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THE STRUCTURE OF Π SPACE (II)

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Abstract

This paper continues the study of [1]. In this paper, the author discusses when a subspace of Π space is a complete subspace, and gives the structure of general linear subspaces (in particular, that of closed linear subspaces).

We adopt the terminologies and notations used in [1].

§ 1. The sufficient conditions for a subspace to be a complete subspace

According to the definition given in [1], a linear subspace L of Π is called a complete subspace if it is closed and is also a Π-space with inner product $(\cdot, \cdot)$.

Suppose that L is a closed linear subspace. In virtue of [1], Lemma 1.3, $L=L_0^-\oplus L_A\oplus L_0^+$. It follows from [1], Corollary 2.5 and Theorem 2.6 that L is a complete subspace if and only if

$$L_A\oplus L_{A^*}=\Pi^{(0)},\ \Pi^{(0)}=\overline{\mathscr{D}(A)\oplus\mathscr{R}(A)}. \tag{1.1}$$

However (1.1) holds if and only if $1\in\rho(A^*A)\cap\rho(AA^*)$, which is equivalent to $1\in\rho(A^*A)$ or $1\in\rho(AA^*)$. Thus the condition for L to be a complete subspace is $1\in\rho(A^*A)$. This condition is easy to judge. Conversely, if L is a Π-space with inner product $(\cdot, \cdot)$, when is L a complete subspace? i. e., when is L closed? It is the main content to be discussed in this section.

First, we use some results of [1] to give the general form of nondegenerate closed subspaces.

Theorem 1.1. (*The general form of non-degenerate closed subspaces*) *L is a non-degenerate closed subspace of Π if and only if the decomposition*

$$L=N\oplus P \tag{1.2}$$

holds, where N and P are negative and positive closed subspaces respectively, and there exist complete subspaces Π', Π'' such that

$$\Pi=\Pi'\oplus\Pi'',\ N\subset\Pi',\ P\subset\Pi''. \tag{1.3}$$

Proof Sufficiency. The result follows from [1], Corollary 2.7.

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Necessity. In view of [1], Corollary 2.2, $L=L_0^-\oplus L_A\oplus L_0^+$, and A, defined in $\Pi^{(0)}=\overline{\mathscr{D}(A)}\oplus\overline{\mathscr{R}(A)}$, is a one to one closed operator from $\mathscr{D}(A)$ onto $\mathscr{R}(A)$. Since L is non-degenerate, L_A is also non-degenerate. Thus $1\bar{\in}\sigma_p(A^*A)\cup\sigma_p(AA^*)$.

For any $0<\alpha<1$, by E_1' and E_2' we denote the spectral subspaces of A^*A corresponding to $[0,\alpha]$ and (α,∞) respectively. $E_1'\oplus E_2'=H_-$. Let

$$A_1'=A|_{E_1'},\quad A_2'=A|_{E_2'\cap\mathscr{D}(A)},\quad N'=L_{A_1'}=\{\{x,\ Ax\}\,|\,x\in E_1'\}.$$

With an argument like the proof of sufficiency of [1], Theorem 2.6, we have

$$\mathscr{D}(A_1')=E_1',\quad \mathscr{D}(A_2')=E_2'\cap\mathscr{D}(A),$$

$$\mathscr{R}(A_1')\perp\mathscr{R}(A_2'),\quad A=A_1'\oplus A_2'.$$

Denote the spectral subspaces corresponding to $[0,\ 1)$ and $[1,\ \infty)$ by E_1 and E_2 respectively, $E_1\oplus E_2=H_-$. From what have been proved we see that $Ax\perp Ay$ for any $x\in E_1'$, $y\in E_2\cap\mathscr{D}(A)$. Letting $\alpha\to1$, we have $Ay\perp AE_1$. Thus

$$N=\{\{x,\ Ax\}\,|\,x\in E_1\}$$

and

$$P=\{\{x,\ Ax\}\,|\,x\in E_2\cap\mathscr{D}(A)\}$$

are respectively negative and positive subspaces, N is closed. That P is closed can be deduced in a way similar to the proof of [1], Theorem 2.6. From $\mathscr{D}(A_1)\perp\mathscr{D}(A_2)$, $\mathscr{R}(A_1)\perp\mathscr{R}(A_2)$, (1.2) follows. Now put

$$\Pi'=E_1\oplus\overline{\mathscr{R}(A_1)},\quad \Pi''=E_2\oplus\overline{\mathscr{R}(A_2)}.$$

It is easy to see that (1.3) holds. The proof is complete.

Lemma 1.2. *Let L be a negative (or positive) subspace of $(\Pi,\ (\cdot,\ \cdot))$, which is a Hilbert space according to $-(\cdot,\ \cdot)$ (or $(\cdot,\ \cdot)$). Then L is a complete subspace of Π if and only if $\overline{L}$ is still a negative (or positive) subspace, that is to say, if and only if $\overline{L}$ is non-degenerate.*

Proof The necessity is obvious. Let's prove the sufficiency.

