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THE GENERAL SCHEME OF THE GENERALIZED WRONSKIAN TECHNIQUE

ZENG YONGBO (曾云波) CHEN DANGYUAN (陈登远)
(*China University of Science and Technology*)

Abstract

The general scheme of the generalized Wronskian technique is given and is applied to four energy-dependent potential's eigenvalue problems to generate four classes of nonlinear evolution equations solvable by the inverse spectral transform.

§ 1. Introduction

In a series of remarkable papers by Calogero and Degasperis, a generalized Wronskian technique was developed and applied to the Schrödinger and the generalized Zakharov-Shabat eigenvalue problems. In this paper we will give the general scheme of the generalized Wronskian technique and apply the scheme to the following four energy-dependent potential's eigenvalue problems to generate four classes of nonlinear evolution equations (NLEE's) solvable by the inverse spectral transform (IST).

We consider four eigenvalue problems as follows:

$$\phi_x=\begin{pmatrix} -i\xi^2 & \xi q \\ \xi r & i\xi^2 \end{pmatrix}\phi,\ \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{1.1}$$

$$\phi_x=\begin{pmatrix} -i\xi & \xi q \\ r & i\xi \end{pmatrix}\phi, \tag{1.2}$$

$$\phi_x=-i\xi\begin{pmatrix} s & r-iq \\ r+iq & -s \end{pmatrix}\phi, \tag{1.3}$$

(In (1.3), $r(x)$, $q(x)$ and $s(x)$ are real functions and satisfy $r^2+q^2+s^2=1$, and $\lim\limits_{|x|\to\infty} s(x)=1$.)

$$\phi_x=\begin{pmatrix} -i\xi & r+\xi q \\ 1 & i\xi \end{pmatrix}\phi. \tag{1.4}$$

These eigenvalue problems can be written in the form

$$\phi_x=R\phi,\ R(x,\ \xi)=-\begin{pmatrix} -i\xi^{m_0}p_1 & p_2 \\ p_3 & i\xi^{m_0}p_1 \end{pmatrix}. \tag{1.5}$$

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The potential $\begin{pmatrix} r \\ q \end{pmatrix}$ of concern in (1.5) might depend on other variables besides x. Assume now that r, q have any order continuous derivative which occur below (in Equation (2.24)) and all vanish at infinity. (abbreviated as (1.6))

Finally, we note that, by the way, setting $\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \xi\phi_1 \\ \phi_2 \end{pmatrix}$ and $\xi' = \xi^2$, (1.1) can be transformed to (1.2) and vice versa.

§ 2. General scheme and fundamental formulae

2.1 Matrix solution

For every real ξ^{m_0}, for (1.1), (1.2) and (1.3) there exists one matrix solution $\Phi(x, \xi)$ defined by the asymptotic boundary conditions

$$\Phi(x, \xi) \xrightarrow[x\to+\infty]{} \begin{pmatrix} e^{-i\xi^{m_0}x} & 0 \\ 0 & e^{i\xi^{m_0}x} \end{pmatrix} \begin{pmatrix} 1 & \alpha^{(-)}(\xi) \\ \alpha^{(+)}(\xi) & 1 \end{pmatrix}, \tag{2.1.1}$$

$$\Phi(x, \xi) \xrightarrow[x\to-\infty]{} \begin{pmatrix} e^{-i\xi^{m_0}x} & 0 \\ 0 & e^{i\xi^{m_0}x} \end{pmatrix} \begin{pmatrix} \beta^{(+)}(\xi) & 0 \\ 0 & \beta^{(-)}(\xi) \end{pmatrix}, \tag{2.1.2}$$

while for (1.4) there exists one matrix solution which satisfies

$$\Phi(x, \xi) \xrightarrow[x\to+\infty]{} \begin{pmatrix} -2i\xi e^{-i\xi x} & 0 \\ e^{-i\xi x} & e^{i\xi x} \end{pmatrix} \begin{pmatrix} 1 & \alpha^{(-)}(\xi) \\ \alpha^{(+)}(\xi) & 1 \end{pmatrix}, \tag{2.1.1'}$$

$$\Phi(x, \xi) \xrightarrow[x\to-\infty]{} \begin{pmatrix} -2i\xi e^{-i\xi x} & 0 \\ e^{-i\xi x} & e^{i\xi x} \end{pmatrix} \begin{pmatrix} \beta^{(+)}(\xi) & 0 \\ 0 & \beta^{(-)}(\xi) \end{pmatrix}. \tag{2.1.2'}$$

Here $\alpha^{(\pm)}(\xi)$ and $\beta^{(\pm)}(\xi)$ are respectively the reflection coefficient and the transmission coefficient. Suppose that the $\beta^{(+)}(\xi)$ and $\beta^{(-)}(\xi)$ have only finite simple poles $\lambda_j^{(+)}$ and $\lambda_j^{(-)}$ in the upper and the lower half $\lambda = \xi^{m_0}$ plane respectively, and $\pm\mathrm{Im}\ \lambda_j^{(\pm)} > 0$ $j = 1, \cdots, N^{(\pm)}$. The scattering data of the problem (1.5) is

$$\{\alpha^{(\pm)}(\xi),\ -\infty < \xi^{m_0} < +\infty;\ \lambda_j^{(\pm)},\ \rho_j^{(\pm)},\ \pm\mathrm{Im}\,\lambda_j^{(\pm)} > 0\ j = 1, \cdots, N^{(\pm)}\}. \tag{2.2}$$

