

in which

$$M = \sup_{\substack{1\leqslant i,j\leqslant n \\ U_k\in[U_k^{(1)}, U_k^{(2)}] \\ (k=1,\cdots,n)}} \left|\frac{\partial \lambda_i}{\partial U_j}(U)\right|, \tag{2.18}$$

then for the smooth solution $U=U(t, x)$ on $R(T)$ $(T\leqslant T_1)$ it holds that

$$m(T) = \sup_{\substack{1\leqslant i\leqslant n \\ (t,x)\in R(T)}} \left|\frac{\partial U_i}{\partial x}(t, x)\right| \leqslant \frac{1}{nMT_1}. \tag{2.19}$$

Proof From (2.13) and (2.14) we have

$$\frac{\partial x_i}{\partial \alpha}(t, \alpha) \geqslant e^{-nMtm(t)}(1-\eta Mte^{nMtm(t)}). \tag{2.20}$$

Introduce the set

$$\Gamma = \Big\{t \,|\, 0\leqslant t\leqslant T, \ 1-\eta Mte^{nMtm(t)} \geqslant \frac{1}{2}\Big\}. \tag{2.21}$$

Obviously, Γ is a closed set in $[0, T]$. We point out that Γ is also a relatively open set in $[0, T]$. In fact, if $t_0\in\Gamma$, then $[0, t_0]\in\Gamma$, hence for any $t\in[0, t_0]$,

$$\frac{\partial x_i}{\partial \alpha} \geqslant \frac{1}{2}e^{-nMtm(t)}.$$

Thus, it follows from (2.12) that for $t\in[0, t_0]$

$$\left|\frac{\partial U_i}{\partial x}(t, x)\right| \leqslant 2\eta e^{nMtm(t)}, \tag{2.22}$$

hence

$$m(t) \leqslant 2\eta e^{nMtm(t)} \leqslant 2\eta e^{nMT_1m(t)}. \tag{2.23}$$

Consider the function

$$\eta(m)=\frac{m}{2}e^{-nMT_1m}. \tag{2.24}$$

For $m\geqslant 0$, $\eta(m)$ attains its maximum $\frac{1}{2enMT_1}$ at $m=\frac{1}{nMT_1}$ and

$$\eta(0)=\eta(+\infty)=0.$$

Noticing (2.23) and $m(0)=\eta$, it follows from (2.17) that for $t\in[0, t_0]$,

$$m(t)<\frac{1}{nMT_1}, \tag{2.25}$$

then
$$1-\eta Mt_0e^{nMt_0m(t_0)}\geqslant 1-\eta MT_1e>1-\frac{1}{2n}>\frac{1}{2}.$$

From this it is easy to see that Γ is open in $[0, T]$.

Hence, $\Gamma=[0, T]$. Taking $t_0=T$ in (2.25), we get immediately (2.19). The proof of the lemma is completed.

For $t>T_1$, assumption (2.4) gives

$$\frac{\partial x_i}{\partial\alpha}(t, \alpha)=e^{F(t,\alpha)}. \tag{2.26}$$

Now we use the method in [7] to estimate $F(t, \alpha)$. Passing through a point $(t, x_i(t, \alpha))$ on the i-th characteristic curve $x=x_i(t, \alpha)$, we draw the j-th $(j\neq i)$ characteristic curve downwards which intersects the x-axis at the point $(0, y_j^i(t, \alpha))$, then

$$U_j(s, x_i(s, \alpha))=\varphi_j(y_j^i(s, \alpha)). \tag{2.27}$$

Thus, from the j-th equation of (2.1) we get

$$\frac{\partial U_j}{\partial x}(s, x_i(s, \alpha))=\frac{1}{\lambda_i-\lambda_j}\frac{dU_j}{ds}(s, x_i(s, \alpha))=\frac{1}{\lambda_i-\lambda_j}\varphi_j'(y_j^i(s, \alpha))\frac{dy_j^i(s, \alpha)}{ds}, \tag{2.28}$$

hence

$$F(t, \alpha)=\sum_{j\neq i}\int_{\alpha}^{y_j^i(t,\alpha)}\frac{\partial\lambda_i}{\partial U_j}\frac{\varphi_j'(y)}{\lambda_i-\lambda_j}dy. \tag{2.29}$$

Noticing that there exists a constant $C_1>0$ such that

$$|y_j^i(t, \alpha)-\alpha|\leqslant C_1t, \tag{2.30}$$

From (2.29) we get

$$|F(t, \alpha)|\leqslant C_0t, \tag{2.31}$$

where $C_0>0$ is a constant.

Thus, from (2.26), (2.31) and (2.12) it follows that for the smooth solution on $R(T)$ it holds that

$$\sup_{x\in\mathbb{R}^1}\left|\frac{\partial U_i}{\alpha x}(t, x)\right|\leqslant C_2e^{C_0t},\ i=1, \cdots, n,\ \forall t\in(T_1, T], \tag{2.32}$$

where C_2 is a positive constant.