With no loss of generality, we assume that L is negative. Since L is a Hilbert space with $-(\cdot,\ \cdot)$, it will suffice to prove $L=\overline{L}$.

Since on $\overline{L}$ the induced topology of Π space is stronger than that induced by $-(\cdot,\ \cdot)$, and L is dense in $\overline{L}$ according to the topology on Π, we see that L is dense in $\overline{L}$ according to the topology induced by $-(\cdot,\ \cdot)$. Thus for any $x\in\overline{L}$, there exist $x_n\in L$ $(n=1,\ 2,\ \cdots)$ such that

$$\lim_{n\to\infty}[-(x_n-x,\ x_n-x)]=0. \tag{1.4}$$

It is easy to see from (1.4) that $\{x_n\}$ is a Cauchy sequence corresponding to $-(\cdot,\ \cdot)$. Since L is a Hilbert space with inner product $-(\cdot,\ \cdot)$, there exists $x_0\in L$ such that

$$\lim_{n\to\infty}[-(x_n-x_0,\ x_n-x_0)]=0. \tag{1.5}$$

By (1.4) and (1.5), we have

$$-(x-x_0,\ x-x_0)=0. \tag{1.6}$$

However $x-x_0\in\overline{L}$. Since $\overline{L}$ is a negative subspace (that is to say, L is non-degenerate), we have $x=x_0\in L$, i. e. $L=\overline{L}$. The proof is complete.

Theorem 1.3. *Let L be a linear subspace of $(\Pi,\ (\cdot,\ \cdot))$. If $(L,\ (\cdot,\ \cdot))$ is a Π-space, then L is a complete subspace of Π if and only if $\overline{L}$ is non-degenerate.*

Proof The necessity is obvious. We show the sufficiency.

As a Π-space, $(L,\ (\cdot,\ \cdot))$ has a regular decomposition

$$L=N\oplus P, \tag{1.7}$$

where N and P are Hilbert spaces with $-(\cdot,\ \cdot)$ and $(\cdot,\ \cdot)$ respectively.

First, we prove that $\overline{N}$ is a negative subspace. Suppose it is not. Then there must be a neutral vector z in the semi-negative subspace $\overline{N}$. Since neutral vectors in a semi-negative (or semi-positive) subspace should be orthogonal to this subspace, we have $z\perp\overline{N}$, so $z\perp N$. Besides, we see $\overline{N}\perp P$ because $N\perp P$, therefore $z\perp P$. Thus $z\in\overline{L}$, and $z\perp\overline{L}$ (since $z\perp L$), which contradicts to the assumption that $\overline{L}$ is non-degenerate. Therefore $\overline{N}$ is a negative subspace.

Similarly, one can prove that $\overline{P}$ is non-degenerate.

In view of lemma 1.2, both N and P are complete subspaces of Π. From [1], Corollary 2.8, it follows immediately that L is a complete subspace of Π. The proof is complete.

§ 2. The standard decomposition of closed subspaces

Definition 2.1. *Let L be a linear subspace of Π space. If there is a decomposition*

$$L=N\oplus Z\oplus P \tag{2.1}$$

where the subspaces N, Z, P are negative, neutral, positive respectively, then we call (2.1) *a standard decomposition of L. If L is a closed linear subspace, and N, Z, P in* (2.1) *are all closed, then* (2.1) *is called a standard decomposition of closed subspace L.*

We have introduced a standard decomposition for Π space before[2]. That is, $\Pi=N\oplus\{Z+Z^*\}\oplus P$ where Z^* and Z are a pair of dual subspaces. $\{Z+Z^*\}$ is a complete subspace.

Definition 2.2. *Let L be a closed linear subspace of Π, which has a standard decomposition* (2.1). *If there exists a standard decomposition of Π, $\Pi=N_1\oplus\{Z_1+Z_1^*\}\oplus P_1$, such that $Z_1=Z$, $P_1\supset P$, $N_1\supset N$, then we say that the standard decomposition of L can be extended, and we call $\Pi=N_1\oplus\{Z+Z^*\}\oplus P_1$ an extension of standard decomposition $L=N\oplus Z\oplus P$.*

The standard decomposition will play a fundamental role in operator theory. Thus, in this section we mainly show when a closed subspace has a standard decomposition, and when it has a standard decomposition which can be extended.

Theorem 2.1. *L is a closed linear subspace of Π if and only if there exists the following decomposition*

$$\Pi=\Pi^{(1)}\oplus\Pi^{(2)}\oplus\Pi^{(3)}$$
$$L=N\oplus Z\oplus P \tag{2.2}$$
$$N\subset\Pi^{(1)},\ Z\subset\Pi^{(2)},\ P\subset\Pi^{(3)},$$

where $\Pi^{(i)}$ $(i=1, 2, 3)$ are all complete subspaces of Π, N, Z, P are closed subspace, and Z is a maximal semi-positive as well as a maximal semi-negative subspace of $\Pi^{(2)}$.

Proof From [1], Corollary 2.7, the sufficiency follows evidently.