In the following, we use σ_0, σ_1, σ_2 and σ_3 to express the usual Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

Now, two sets of potentials $\begin{pmatrix} r \\ q \end{pmatrix}$ and $\begin{pmatrix} r' \\ q' \end{pmatrix}$ and their corresponding R, Φ and R', Φ' of (1.5) have been given. By virtue of (2.1) it is easily seen that $\Phi'^T(x, \xi) F(x, \xi) \cdot \Phi(x, \xi) |_{x=-\infty}^{+\infty}$ will not contain the term related to x only when F is a simple linear combination of $i\sigma_2$ and $i\sigma_2 R(\infty, \xi)$. Actually, from (2.1) we have

$$\Phi'^T(x, \xi) i\sigma_2 \Phi(x, \xi)\Big|_{x=-\infty}^{\infty} = -l(\xi)\begin{pmatrix} \alpha^{(+)'}(\xi)-\alpha^{(+)}(\xi) & \alpha^{(+)'}(\xi)\alpha^{(-)}(\xi)+\beta^{(+)'}(\xi)\beta^{(-)}(\xi)-1 \\ 1-\alpha^{(+)}(\xi)\alpha^{(-)'}(\xi)-\beta^{(+)}(\xi)\beta^{(-)'}(\xi) & \alpha^{(-)}(\xi)-\alpha^{(-)'}(\xi) \end{pmatrix} \tag{2.4}$$

$$\Phi^T(x, \xi) i\sigma_2 R(x, \xi)\Phi(x, \xi)\Big|_{x=-\infty}^{+\infty} = 2i\xi^{m_0} l(\xi)\begin{pmatrix} \alpha^{(+)}(\xi) & \alpha^{(+)}(\xi)\alpha^{(-)}(\xi) \\ \alpha^{(+)}(\xi)\alpha^{(-)}(\xi) & \alpha^{(-)}(\xi) \end{pmatrix}, \tag{2.5}$$

where $l(\xi)$ is related to the eigenvalue problem, for (1.1), (1.2) and (1.3) $l(\xi)=1$, and for (1.4) $l(\xi)=-2i\xi$.

2.2 Basic formulae

The following generalized Wronskian relation is satisfied by virtue of equation (1.5) for an arbitrary matrix function $F(x)$ (at least once differentiable):

$$\int_{-\infty}^{\infty}\Phi'^T(x, \xi)[F_x(x)+R'^T(x, \xi)F(x)+F(x)R(x, \xi)]\Phi(x, \xi)dx = \Phi'^T(x, \xi)F(x)\Phi(x, \xi)\Big|_{x=-\infty}^{\infty}. \tag{2.6}$$

Since $\mathrm{tr}R=0$, we immediately obtain

$$R^T i\sigma_2 = -i\sigma_2 R. \tag{2.7}$$

By using equations (2.4), (2.7) and choosing specially $F(x)=i\sigma_2$ in equation (2.6), we get

$$\int_{-\infty}^{\infty}\Phi'^T(x, \xi) i\sigma_2(R'-R)\Phi(x, \xi)dx = l(\xi)\begin{pmatrix} \alpha^{(+)'}(\xi)-\alpha^{(+)}(\xi) & \alpha^{(+)'}(\xi)\alpha^{(-)}(\xi)+\beta^{(+)'}(\xi)\beta^{(-)}(\xi)-1 \\ 1-\alpha^{(+)}(\xi)\alpha^{(-)'}(\xi)-\beta^{(+)}(\xi)\beta^{(-)'}(\xi) & \alpha^{(-)}(\xi)-\alpha^{(-)'}(\xi) \end{pmatrix}. \tag{2.8}$$

We now consider two special cases of Equation (2.8). First, taking $\begin{pmatrix} r \\ q \end{pmatrix}=\begin{pmatrix} r(x, y) \\ q(x, y) \end{pmatrix}$ and $\begin{pmatrix} r' \\ q' \end{pmatrix}=\begin{pmatrix} r(x, y+\Delta y) \\ q(x, y+\Delta y) \end{pmatrix}$ in Equation (2.8), we obtain

$$\alpha^{(+)}(\xi, y)\alpha^{(-)}(\xi, y)+\beta^{(+)}(\xi, y)\beta^{(-)}(\xi, y)=1, \tag{2.9}$$

$$\int_{-\infty}^{\infty}\Phi^T(x, \xi, y) i\sigma_2 R_y(x, y, \xi)\Phi(x, \xi, y)dx = l(\xi)\begin{pmatrix} \alpha_y^{(+)}(\xi, y) & \alpha_y^{(+)}(\xi, y)\alpha^{(-)}(\xi, y)+\beta_y^{(+)}(\xi, y)\beta^{(-)}(\xi, y) \\ -\alpha^{(+)}(\xi, y)\alpha_y^{(-)}(\xi, y)-\beta^{(+)}(\xi, y)\beta_y^{(-)}(\xi, y) & -\alpha_y^{(-)}(\xi, y) \end{pmatrix}. \tag{2.10}$$

Because $i\sigma_2 R_y$ contains only the terms related to r_y and q_y, we can introduce a formal linear operator η that transforms a spinor into a symmetric matrix

$$\eta\begin{pmatrix} r_y \\ q_y \end{pmatrix}=\frac{1}{k_1\xi^{m_1}} i\sigma_2 R_y, \tag{2.11}$$

where $k_1\xi^{m_1}$ is the coefficient of r in R, $k_2\xi^{m_2}$ (or $-k_2\xi^{m_2}$) is the coefficientof q in R.