The combination of (2.19) and (2.32) gives the uniform estimation for the C^1 norm of the solution on $R(T)$ $(T\leqslant T_0)$, then the Cauchy problem (2.5) possesses a nontrivial global smooth solution on $t\geqslant 0$. It is the same for $t\leqslant 0$. Therefore, the theorem follows.

Theorem 6. *Suppose that i–th characteristic value is genuinely nonlinear in the sense of P. D. Lax:*

$$\frac{\partial\lambda_i}{\partial U_i}\neq 0 \tag{2.33}$$

on the domain under consideration. Then for any bounded GSSWS of system (2.1) (if it exists), either

$$U_i\equiv\text{const.} \tag{2.34}$$

or for any fixed $t\in\mathbf{R}^1$, *there are at least two functions from* $\frac{\partial U'}{\partial x}$ *and* $\frac{\partial U''}{\partial x}$, *which are not absolutely integrable on* $\mathbf{R}^1$.

Proof Let $\varphi(x)$ be the corresponding initial data of this GSSWS. If the theorem does not hold, without loss of generality, we may assume that

(i)
$$\frac{\partial\lambda_i}{\partial U_i}>0, \tag{2.35}$$

(ii) there exists α_0 such that

$$\varphi_i'(\alpha_0)=\frac{\partial U_i}{\partial x}(0,\ \alpha_0)>0, \tag{2.36}$$

and (iii) $\varphi_j'(x)$ is absolutely integrable on $\mathbf{R}^1$ for $j\neq i,\ n$.

Define the function $h(U)$ by

$$\frac{\partial h}{\partial U_n}=\frac{1}{\lambda_i-\lambda_n}\frac{\partial\lambda_i}{\partial U_n}. \tag{2.37}$$

From (2.14) and (2.28) we get

$$F(t,\ \alpha)=\int_0^t\frac{dh}{ds}ds+\sum_{j\neq i,n}\int_\alpha^{y_j^i(t,\alpha)}\left(\frac{\frac{\partial\lambda_i}{\partial U_j}}{\lambda_i-\lambda_j}-\frac{\partial h}{\partial U_j}\right)\varphi_j'(y)dy. \tag{2.38}$$

Noticing assumption (iii), it is easy to see from (2.38) that $F(t,\ \alpha)$ is bounded.

$$|F(t,\ \alpha)|\leqslant F_0, \tag{2.39}$$

where F_0 is a positive constant.

Let

$$A=\operatorname{Inf}\frac{\partial\lambda_i}{\partial U_i}>0. \tag{2.40}$$

Using (2.36) we have

$$1+\varphi_i'(\alpha_0)\int_0^t\frac{\partial\lambda_i}{\partial U_i}e^{-F(\tau,\alpha_0)}d\tau\leqslant 1+\varphi_i'(\alpha_0)Ae^{-F_0}t.$$

Then, from (2.13) it follows that when t increases, the denominator $\frac{\partial x_i}{\partial\alpha}(t,\ \alpha_0)$ of (2.12) must change its sign in a finite time, hence the soltuion of the corresponding Cauchy problem on $t\geqslant 0$ must develop singularities in a finite time. Therefore we obtain the theorem by contradiction.

Theorem 1 is a simple corollary of Theorem 6.

§ 3. Remark on Non-Strictly Hyperbolic Systems

So far all results are obtained for strictly hyperbolic systems. For non-strictly hyperbolic systems, the situation will be quite different. As a simple example, consider the following system

$$\begin{cases} \frac{\partial r}{\partial t}+\lambda(r,\ s)\frac{\partial r}{\partial x}=0, \\ \frac{\partial s}{\partial t}+\lambda(r,\ s)\frac{\partial s}{\partial x}=0, \end{cases} \tag{3.1}$$

which possesses only one family of characteristic curves. We have

Theorem 7. *Suppose that on the domain under consideration*

$$\frac{\partial \lambda}{\partial r}\neq 0,\ \frac{\partial \lambda}{\partial s}\neq 0, \tag{3.2}$$

then system (3.1) *always admits nontrivial GSSWS.*

Proof It is easily proved (cf. [1]) that the Cauchy problem

$$\begin{cases} (3.1), \\ t=0;\ r=r_0(x),\ s=s_0(x) \end{cases} \tag{3.3}$$

possesses a GSSWS iff

$$\lambda(r_0(\alpha),\ s_0(\alpha))\equiv\text{const.},\ \forall\ \alpha\in\mathbb{R}^1. \tag{3.4}$$

Under assumption (3.2) it is quite easy to find a pair of functions $r_0(\alpha)$ and $s_0(\alpha)$ such that both $r_0(\alpha)$ and $s_0(\alpha)$ are not constants and that (3.4) holds. The proof of Theorem 7 is completed.

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