Now we prove the necessity. From [1], Corollary 2.2, we may assume that $L=L_A$, and under the regular decomposition $\Pi=H_-\oplus H_+$, A is a closed operator which is dense defined, one to one, and whose range is dense in H_+.

Denote the eigensubspaces of A^*A and AA^* corresponding to 1 by E and F respectively (possibly, they only contain zero vector). It is evident that E and F are closed subspaces in H_- and H_+ respectively. Put $\Pi^{(2)}=E\oplus F$, which is a complete subspace. For any $x\in E$, let $y=Ax$. It is easy to see that

$$Z_L=L\cap L^\perp=\{\{x, Ax\}\,|\,x\in E\}=\{\{A^*y, y\}\,|\,y\in F\}. \tag{2.3}$$

Take $Z=Z_L$. Evidently, Z is not only a maximal semi-positive subspace of $\Pi^{(2)}$ but also a maximal semi-negative one (refer to [1], Lemma 1.1). Z is naturally a closed subspace of $\Pi^{(2)}$. Since $\Pi^{(2)}$ is a complete subspace, of course, Z is a closed subspace of Π.

Put $\Pi'=\Pi\ominus\Pi^{(2)}=(H_-\ominus E)\oplus(H_+\ominus F)$. Denote

$$L'=\{\{x, Ax\}\,|\,x\in(H_-\ominus E)\cap\mathscr{D}(A)\}. \tag{2.4}$$

Now we porve $L'\subset\Pi'$. Since $Z\perp L$, $x\perp y$ for any $x\in(H_-\ominus E)\cap\mathscr{D}(A)$, $y\in E$, we have

$$0=(\{x, Ax\}, \{y, Ay\})=(Ax, Ay). \tag{2.5}$$

It follows from (2.5) that $A((H_-\ominus E)\cap\mathscr{D}(A))\perp F$, i. e. $L'\subset\Pi'$.

By [1], Corollary 2.7, L' is a closed linear subspace of Π', and

$$L=L'\oplus Z. \tag{2.6}$$

Using Theorem 1.1 for $L'\subset\Pi'$, we obtain $\Pi'=\Pi^{(1)}\oplus\Pi^{(3)}$, $L'=N\oplus P$, $\Pi^{(1)}\supset N$ and $\Pi^{(3)}\supset P$. The proof is complete.

Theorem 2.2. *Let L be a closed linear subspace of Π. If $L=N_L\oplus Z_L\oplus P_L$ is a standard decomposition of L, then it can be extended if and only if N_L, P_L are complete subspaces of Π.*

Proof Necessity. Suppose that the standard decomposition $\Pi=N\oplus\{Z+Z^*\}\oplus P$ is an extension of the decomposition $L=N_L\oplus Z_L\oplus P_L$. From $Z_L=L\cap L^\perp$, $N_L=N\cap L$, $P_L=P\cap L$, it follows that Z_L, N_L, P_L are all closed subspaces, so that $N|_L$, P_L are closed subspaces of complete subspaces (as Π-space) N, P. Since N, P are Hilbert spaces corresponding to $-(\cdot, \cdot)$, $(\cdot, \cdot)$ respectively, the closed subspaces of N and

P, N_L and P_L, are also Hilbert spaces with inner products $-(\cdot,\cdot)$, $(\cdot,\cdot)$, i. e. N_L, P_L are complete subspaces of Π.

Sufficiency. Since N_L is a complete subspace, so is $N_L^\perp$ (see [1], Corollary 2.8), and $\Pi=N_L\oplus N_L^\perp$ (see [1], Theorem 2.6). Moreover, since P_L is a complete subspace and $P_L\subset N_L^\perp$, P_L is a complete subspace of Π-space $N_L^\perp$. Therefore

$$N_L^\perp=P_L\oplus\Pi',\ Z_L\subset\Pi'.$$

Let $\Pi'=H'_-\oplus H'_+$ be a regular decomposition (using [1], Theorem 2.6 and Corollary 2.8 for Π-space $N_L^\perp$, it follows that Π' is a Π-space, thus the regular decomposition exists). Let $P'_\pm$ be projections from Π' onto $H'_\pm$. Take $\Pi^{(2)}=(P'_-Z_L)\oplus(P'_+Z_L)$. Since $Z_L=L\cap L^\perp$ is a closed subspace, and so is a closed subspace of Π-space Π', $P'_\pm Z_L$ are closed subspaces. Write $Z=Z_L$, $Z^*=J'Z$ where $J'=P'_+-P'_-$. We have

$$\Pi^{(2)}=\{Z+Z^*\},\ Z=Z_L,$$
$$\Pi'=\Pi^{(2)}\oplus(\Pi'\ominus\Pi^{(2)}),\ \Pi'\ominus\Pi^{(2)}=(H'_-\ominus P'_-Z_L)\oplus(H'_+\ominus P'_+Z_L). \tag{2.7}$$

Put $N=N_L\oplus(H'_-\ominus P'_-Z_L)$, $P=P_L\oplus(H'_+\ominus P'_+Z_L)$. It follows that the above standard decomposition $\Pi=N\oplus\{Z+Z^*\}\oplus P$ is an extension of $L=N_L\oplus Z_L\oplus P_L$. The proof is complete.