With this definition, equation (2.10) can be rewritten as follows.

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi,y)\eta\begin{pmatrix}r_y\\ q_y\end{pmatrix}\Phi(x,\xi,y)dx$$
$$=\frac{l(\xi)}{k_1\xi^{m_1}}\begin{pmatrix}\alpha_y^{(+)}(\xi,y) & \alpha_y^{(+)}(\xi,y)\alpha^{(-)}(\xi,y)+\beta_y^{(+)}(\xi,y)\beta^{(-)}(\xi,y)\\ -\alpha^{(+)}(\xi,y)\alpha_y^{(-)}(\xi,y)-\beta^{(+)}(\xi,y)\beta_y^{(-)}(\xi,y) & -\alpha_y^{(-)}(\xi,y)\end{pmatrix}. \tag{2.12}$$

Second, taking $\begin{pmatrix}r(x,y)\\ q(x,y)\end{pmatrix}=\begin{pmatrix}r(x)\\ q(x)\end{pmatrix}$,

$$\begin{pmatrix}r'(x,y)\\ q'(x,y)\end{pmatrix}=\begin{pmatrix}(1+y)^{\widetilde{m}_1}r((1+y)^{m_0}x)\\ (1+y)^{\widetilde{m}_2}q((1+y)^{m_0}x)\end{pmatrix},$$

where $\widetilde{m}_1=m_0-m_1$, $\widetilde{m}_2=m_0-m_2$ for (1.1), (1.2) and (1.3), and $\widetilde{m}_1=2$, $\widetilde{m}_2=1$ for (1.4), we have

$$\alpha^{(\pm)\prime}(\xi,y)=\alpha^{(\pm)}\left(\frac{\xi}{1+y}\right),$$

$$\beta^{(\pm)\prime}(\xi,y)=\beta^{(\pm)}\left(\frac{\xi}{1+y}\right).$$

Substituting above results into (2.8) and keeping terms linear in Δy, we obtain

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi)\eta\begin{pmatrix}\widetilde{m}_1r+m_0xr_x\\ \widetilde{m}_2q+m_0xq_x\end{pmatrix}\Phi(x,\xi)dx$$
$$=\frac{-\xi l(\xi)}{k_1\xi^{m_1}}\begin{pmatrix}\alpha_\xi^{(+)}(\xi) & \alpha_\xi^{(+)}(\xi)\alpha^{(-)}(\xi)+\beta_\xi^{(+)}(\xi)\beta^{(-)}(\xi)\\ -\alpha^{(+)}(\xi)\alpha_\xi^{(-)}(\xi)-\beta^{(+)}(\xi)\beta_\xi^{(-)}(\xi) & -\alpha_\xi^{(-)}(\xi)\end{pmatrix}, \tag{2.13}$$

where we have used the definition of the operator η and (2.9).

Moreover, to set $F=i\sigma_2R$, $R'=R$ in equation (2.6) and use equations (2.5), (2.7) and (2.11), we obtain

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi)\eta\begin{pmatrix}r_x\\ q_x\end{pmatrix}\Phi(x,\xi)dx$$
$$=\frac{2i\xi^{m_0}l(\xi)}{k_1\xi^{m_1}}\begin{pmatrix}\alpha^{(+)}(\xi) & \alpha^{(+)}(\xi)\alpha^{(-)}(\xi)\\ \alpha^{(+)}(\xi)\alpha^{(-)}(\xi) & \alpha^{(-)}(\xi)\end{pmatrix}. \tag{2.14.1}$$

For the eigenvalue problems (1.1) and (1.2), to set $F=i\sigma_2R(+\infty,\xi)$ in equation (2.6) and use (2.5), (2.7) and (2.11), we get in addition

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi)\eta\begin{pmatrix}r\\ -q\end{pmatrix}\Phi(x,\xi)dx$$
$$=\frac{l(\xi)}{k_1\xi^{m_1}}\begin{pmatrix}\alpha^{(+)}(\xi) & \alpha^{(+)}(\xi)\alpha^{(-)}(\xi)\\ \alpha^{(+)}(\xi)\alpha^{(-)}(\xi) & \alpha^{(-)}(\xi)\end{pmatrix}. \tag{2.14.2}$$