Theorem 2.3. *Let L be a closed linear subspace of Π. Then there exists a standard decomposition $L=N_L\oplus Z_L\oplus P_L$ which can be extended to a standard decomposition of the whole space, $\Pi=N\oplus\{Z+Z^*\}\oplus P$, if and only if there exists a regular decomposition $\Pi=H_-\oplus H_+$, under which $L=L_0^-\oplus L_A\oplus L_0^+$, $1\in\rho(A^*A)$ or 1 is one of isolated spectral points of A^*A.*

Proof Sufficiency. Suppose that under the regular decomposition $\Pi=H_-\oplus H_+$, $L=L_0^-\oplus L_A\oplus L_0^+$. Evidently $Z_L=L\cap L^\perp=\{\{x,\ Ax\}\,|\,A^*Ax=x,\ x\in\mathscr{D}(A)\}$. (if $1\in\rho(A^*A)$, then $Z_L=\{0\}$). Put $Z=Z_L$, $Z^*=JZ_L$ where $J=P_+-P_-$. Take $N_L=\{\{x,\ Ax\}\,|\,x\in E_1\}$, $P_L=\{\{x,\ Ax\}\,|\,x\in(H_-\ominus E_-)\cap\mathscr{D}(A)\}$, where E_1 is the spectral subspace of A^*A corresponding to $[0,\ 1)$. Evidently, if we take

$$N=N_L\oplus(H_-\ominus\mathscr{D}(A)),\ P=P_L\oplus(H_+\ominus\mathscr{R}(A)),\ Z=Z_L, \tag{2.8}$$

then $\Pi=N\oplus\{Z+Z^*\}\oplus P$ is a standard decomposition which is an extension of $L=N_L\oplus Z_L\oplus P_L$.

Necessity. Suppose that $\Pi'=\{Z+Z^*\}=H'_-\oplus H'_+$ is a regular decomposition. Put $H_-=N\oplus H'_-$, $H_+=P\oplus H'_+$. $\Pi=H_-\oplus H_+$ is obviously a regular decomposition. Since N_L and P_L are complete subspaces of N and P respectively, under the decomposition $\Pi=H_-\oplus H_+$, it is easy to see that if L is represented by $L=L_0^-\oplus L_A\oplus L_0^+$, then when $Z_L=\{0\}$ we have $L_A=\{0\}$, thus $H_-=N$, $H_+=P$, when $Z_L\neq\{0\}$, 1 is an isolated spectral point of A^*A, A is a unitary operator from $\mathscr{D}(A)$ onto $\mathscr{R}(A)$ because of the particularity of the regular decomposition $\Pi=H_-\oplus H_+$. The proof is complete.

It should be noted that the examples can be given easily to illustrate that in the decomposition of closed subspace L, $L=N'\oplus Z_L\oplus P'$, N' and P' needn't be closed. But for the regular decomposition of L (i. e. N', P' are closed subspaces), we have the following results.

Theorem 2.4. *Suppose that L is a closed subspace of Π, and there exists a standard decomposition $L=N_L\oplus Z_L\oplus P_L$ where N_L, P_L are complete subspace. Then for any decomposition of L, $L=N'\oplus Z_L\oplus P'$, if $N'\oplus P'$ is (or both N' and P' are) closed subspace, N' and P' are complete subspaces.*

Proof First, we prove the result in case that $N'\oplus P'$ is closed. By Theorem 2.2, for the decomposition $L=N_L\oplus Z_L\oplus P_L$ there exists an extension $\Pi=N\oplus\{Z+Z^*\}\oplus P(Z=Z_L,\ P\supset P_L,\ N\supset N_L)$, which is a standard decomposition of the whole space. Let $\Pi^{(1)}=N\oplus P$, $\Pi^{(2)}=\{Z+Z^*\}$, $L'=N'\oplus P'$. It is apparent that for any $x'\in L'$ there exists unique $x\in\Pi^{(1)}$, $z\in Z$ such that

$$x'=x+z. \tag{2.9}$$

If $x=0$, we have $z=0$ (otherwise $x'=z$, which contradicts the assumption $L'\cap Z=\{0\}$). Since L' and L are linear subspaces, it is easy to see that there exists a linear operator B: $\Pi^{(1)}\to Z$ such that

$$x'=x+Bx \qquad (\text{i. e. } z=Bx). \tag{2.10}$$

Evidently $\mathscr{D}(B)=N_L\oplus P_L\subset\Pi^{(1)}$. Using the assumption that L' is closed we see that B is a closed operator from Hilbert space $(\Pi^{(1)}, [\cdot,\cdot]^{(1)})$ into Hilbert space $(\Pi^{(2)}, [\cdot,\cdot]^{(2)})$, where $[\cdot,\cdot]^{(i)}$ $(i=1, 2)$ are respectively products induced by some regular decompositions of $\Pi^{(i)}$. Therefore B is bounded.

Now let's prove that L' has a regular decomposition (i. e. L' is a Π-space). Denote

$$N''=\{x+Bx\,|\,x\in N_L\},\ P''=\{y+By\,|\,y\in P_L\}. \tag{2.11}$$

Apparently, N'' and P'' are negative and positive subspaces respectively, $L'=N''\oplus P''$.

For any sequence $\{x_n+Bx_n\}\subset N''(n=1, 2, \cdots)$, since

$$-((x_n-x_m)+B(x_n-x_m),\ (x_n-x_m)+B(x_n-x_m))=-(x_n-x_m,\ x_n-x_m), \tag{2.12}$$

we see that $\{x_n+Bx_n\}$ is a Cauchy sequence corresponding to $-(\cdot,\cdot)$ if and only if $\{x_n\}$ is a Cauchy sequence in N corresponding to $-(\cdot,\cdot)$. In view of the fact that N is a complete subspace, there exists $x\in N$ such that

$$\lim_{n\to\infty}-(x_n-x,\ x_n-x)=0. \tag{2.13}$$

Since B is a bounded linear operator, it follows from (2.13) that according to $-(\cdot,\cdot)$, $\{x_n+Bx_n\}$ converges to $x+Bx\in N''$, i. e. N'' is a Hilbert space with inner product $-(\cdot,\cdot)$. Similarly, one can show that P'' is a Hilbert space with inner product $(\cdot,\cdot)$.

Since $L'=N'\oplus P'$ is a closed subspace, as well as a Π-space., it is a complete subspace. Since $L'=N'\oplus P'$, both N' and P' are complete subspaces of L' by [1], Theorem 2.6, therefore they are complete subspaces of Π by [1], Lemma 2.1.

Next, we discuss the case that both N' and P' are closed suspaces. Replacing L' discussed above by N', P' respectively, we obtain that for any $n'\in N'$, $p'\in P'$, there exist unique n, $p\in \Pi^{(1)}$, and linear operators B_1, B_2: $\Pi^{(1)}\to Z$ such that

$$n'=n+B_1n,\ p'=p+B_2p. \tag{2.14}$$

Denote $N''=\{n|n'=n+B_1n,\ n'\in N'\}$, $P''=\{p|p'=p+B_2p,\ p'\in P'\}$. Obviously, N'', $P''\subset\Pi^{(1)}$, $\mathscr{D}(B_1)=N''$, $\mathscr{D}(B_2)=P''$. From $N'\perp P'$ and $Z\perp N\oplus P$ we get

$$N''\perp P''. \tag{2.15}$$

For any $x\in\Pi^{(1)}\cap L$, there must be $n'\in N'$, $p'\in P'$, $z'\in Z$ such that $x=n'+p'+z'$. It follows from (2.14) that

$$x=n+p+B_1n+B_2p+z'.$$

Since $\Pi^{(1)}\cap\Pi^{(2)}=\{0\}$, we have $x=n+p$, $B_1n+B_2p+z'=0$. From $x=n+p$ we have

$$N_L\oplus P_L=N''\oplus P''. \tag{2.16}$$

By [1], Theorem 2.6, N'' and P'' are complete subspaces of $\Pi^{(1)}$, thus they are complete subspaces of Π.

Now N'' is a complete subspace, it is, of course, a closed subspace. Besides, since N' is a closed subspace, the closed operator B_1 is bounded. In the same way, we can show that B_2 is a bounded linear operator.

Since N'', P'' are complete subspace, B_1, B_2 are bounded, and N', P' are closed, it follows that N', P' are complete subspaces. The proof is complete.

Corollary 2.5. *Let L be a closed linear subspace. If there exists a standard decomposition of L, which can be extended to a standard decomposition of the whole space, then all the standard decompositions of L can be extended to standard decompositions of the whole space.*

This corollary is obvious.

Theorem 2.6. *Let L be a closed linear subspace of Π. If there exists a regular decomposition $\Pi=H_-\oplus H_+$, under which $L=L_0^-\oplus L_A\oplus L_0^+$, such that $1\in\rho(A^*A)$ or 1 is an isolated spectral point of A^*A, then under any regular decomposition $\Pi=H'_-\oplus H'_+$, for the representation of L: $L=L_0'^-\oplus L_{A'}\oplus L_0'^+$, we have $1\in\rho(A'^*A')$ or 1 is an isolated spectral point of A'^*A' correspondingly.*

Proof If under the regular decomposition $\Pi=H_-\oplus H_+$, $1\in\rho(A^*A)$, then we have $L=(L_0^-\oplus N_L)\oplus(L^+\oplus P_L)$, where

$N_L=\{\{x,\ Ax\}|x\in E_1,\ E_1$ is the spectral subspace of A^*A corresponding to $[0,\ 1)\}$,

$P_L=\{\{x,\ Ax\}|x\in E_2\cap\mathscr{D}(A),\ E_2$ is the spectral subspace of A^*A corresponding to $(1,\ \infty)\}$, (2.17)

L is evidently a complete subspace of Π. Thus $\Pi=L\oplus L^\perp$. Therefore under any regular decomposition $\Pi=H'_-\oplus H'_+$, if $L=L_0'^-\oplus L_{A'}\oplus L_0'^+$, we must have $1\in\rho(A'^*A')$ (see [1], Lemmas 1.3, 1.2). Thus the only case we should discuss is that when 1 is an isolated spectral point.