2.3 General relations

We indicate anyone of $\begin{pmatrix} r_y \\ q_y \end{pmatrix}$, $\begin{pmatrix} \widetilde{m}_1 r + m_0 x r_x \\ \widetilde{m}_2 q + m_0 x q_x \end{pmatrix}$ and $\begin{pmatrix} r_x \\ q_x \end{pmatrix}$ $\left(\text{or } \begin{pmatrix} r \\ -q \end{pmatrix}\right)$ with the notation $\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}$. Now we start with (2.12), (2.13) and (2.14), and use equation (2.6) to get the general relations. Setting

$$F^{(n)}(x, \xi) = \begin{pmatrix} a_0^{(n)}(x) + a_1^{(n)}(x)\xi & b_0^{(n)}(x) + b_1^{(n)}(x)\xi \\ b_0^{(n)}(x) + b_1^{(n)}(x)\xi & d_0^{(n)}(x) + d_1^{(n)}(x)\xi \end{pmatrix},$$

$$F^{(n)}(+\infty, \xi) = 0, \tag{2.15}$$

we require $\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}$ and $F^{(n)}$ to satisfy the recursion formula

$$\eta \begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix} - \xi^{m_0}\eta \begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix} = F_x^{(n)}(x, \xi) + R^T(x, \xi)F^{(n)}(x, \xi) + F^{(n)}(x, \xi)R(x, \xi). \tag{2.16}$$

We then introduce a integro-differential matrix operator L through the recursion formula (2.16) (see 3.) so that

$$\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix} = L \begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix} = L^n \begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}. \tag{2.17}$$

Simultaneously, $F^{(n)}$ can be expressed in terms of $\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}$ through (2.16), and we have (see 3.)

$$F^{(n)}(-\infty, \xi) = \frac{b^{(n)}}{\xi^{m_0}} \sigma_2 R(+\infty, \xi),$$

$$\Phi^T(x, \xi)F^{(n)}(x, \xi)\Phi(x, \xi)\,|_{x=-\infty} = l(\xi) b^{(n)} \beta^{(+)}(\xi)\beta^{(-)}(\xi)\sigma_1. \tag{2.18}$$

Using (2.6), (2.15), (2.16) and (2.18), we get

$$\int_{-\infty}^{\infty} \Phi^T(x, \xi)\eta \begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix} \Phi(x, \xi)dx$$

$$= \xi^{m_0} \int_{-\infty}^{\infty} \Phi^T(x, \xi)\eta \begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix} \Phi(x, \xi)dx - l(\xi) b^{(n)} \beta^{(+)}(\xi)\beta^{(-)}(\xi)\sigma_1.$$

From the above equation and (2.17), it yields

$$\int_{-\infty}^{\infty} \Phi^T(x, \xi)\eta \left(L^n \begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right) \Phi(x, \xi)dx$$

$$= \xi^{nm_0} \int_{-\infty}^{\infty} \Phi^T(x, \xi)\eta \begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix} \Phi(x, \xi)dx - l(\xi)\sum_{j=1}^{n} (\xi^{m_0})^{n-j} b^{(j)} \beta^{(+)}(\xi)\beta^{(-)}(\xi)\sigma_1. \tag{2.19}$$

The structure of eq. (2.19) implies immediately the more general formula

$$\int_{-\infty}^{\infty} \Phi^T(x, \xi)\eta \left(f(L) \begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right) \Phi(x, \xi)dx = f(\xi^{m_0}) \int_{-\infty}^{\infty} \Phi^T(x, \xi)\eta \begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix} \Phi(x, \xi)dx$$

$$+ \tilde{f} l(\xi)\beta^{(+)}(\xi)\beta^{(-)}(\xi)\sigma_1, \tag{2.20}$$

where $f(z)$ is an arbitrary entire function.

$$f(z)=\sum_{n=0}^{\infty}\gamma_n z^n,$$

$$\tilde{f}=-\sum_{n=0}^{\infty}\gamma_n\sum_{j=1}^{n}(\xi^{m_0})^{n-j}b^{(j)}.$$

By means of (2.20) and (2.12) we obtain

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi,y)\eta\left(h(L)\begin{pmatrix}r_y\\ q_y\end{pmatrix}\right)\Phi(x,\xi,y)dx$$
$$=\frac{l(\xi)h(\xi^{m_0})}{k_1\xi^{m_1}}\begin{pmatrix}\alpha_y^{(+)}(\xi,y) & \alpha_y^{(+)}(\xi,y)\alpha^{(-)}(\xi,y)+\beta_y^{(+)}(\xi,y)\beta^{(-)}(\xi,y)\\ -\alpha^{(+)}(\xi,y)\alpha_y^{(-)}(\xi,y)-\beta^{(+)}(\xi,y)\beta_y^{(-)}(\xi,y) & -\alpha_y^{(-)}(\xi,y)\end{pmatrix}+\tilde{h}l(\xi)\beta^{(+)}(\xi,y)\beta^{(-)}(\xi,y)\sigma_1. \tag{2.21}$$

From (2.20) and (2.13), we get

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi)\eta\left(g(L)\begin{pmatrix}\tilde{m}_1 r+m_0 x r_x\\ \tilde{m}_2 q+m_0 x q_x\end{pmatrix}\right)\Phi(x,\xi)dx$$
$$=\frac{-\xi l(\xi)g(\xi^{m_0})}{k_1\xi^{m_1}}\begin{pmatrix}\alpha_\xi^{(+)}(\xi) & \alpha_\xi^{(+)}(\xi)\alpha^{(-)}(\xi)+\beta_\xi^{(+)}(\xi)\beta^{(-)}(\xi)\\ -\alpha^{(+)}(\xi)\alpha_\xi^{(-)}(\xi)-\beta^{(+)}(\xi)\beta_\xi^{(-)}(\xi) & -\alpha_\xi^{(-)}(\xi)\end{pmatrix}+\tilde{g}l(\xi)\beta^{(+)}(\xi)\beta^{(-)}(\xi)\sigma_1. \tag{2.22}$$