Let

$$Z_L=\{\{x,\ Ax\}\,|\,x=A^*Ax,\ x\in\mathscr{D}(A)\}. \tag{2.18}$$

Since 1 is an isolated spectral point of A^*A, $L=N_L\oplus Z_L\oplus P_L$ is a standard decomposition, and both N_L and P_L are complete subspaces of Π. By Theorem 2.4 and Corollary 2.5, it follows that any standard decomposition of L can be extended to standard decomposition of the whole space.

Suppose that $\Pi=H'_-\oplus H'_+$ is regular decomposition, $L=L_0'^-\oplus L_{A'}\oplus L_0'^+$. Since $Z_L=L\cap L^\perp\neq\{0\}$, we have $1\in\sigma_p(A'^*A')$ and

$$Z_L=\{\{x,\ A'x\}\,|\,x=A'^*A'x,\ x\in\mathscr{D}(A')\}. \tag{2.19}$$

The remainder is to prove that 1 is an isolated spectral point of A'^*A'. Put

$N'=\{\{x,\ A'x\}\,|\,x\in E_1',\ E_1'$ is the spectral subspace of A'^*A' corresponding to $[0,\ 1-0]\}$,

$P'=\{\{x,\ A'x\}\,|\,x\in E_2'\cap\mathscr{D}(A'),\ E_2'$ is the spectral subspace of A'^*A' corresponding to $[1+0,\ \infty)\}$.

Since A' is a closed operator, N' and P' are closed subspaces, $N'\perp P'$ (see Theorem 1.1), it follows that

$$L=(L_0'^-\oplus N')\oplus Z_L\oplus(L_0'^+\oplus P'),$$

which is a standard decomposition of L. Therefore $L_0'^-\oplus N'$, $L_0'^+\oplus P'$ are complete subspaces of Π, which implies that N', P' are complete subspaces of Π, thus there exists a constant α, $1>\alpha\geqslant0$, such that $\|A'|_{E_1'}\|\leqslant\alpha<1$. Similarly, one can prove that for any $x\in E_2'\cap\mathscr{D}(A')$, $x\neq0$, $\|A'x\|\geqslant(1+\beta)\|x\|$ holds, where β is some positive number, i. e. 1 is an isolated spectral point of A'^*A'. The proof is complete.

Summing up the above results, we obtain the following theorem.

Theorem 2.7. *Let L be a closed subspace of Π. Then the following propositions are equivalent to each other.*

(i) *There is a standard decomposition $L=N_L\oplus Z_L\oplus P_L$, where N_L, P_L are complete subspaces.*

(ii) *In any standard decomposition $L=N'\oplus Z_L\oplus P'$, N', P' are complete subspaces.*

(iii) *There exists a standard decomposition $L=N_L\oplus Z_L\oplus P_L$, which can be extended to a standard decomposition of the whole space Π.*

(iv) *All the standard decompositions of L, $L=N'\oplus Z_L\oplus P'$, can be extended to standard decompositions of the whole space Π.*

(v) *There exists a regular decomposition $\Pi=H_-\oplus H_+$, $L=L_0^-\oplus L_A\oplus L_0^+$, such that $1\in\rho(A^*A)$ or 1 is an isolated spectral point of A^*A.*

(vi) *For any regular decomposition*

$$\Pi=H'_-\oplus H'_+,\ L=L_0'^-\oplus L_{A'}\oplus L_0'^+,$$

it holds that $1\in\rho(A'^*A')$ or 1 is an isolated spectral point of A'^*A'.

§ 3. The decompositions of subspaces

Now we discuss general subspaces, which needn't be closed.

Lemma 3.1. *Suppose that L is a negative (or positive) subspace of Π, which is a Hilbert space with $-(\cdot, \cdot)$ (or $(\cdot, \cdot)$). For any decomposition of $\bar{L}$, $\bar{L}=N\oplus Z(Z=\bar{L}\cap\bar{L}^{\perp})$, if N is a closed subspace, then N is a complete subspace. Moreover there must exist a decomposition such that N is a closed subspace.*

Proof We only consider the case that L is a negative subspace. $\bar{L}$ is then a closed semi-negative subspace. By Theorem 2.1, there exists a decomposition $\bar{L}=N\oplus Z$ where N, Z are closed subspaces, $Z=\bar{L}\cap\bar{L}^{\perp}$, and N, Z are contained in complete subspaces $\Pi^{(1)}$, $\Pi^{(2)}$ respectively, besides, $\Pi=\Pi^{(1)}\oplus\Pi^{(2)}$. First, we prove that N is a complete subspace for this particular decomposition.