For (1.1) and (1.2), by means of equations (2.20) and (2.14.2), we have

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi)\eta\left(f(L)\begin{pmatrix}r\\ -q\end{pmatrix}\right)\Phi(x,\xi)dx$$
$$=\frac{l(\xi)f(\xi^{m_0})}{k_1\xi^{m_1}}\begin{pmatrix}\alpha^{(+)}(\xi) & \alpha^{(+)}(\xi)\alpha^{(-)}(\xi)\\ \alpha^{(+)}(\xi)\alpha^{(-)}(\xi) & -\alpha^{(-)}(\xi)\end{pmatrix}+\tilde{f}l(\xi)\beta^{(+)}(\xi)\beta^{(-)}(\xi)\sigma_1. \tag{2.23.1}$$

For (1.3) and (1.4), using (2.20) and (2.14.1), we get

$$\int_{-\infty}^{\infty}\Phi^T(x,\xi)\eta\left(f(L)\begin{pmatrix}r_x\\ q_x\end{pmatrix}\right)\Phi(x,\xi)dx$$
$$=\frac{2i\xi l(\xi)f(\xi)}{k_1\xi^{m_1}}\begin{pmatrix}\alpha^{(+)}(\xi) & \alpha^{(+)}(\xi)\alpha^{(-)}(\xi)\\ \alpha^{(+)}(\xi)\alpha^{(-)}(\xi) & -\alpha^{(-)}(\xi)\end{pmatrix}+\tilde{f}l(\xi)\beta^{(+)}(\xi)\beta^{(-)}(\xi)\sigma_1. \tag{2.23.2}$$

The three functions $f(z)$, $g(z)$ and $h(z)$ are arbitrary entire functions.

2.4 NLEE's solvable by the IST.

Assume now that, in (1.5), the potential $\begin{pmatrix}r\\ q\end{pmatrix}$ depends on other two variables t, y besides x. The arbitrary entire functions $f(z)$, $g(z)$ and $h(z)$ might also depend on t and y besides z. We can write an equation similar to (2.10) with the variable t playing the role of y. By taking a simple linear combination of this equation and

Equations (2.21), (2.22) and (2.23), it immediately follows that validity of the nonlinear evolution equations for the potential $\begin{pmatrix} r(x, y, t) \\ q(x, y, t) \end{pmatrix}$

$$\begin{pmatrix} r_t \\ q_t \end{pmatrix} = h(L, y, t)\begin{pmatrix} r_y \\ q_y \end{pmatrix} + f(L, y, t)\begin{pmatrix} r_1 \\ q_1 \end{pmatrix} + g(L, y, t)\begin{pmatrix} \widetilde{m}_1 r + m_0 x r_x \\ \widetilde{m}_2 q + m_0 x q_x \end{pmatrix}, \tag{2.24}$$

where $\begin{pmatrix} r_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} r \\ -q \end{pmatrix}$ for (1.1), (1.2),

$\left(\text{and } L\begin{pmatrix} r \\ -q \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} r_x \\ q_x \end{pmatrix}\right)$, and $\begin{pmatrix} r_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} r_x \\ q_x \end{pmatrix}$ for (1.3), (1.4),

implies validity of the linear equations for $\alpha^{(\pm)}$, $\beta^{(\pm)}$

$$\alpha_t^{(\pm)}(\xi, y, t) = h(\xi^{m_0}, y, t)\alpha_y^{(\pm)}(\xi, y, t) \pm A(\xi^{m_0}) f(\xi^{m_0}, y, t)\alpha^{(\pm)}(\xi, y, t) - \xi g(\xi^{m_0}, y, t)\alpha_\xi^{(\pm)}(\xi, y, t), \tag{2.25}$$

$$\beta_t^{(\pm)}(\xi, y, t) = h(\xi^{m_0}, y, t)\beta_y^{(\pm)}(\xi, y, t) - \xi g(\xi^{m_0}, y, t)\beta_\xi^{(\pm)}(\xi, y, t) \pm \widetilde{E}\beta^{(\pm)}(\xi, y, t), \tag{2.26}$$

where $A(z) = 1$ for (1.1), (1.2), and $A(z) = 2iz$ for (1.3), (1.4),

$$\widetilde{E} = k_1 \xi^{m_1}(\widetilde{h} + \widetilde{g} + \widetilde{f}).$$

It is easy to show $\widetilde{f} = 0$.