Since N is a closed subspace, it will suffice to prove that N is a Hilbert space with inner product $-(\cdot, \cdot)$. Suppose that $\Pi^{(i)}=H_{-}^{(i)}\oplus H_{+}^{(i)}$, $(i=1, 2)$, are regular decompositions, which induce products $[\cdot, \cdot]^{(i)}$. Take the regular decomposition of Π, $\Pi=(H_{-}^{(1)}\oplus H_{-}^{(2)})\oplus(H_{+}^{(1)}\oplus H_{+}^{(2)})$, which induce product $[\cdot, \cdot]$. Obviously, $[\cdot, \cdot]=[\cdot, \cdot]^{(1)}+[\cdot, \cdot]^{(2)}$. Later on we shall use these norms to calculate.

Suppose that $\{x_n\}\subset N$, and it is a Cauchy sequence corresponding to $-(\cdot, \cdot)$. Since $N\subset\bar{L}$, there exist $\{y_n\}\subset L$ such that

$$\|y_n-x_n\|\leqslant\frac{1}{n}, n=1, 2, \cdots, \tag{3.1}$$

therefore $\{y_n\}$ is a Cauchy sequence corresponding to $-(\cdot, \cdot)$. By assumption, there exists $y\in L$ such that

$$\lim_{n\to\infty}-(y_n-y, y_n-y)=0. \tag{3.2}$$

Since $y_n\in L\subset\bar{L}=N\oplus Z$, there exist unique decompositions $y_n=x_n'+z_n$ where $x_n'\in N$, $z_n\in Z$, $n=1, 2, \cdots$. We notice that $N\subset\Pi^{(1)}$, $Z\subset\Pi^{(2)}$. Using (3.1), it follows that

$$\|z_n\|\leqslant\frac{1}{n},\ \|x_n'-x_n\|\leqslant\frac{1}{n},\ n=1, 2, \cdots. \tag{3.3}$$

Denote $y=x+z$, $x\in N$, $z\in Z$. By (3.2), (3.3), we have

$$\lim_{n\to\infty}-(x_n-x, x_n-x)=0. \tag{3.4}$$

Therefore N is a Hilbert space with inner product $-(\cdot, \cdot)$, thus N is a complete subspace.

By Theorem 2.4, it follows that for any standard decomposition of $\bar{L}, \bar{L}=N\oplus Z$, N is a complete subspace. The proof is complete.

Theorem 3.2. *Let L be a linear subspace of Π. If $(L, (\cdot, \cdot))$ is a Π-space, then in any standard decomposition of $\bar{L}$, $\bar{L}=N_{\bar{L}}\oplus Z_{\bar{L}}\oplus P_{\bar{L}}$, both $N_{\bar{L}}$ and $P_{\bar{L}}$ are complete* *subspaces.*

Proof By Theorem 2.4, it will suffice to show the theorem for a particular decomposition of $\bar{L}$. Take a standard decomposition $\bar{L}=N_{\bar{L}}\oplus Z_{\bar{L}}\oplus P_{\bar{L}}$ such that there exist complete subspaces $\Pi^{(i)}$ $(i=1, 2)$, $\Pi=\Pi^{(1)}\oplus\Pi^{(2)}$, $Z_{\bar{L}}\subset\Pi^{(2)}$ and $N_{\bar{L}}\oplus P_{\bar{L}}\subset\Pi^{(1)}$ Evidently, such a particular decomposition exists.

By the hypothesis, L is a Π-space with inner product $(\cdot, \cdot)$, thus there exist regular decompositions. Suppose that $L=L_-\oplus L_+$ is a regular decomposition. Using Lemma 3.1 to L_-, it follows that there exist decompositions

$$\bar{L}_-=N\oplus Z_-,\ Z_-=\bar{L}_-\cap\bar{L}_-^{\perp},\tag{3.5}$$

where N is a complete subspace of Π.

Since $L_+\perp\bar{L}_-$, L_+ is a subspace of complete subspace $N^{\perp}$ (refer to [1], Theorem 2.5), of course, $Z_-\subset N^{\perp}$. In Π-space $N^{\perp}$, using Lemma 3.1 for L^+, there exists decomposition

$$(N^{\perp}\supset)\bar{L}_+=Z_+\oplus P,\ Z_+=\bar{L}_+\cap L_+^{\perp}=\bar{L}_+\cap\bar{L}^{\perp}\cap N^{\perp},$$

where P is a complete subspace of $N^{\perp}$, so is also a complete subspace of Π. Thus there exists decomposition

$$\begin{gathered}\Pi=\Pi^{(1)}\oplus\Pi^{(2)}\oplus\Pi^{(3)}\\ \Pi^{(1)}=N,\ \Pi^{(3)}=P,\ \Pi^{(2)}=\Pi\ominus(N\oplus P),\ Z_+\subset\Pi^{(2)}.\end{gathered}\tag{3.6}$$