In the similar way we also obtain the equations for $\lambda_j^{(\pm)}(y, t)$ and $\rho_j^{(\pm)}(y, t)$

$$\lambda_{jt}^{(\pm)}(y, t) = h(\lambda_j^{(\pm)}, y, t)\lambda_{jy}^{(\pm)}(y, t) + m_0 \lambda_j^{(\pm)}(y, t) g(\lambda_j^{(\pm)}, y, t), \tag{2.27}$$

$$\rho_{jt}^{(\pm)}(y, t) = h(\lambda_j^{(\pm)}, y, t)\rho_{jy}^{(\pm)}(y, t) + \Big[\frac{\partial}{\partial z} h(\lambda_j^{(\pm)}, y, t)\lambda_{jy}^{(\pm)}(y, t) \pm A(\lambda_j^{(\pm)}) f(\lambda_j^{(\pm)}, y, t) + m_0 g(\lambda_j^{(\pm)}, y, t) + m_0 \lambda_j^{(\pm)}(y, t)\frac{\partial}{\partial z} g(\lambda_j^{(\pm)}, y, t)\Big]\rho_j^{(\pm)}(y, t), \tag{2.28}$$

where

$$\frac{\partial}{\partial z} h(\lambda_j^{(\pm)}, y, t) = \frac{\partial}{\partial z} h(z, y, t)\big|_{z=\lambda_j^{(\pm)}}.$$

Now using the results discribed above, we can solve the Cauchy problem of the equations (2.24) by the inverse spectral transform.

By the way, for the eigenvalue problem

$$\phi_x = \begin{pmatrix} -i\xi & q \\ r & i\xi \end{pmatrix}\phi, \tag{2.29}$$

the procedure described above for (1.1) and (1.2) can be applied to (2.29), but the term $k_0\begin{pmatrix} xr \\ -xq \end{pmatrix}$ must be added to (2.24) because for (2.29) we have

$$\begin{pmatrix} \widetilde{m}_1 r + m_0 x r_x \\ \widetilde{m}_2 q + m_0 x q_x \end{pmatrix} = \begin{pmatrix} r + x r_x \\ q + x q_x \end{pmatrix} = 2iL\begin{pmatrix} xr \\ -xq \end{pmatrix},$$

and

$$-2i\int_{-\infty}^{\infty} \Phi^T(x, \xi)\eta\begin{pmatrix} xr \\ -xq \end{pmatrix}\Phi(x, \xi)dx$$

$$=\begin{pmatrix} \alpha_\xi^{(+)}(\xi) & \alpha_\xi^{(+)}(\xi)\alpha^{(-)}(\xi)+\beta_\xi^{(+)}(\xi)\beta^{(-)}(\xi) \\ -\alpha^{(+)}(\xi)\alpha_\xi^{(-)}(\xi)-\beta^{(+)}(\xi)\beta_\xi^{(-)}(\xi) & -\alpha_\xi^{(-)}(\xi) \end{pmatrix}. \tag{2.30}$$

If we take $\begin{pmatrix} r' \\ q' \end{pmatrix}=\begin{pmatrix} e^{-2iyx}r(x) \\ e^{2iyx}q(x) \end{pmatrix}$ for (2.29), we will have $\alpha^{(\pm)'}(\xi, y)=\alpha^{(\pm)}(\xi+y)$, $\beta^{(\pm)'}(\xi, y)=\beta^{(\pm)}(\xi+y)$. By substituting the above results into (2.8) we immediately get (2.30).

§ 3. The expression of L and $\tilde{E}$

It is seen from 2 that the scheme of the generalized Wronskian technique applied to the eigenvalue problems (1.1)—(1.4) is the same, and the results thus obtained are similar except for the expression of the operator L and $\tilde{E}$.

3.1 For (1.1)[3] the operator η is defined by virtue of equation (2.11) as

$$\eta\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}. \tag{3.1}$$

From (2.16), (3.1) and (2.15), it yields

$$a_1^{(n)}=d_1^{(n)}=b_0^{(n)}=0,$$
$$b_1^{(n)}=I(q(x)a_0^{(n)}+r(x)d_0^{(n)}), \tag{3.2}$$

$$\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=\begin{pmatrix} 2i-2rIq & 2rIr \\ 2qIq & -2i-2qIr \end{pmatrix}\begin{pmatrix} a_0^{(n)} \\ -d_0^{(n)} \end{pmatrix}=L_1\begin{pmatrix} a_0^{(n)} \\ -d_0^{(n)} \end{pmatrix}, \tag{3.3}$$

$$\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}=D\begin{pmatrix} a_0^{(n)} \\ -d_0^{(n)} \end{pmatrix}, \tag{3.4}$$

where $$D=\frac{\partial}{\partial x},\quad I=\int_x^\infty \cdots dx.$$

Using (3.3) and (3.4), we obtain

$$\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}=DL_1^{-1}\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=L\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=L^n\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}, \tag{3.5}$$

$$L=DL_1^{-1}=D\begin{pmatrix} \frac{1}{2i}-\frac{1}{2}rIq & -\frac{1}{2}rIr \\ -\frac{1}{2}qIq & -\frac{1}{2i}-\frac{1}{2}qIr \end{pmatrix}. \tag{3.6}$$

By means of (3.4)and (3.5), (3.2) becomes

$$b_1^{(n)}(-\infty)=\int_{-\infty}^{\infty}\left(\begin{pmatrix} -q \\ r \end{pmatrix},\ IL^n\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right)dx, \tag{3.7}$$

where $$\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)=u_1v_1+u_2v_2.$$

Furthermore, it is evident from (1.6), (3.5) and (3.6) that

$$\lim_{|x|\to\infty}\begin{pmatrix} h_1^{(j)} \\ h_2^{(j)} \end{pmatrix}=0, \quad j=0,\ 1,\ \cdots,\ n. \tag{3.8}$$