Since $Z_-\perp L_+$, $Z_-\perp\bar{L}_+$. Hence $Z_-\perp Z_+$, $Z_-\perp P$. Therefore $Z_-\subset\Pi^{(2)}$, and Z_-+Z_+ is still a neutral subspace contained in $\Pi^{(2)}$. Denote $Z=\overline{Z_-+Z_+}$. Apparently

$$\bar{L}_-+\bar{L}_+\subset N\oplus Z\oplus P.\tag{3.7}$$

Since $N\oplus Z\oplus P$ is a closed linear subspace of Π, we have

$$\bar{L}\subset N\oplus Z\oplus P.\tag{3.8}$$

On the other hand, because $N\oplus Z_-=\bar{L}_-$, $P\oplus Z_+=\bar{L}_+$, hence N, P, Z_-+Z_+ are all linear subspaces of $\bar{L}$. Thus $Z\subset\bar{L}$. So we have

$$\bar{L}=N\oplus Z\oplus P,$$

where N, P are complete subspaces. By Theorem 2.4, Theorem 3.2 follows. The proof is complete.

§ 4. Dual families

Definition 4.1. *Let* $\{z_\lambda, \lambda\in\Lambda\}$, $\{z_\lambda^*, \lambda\in\Lambda\}$ *be two families of vectors of* Π. *Suppose that for any* λ, $\mu\in\Lambda$,

$$(z_\lambda, z_\mu^*)=\delta_{\lambda\mu},\tag{4.1}$$

where $\delta_{\lambda\mu}$ *is Kroneckr functions* (i. e. $\delta_{\lambda\mu}=0$ *if* $\lambda\neq\mu$, $\delta_{\lambda\mu}=1$ *if* $\lambda=\mu$). *Then* $\{z_\lambda, \lambda\in\Lambda\}$, $\{z_\lambda^*, \lambda\in\Lambda\}$ *are called dual families. If for all* $\lambda\in\Lambda$, $z_\lambda=z_\lambda^*$, *then we call* $\{z_\lambda, \lambda\in\Lambda\}$ *a self-dual family.*

From (4.1) it follows that $\{z_\lambda, \lambda\in\Lambda\}$ and $\{z_\lambda^*, \lambda\in\Lambda\}$ are respectively linear independent vector families.

In Hilbert space, any family of orthonormal basis $\{e_\lambda, \lambda\in\Lambda\}$ is self-dual. However, for indefinite inner product space, the degenerate subspaces appear. In general, the concept of "orthonormal basis" relative to the inner product makesno sense. On these occasions, we frequently use the dual families. In particular, for the standard decomposition $\Pi=N\oplus\{Z+Z^*\}\oplus P$, we often treat Z and Z^* by dual families. Indeed, it is a convenient tool in operator theory. Here we give the following theorem which is similar to a result for Hilbert space.

Theorem 4.1. *Let Π be a separable indefinite inner product space. If $\{z_\lambda, \lambda\in\Lambda\}$ is a self-dual family of Π, then the cardinal of Λ is not more than $\aleph_0$.*

Proof Let $\Pi=H_-\oplus H_+$ be a regular decompotition. $L=\overline{\text{span}}\ \{z_\lambda, \lambda\in\Lambda\}$. Evidently, L is a separable closed subspace of Π. Take a countable set $\{x_n\}$ which is dense in L. Since L is spanned by $\{z_\lambda, \lambda\in\Lambda\}$, for each x_i there exist at most countable indexes $\{\lambda^i_j\}\ (j=1, 2, \cdots)$ such that

$$x_i\in\overline{\text{span}}\{z_{\lambda^i_j}, j=1, 2, \cdots\}=L_i, \quad i=1, 2, \cdots. \tag{4.2}$$

Suppose that Λ isn't countable. Take $\lambda\in\Lambda-\{\lambda^i_j, i, j=1, 2, \cdots\}$. Since $z_\lambda\in L$, there exists a subsequence of $\{x_i\}$ which converges to z_λ. But $x_i\in L_i$, so that there exists a subsequence of the countable set $\{\lambda^i_j, i, j=1, 2, \cdots\}$, denoted by $\{\lambda_j, j=1, 2, \cdots\}$, such that a certain sequence $\{\varphi_j\}$ consisting of linear combinations of $\{z_{\lambda_j}\}$, $\left(\varphi_j=\sum_{k=1}^{n_j}\alpha^j_k z_{\lambda_k}, j=1, 2, \cdots\right)$, converges to z_λ, i. e.

$$\|z_\lambda-\varphi_j\|\to 0 \qquad (j\to\infty). \tag{4.3}$$

Since $(\cdot, \cdot)$ is a continuous functional with two variables, we bave

$$1=(z_\lambda, z_\lambda)=(z_\lambda, \lim_{j\to\infty}\varphi_j)=\lim_{j\to\infty}(z_\lambda, \varphi_j)=0,$$

which is a contradiction. Thus the cardinal of Λ is not more than $\aleph_0$.

References

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[2] Yan Shaozong, *Sci. Sinica*, 4(1981).