By using (3.3), (3.6) and (3.8), we have

$$\lim_{x\to-\infty}\begin{pmatrix} a_0^{(n)} \\ -d_0^{(n)} \end{pmatrix}=L_1^{-1}\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=0,$$

and $$F^{(n)}(-\infty,\ \xi)=b_1^{(n)}(-\infty)\xi\sigma_1=\frac{b_1^{(n)}(-\infty)\xi}{\xi^2}\sigma_2 R(+\infty,\ \xi).$$

Then the $b^{(n)}$ in (2.18), $\tilde{f}$ in (2.20) and $\tilde{E}$ in (2.26) can be written as follows:

$$b^{(n)}=b_1^{(n)}(-\infty)\xi=\xi\int_{-\infty}^{\infty}\left(\begin{pmatrix} -q \\ r \end{pmatrix},\ IL^n\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right)dx$$

$$=\frac{\xi}{2i}\int_{-\infty}^{\infty}\left(\begin{pmatrix} q \\ r \end{pmatrix},\ L^{n-1}\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right)dx,$$

$$\tilde{f}=-\frac{\xi}{2i}\int_{-\infty}^{\infty}\left(\begin{pmatrix} q \\ r \end{pmatrix},\ \frac{f(\xi^2,\ y,\ t)-f(L,\ y,\ t)}{\xi^2-L}\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right)dx, \tag{3.9}$$

$$\tilde{E}=-\frac{\xi^2}{2i}\int_{-\infty}^{\infty}\left(\begin{pmatrix} q \\ r \end{pmatrix},\right.$$

$$\left.\frac{h(\xi^2,\ y,\ t)-h(L,\ y,\ t)}{\xi^2-L}\begin{pmatrix} r_y \\ q_y \end{pmatrix}+\frac{g(\xi^2,\ y,\ t)-g(L,\ y,\ t)}{\xi^2-L}\begin{pmatrix} r+2xr_x \\ q+2xq_x \end{pmatrix}\right)dx. \tag{3.10}$$

3.2 For (1.2)[4] the definition of the operator η in (2.11) is

$$\eta\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & -b\xi \end{pmatrix}.$$

From (2.16) and (2.15) we obtain

$$a_1^{(n)}=b_0^{(n)}=d_0^{(n)}=0,$$

$$b_1^{(n)}=I(qa_0^{(n)}+rd_1^{(n)}),$$

$$\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=\begin{pmatrix} 2i-2rIq & 2rIr \\ 2qIq & -2i-2qIr \end{pmatrix}\begin{pmatrix} a_0^{(n)} \\ -d_1^{(n)} \end{pmatrix},$$

$$\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}=D\begin{pmatrix} a_0^{(n)} \\ -d_1^{(n)} \end{pmatrix}.$$

In analogy to 3.1, we get

$$\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}=L\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=L^n\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix},$$

$$L=D\begin{pmatrix} \frac{1}{2i}-\frac{1}{2}rIq & -\frac{1}{2}rIr \\ -\frac{1}{2}qIq & -\frac{1}{2i}-\frac{1}{2}qIr \end{pmatrix}, \tag{3.11}$$

$$b_1^{(n)}(-\infty)=\xi\int_{-\infty}^{\infty}\left(\begin{pmatrix} -q \\ r \end{pmatrix},\ IL^n\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right)dx=\frac{\xi}{2i}\int_{-\infty}^{\infty}\left(\begin{pmatrix} q \\ r \end{pmatrix},\ L^{n-1}\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right)dx,$$

$$\widetilde{E}=-\frac{\xi}{2i}\int_{-\infty}^{\infty}\left(\begin{pmatrix} q \\ r \end{pmatrix},\right.$$

$$\left.\frac{h(\xi,y,t)-h(L,y,t)}{\xi-L}\begin{pmatrix} r_y \\ q_y \end{pmatrix}+\frac{g(\xi,y,t)-g(L,y,t)}{\xi-L}\begin{pmatrix} r+xr_x \\ xq_x \end{pmatrix}\right)dx.$$

3.3 For (1.3)[5], by using $r^2+q^2+s^2=1$, the operator η may be expressed as follows:

$$\eta\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} a+ib & \frac{r}{s}a+\frac{q}{s}b \\ \frac{r}{s}a+\frac{q}{s}b & -a+ib \end{pmatrix}.$$

From (1.6) and (1.5), it yields

$$\begin{cases} a_1^{(n)}=b_1^{(n)}=d_1^{(n)}=0, \\ I(h_1^{(n)}+ih_2^{(n)})=-a_0^{(n)}, \\ I(h_1^{(n)}-ih_2^{(n)})=d_0^{(n)}, \\ I\left(\frac{r}{s}h_1^{(n)}+\frac{q}{s}h_2^{(n)}\right)=-b_0^{(n)}, \\ h_1^{(n-1)}=is(a_0^{(n)}+d_0^{(n)})-2qb_0^{(n)}, \\ h_2^{(n-1)}=s(a_0^{(n)}-d_0^{(n)})+2rb_0^{(n)}, \\ \frac{r}{s}h_1^{(n-1)}+\frac{q}{s}h_2^{(n-1)}=i[(r-iq)a_0^{(n)}+(r+iq)d_0^{(n)}]. \end{cases} \tag{3.12}$$

From (3.12), we get

$$\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=-\begin{pmatrix} 2qI\frac{r}{s}D & -2s+2qI\frac{q}{s}D \\ 2s-2rI\frac{r}{s}D & -2rI\frac{q}{s}D \end{pmatrix}I\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}=L_1\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}. \tag{3.13}$$

Using (3.12) and (3.13), we obtain

$$\begin{pmatrix} h_1^{(n)} \\ h_2^{(n)} \end{pmatrix}=L_1^{-1}\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix}=L\begin{pmatrix} h_1^{(n-1)} \\ h_2^{(n-1)} \end{pmatrix},$$

$$L=\frac{1}{2}D\begin{pmatrix} -rIqD\frac{1}{s} & \frac{1}{s}+rIrD\frac{1}{s} \\ -\frac{1}{s}-qIqD\frac{1}{s} & qIrD\frac{1}{s} \end{pmatrix},$$

and

$$b_0^{(n)}(-\infty)=\int_{-\infty}^{\infty}\frac{1}{s}\left(\begin{pmatrix} r \\ q \end{pmatrix},L^n\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \end{pmatrix}\right)dx.$$

In analogy to 3.1, we have

$$\widetilde{E}=i\xi\int_{-\infty}^{\infty}\frac{1}{s}\left(\begin{pmatrix} r \\ q \end{pmatrix},\frac{h(\xi,y,t)-h(L,y,t)}{\xi-L}L\begin{pmatrix} r_y \\ q_y \end{pmatrix}\right.$$

$$\left.+\frac{g(\xi,y,t)-g(L,y,t)}{\xi-L}L\begin{pmatrix} xr_x \\ xq_x \end{pmatrix}\right)dx.$$

3.4 For (1.4)[6], from (2.11) we have

$$\eta\begin{pmatrix}a\\b\end{pmatrix}=-\begin{pmatrix}0&0\\0&a+\xi b\end{pmatrix}.$$

From (2.16), it yields

$$\begin{cases}a_1^{(n)}=0,\\ b_0^{(n)}=-\frac{1}{2}a_{0x}^{(u)},\\ b_1^{(n)}=ia_0^{(n)},\\ d_0^{(n)}=\frac{1}{2}a_{0xx}^{(n)}-a_0^{(n)}r,\\ d_1^{(n)}=-ia_{0x}^{(n)}-a_0^{(n)}q,\\ h_2^{(n-1)}=2a_{0x}^{(n)},\\ h_1^{(n)}=-\frac{1}{2}a_{0xxx}^{(n)}+2a_{0x}^{(n)}r+a_0^{(n)}r_x,\\ h_2^{(n)}=h_1^{(n-1)}+2a_{0x}^{(n)}q+a_0^{(n)}q_x.\end{cases}\tag{3.15}$$

Using (3.15) and (2.15), we obtain

$$a_0^{(n)}=-\frac{1}{2}Ih_2^{(n-1)},\tag{3.16}$$

$$\begin{pmatrix}h_1^{(n)}\\h_2^{(n)}\end{pmatrix}=L\begin{pmatrix}h_1^{(n-1)}\\h_2^{(n-1)}\end{pmatrix},\tag{3.17}$$

$$L=\begin{pmatrix}0&-\frac{1}{4}D^2+r-\frac{1}{2}r_xI\\1&q-\frac{1}{2}q_xI\end{pmatrix},\tag{3.18}$$

$$\begin{aligned}b_1^{(n)}(-\infty)&=ia_0^{(n)}(-\infty)=-\frac{i}{2}\int_{-\infty}^{\infty}h_2^{(n-1)}(x)dx\\&=-\frac{i}{2}\int_{-\infty}^{\infty}\left(\begin{pmatrix}0\\1\end{pmatrix},\ L^{n-1}\begin{pmatrix}h_1^{(0)}\\h_2^{(0)}\end{pmatrix}\right)dx.\end{aligned}\tag{3.19}$$

From (1.6), (3.17) and (3.18) it is easily seen that

$$\lim_{|x|\to\infty}\begin{pmatrix}h_1^{(j)}\\h_2^{(j)}\end{pmatrix}=0,\quad j=0,\ 1,\ \cdots,\ n.\tag{3.20}$$

Using (3.15), (3.16) and (3.20) we have

$$F^{(n)}(-\infty)=a_0^{(n)}(-\infty)i\sigma_2R(+\infty,\ \xi),$$

$$b^{(n)}=-\frac{i\xi}{2}\int_{-\infty}^{\infty}\left(\begin{pmatrix}0\\1\end{pmatrix},\ L^{n-1}\begin{pmatrix}h_1^{(0)}\\h_2^{(0)}\end{pmatrix}\right)dx$$

$$\begin{aligned}\tilde{E}=\frac{i\xi}{2}\int_{-\infty}^{\infty}\Bigg(\begin{pmatrix}0\\1\end{pmatrix},\ &\frac{h(\xi,\ y,\ t)-h(L,\ y,\ t)}{\xi-L}\begin{pmatrix}r_y\\q_y\end{pmatrix}\\&+\frac{g(\xi,\ y,\ t)-g(L,\ y,\ t)}{\xi-L}\begin{pmatrix}2r+xr_x\\q+xq_x\end{pmatrix}\Bigg)dx.\end{aligned}$$

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