Chin. Ann. of Math.
5B (3) 1984

BVPS FOR DIFFERENTIAL OPERATORS WITH CHARACTERISTIC DEGENERATE SURFACES

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Abstract

The present paper is devoted to the BVPs of Lopatinski's type for differential operators of higher order and mixed type. A sign function, on which the formulation of BVPs and regularity of their solutions rely heavily, is associated with characteristic degenerate surface. The behavior of finite index for such kind of BVPs is found.

§ 1. Introduction

In study of the partial differential equation, $P(x, D)u=f$, in Ω, one of the most fundamental questions is what conditions prescribed on $\partial\Omega$ can ensure the corresponding boundary value problems to be well posed. As is well known, quite complete results have been obtained only for the classical cases, which refer to elliptic, parabolic and hyperbolic equations, but for other cases results obtained are not systematic yet. Without doubt, the problem is very closely related to the type of the operator P. When P is of principal type, a number of BVPs have been investigated by Wenston ([1]). Recently, the study of the case with multiple characteristics has attracted a certain amounts of interests. Baouendi and Goulaouic ([2]), Chi ([3]) successively discussed several kinds of characteristic Cauchy problems for Fuchs type operators. Gu ([4]) studied two kinds of boundary value problems for a class of higher order mixed equations, which are the generalization of Busemann equation to the case of higher order and whose degenerate surfaces are characteristics for themselves. As a continuation of Gu's work, the present paper is devoted to the general BVPs for such kinds of differential operators.

Consider a differential operator of order m

$$P(x, D)=P_m(x, D)+P_{m-1}(x, D)+\cdots, \quad D=\frac{1}{\sqrt{-1}}(\partial_{x_1}, \cdots, \partial_{x_n}),$$

defined in a bounded domain $\Omega\subset R^n$. Here P_m has real coefficients. Assume that all coefficients of P are smooth enough and that the conditions H's are fulfilled.

(H_1) There exists a function $\psi(x)\in C^\infty(\overline{\Omega})$ with $d\psi|_{\overline{\Omega}}\neq 0$ such that $\partial\Omega=S_{-1}\cup S_1$,

Manuscript received February 16, 1982.

say $S_t=\{x|\psi(x)=t\}$, $t\in[-1, 1]$.

(H_2) $p_m(x, d\psi)\neq 0$ outside S_0 and $\psi_{x_j}p^j_m(x, \tau d\psi+\eta)\neq 0$ as $p_m(x, \tau d\psi+\eta)=0$, $\tau\in R^1$, $R^n\ni\eta\not\parallel d\psi$, $(x, \eta)\overline{\in} S_0\times\{0\}$.

(H_3) $p^j_m(x, d\psi)|_{S_0}=0$ $(j=1, \cdots, n)$, $d\psi\cdot d(p_m(x, d\psi))|_{S_0}\neq 0$ and the matrix $(p^{j_1j_2}_m(x, d\psi)|_{S_0})$ has $(n-1)$ positive eigenvalues.

Condition (H_2) implies that $P(x, D)$ is of principal type when $(x, \xi)\overline{\in}\{(x, \rho d\psi(x))|x\in S_0, \rho\in R^1\backslash\{0\}\}$. The first assumption of (H_3) means that S_0 is a multiple characteristic surface, and the latter assumptions of (H_3) show that there is a pair of real characteristics reducing to complex characteristics when point x crosses S_0.

In close analogy with [11], the sign function on S_0 can be defined. Set

$$C_{m-1}(x, D)=\text{ principal part of }\frac{\sqrt{-1}}{2}(P-P^*).$$

Clearly, $C_{m-1}(x, D)$ is just the subprincipal symbol of $\sqrt{-1}\,P$, i. e.,

$$C_{m-1}(x, \xi)=\sqrt{-1}\left(p_{m-1}(x, \xi)-\frac{1}{2\sqrt{-1}}\frac{\partial^2 p_m}{\partial x_j\partial\xi_j}(x, \xi)\right), \tag{1.1'}$$

which is coordinate-invariant on multiple characteristics. A sign function on S_0

$$\operatorname{sgn}(S_0, P)=\operatorname{Re}C_{m-1}(x, d\psi)+\frac{m-1}{2}d\psi\cdot d(p_m(x, d\psi))/|d\psi|^2 \tag{1.1}$$

can be associated. To determine (1.1) uniquely, we always require

$$d\psi\cdot d(p_m(x, d\psi))|_{S_0}>0.$$

(this hypothesis is not serious. If necessary, substitute $-P$ for P or $-\psi(x)$ for $\psi(x)$). It is worth to point out that (1.1) is invariant on S_0 under coordinate transformation, because the second term of (1.1) is invariant too. In fact, $d(p_m(x, d\psi))=\lambda d\psi$, since $p_m(x, d\psi)$ is zero-order tensor and $p_m(x, d\psi)|_{S_0}=0$. Hence $\lambda=d\psi\cdot d(p_m(x, d\psi))/|d\psi|^2$ has the expected behavior.

The boundary value problems we shall discuss are of Lopatinski's type. By means of Condition (H_2) we have

$$\begin{aligned}p_m(x, \tau d\psi+\eta)&=p_m(x, d\psi)\prod_{k=1}^{m_e}(\tau-\lambda^k_+)\prod_{k=1}^{m_e}(\tau-\lambda^k_-)\prod_{k=1}^{m_h}(\tau-\lambda^k)\\&=p_m(x, d\psi)p^+_m(\tau)p^-_m(\tau)p^0_m(\tau), \text{ on } S_a(a=\pm 1).\end{aligned}$$

Here λ^k_+ (λ^k_- or λ^k) are the roots with imaginary parts>0 (<0 or $=0$ respectively). It should be emphasized that $m_e(1)=m_e(-1)+1$, $m_h(1)=m_h(-1)-2$ where $m_e(a)=m_e$, $m_h(a)=m_h$ on S_a. Two kinds of boundary value problems will be studied. Let $\{B_{J_a}\}$ be normal systems of boundary differential operators on S_a of orders r_{J_a} with $J_a\in\mathscr{F}_a$, being subsets of

$$\{0, 1, \cdots, m-1\}\quad (a=1, -1).$$

T_+(as sgn $(S_0, P)>0$):

the principal parts of $B_{J_{-1}}(x, \tau d\psi+\eta)$, mod $p^+_m(\tau)$, on S_{-1},

the principal parts of $B_{J_1}(x, \tau d\psi+\eta)$, mod $p^-_m(\tau)p^0_m(\tau)$, on S_1

are linearly independent, respectively. (1.2)

T_- (as $\mathrm{sgn}(S_0, P)<0$):

the principal parts of $B_{J_{-1}}(x, \tau d\psi+\eta)$, mod $p_m^+(\tau)p_m^0(\tau)$, on S_{-1},

the principal parts of $B_{J_1}(x, \tau d\psi+\eta)$, mod $p_m^-(\tau)$, on S_1

are linearly independent, respectively. (1.3)

Note that $\mathscr{F}_a$ in T_+ does not equel to $\mathscr{F}_a$ in T_-. In the other hand, if there does not occur the multiple characteristic surface S_0, i. e.,

$(H_2)'$ $\quad p_m(x, d\psi)|_{\bar{\Omega}}\neq 0$ and $\psi_{x_j}p_m^j(x, \tau d\psi+\eta)|_{\bar{\Omega}}\neq 0$ as $p_m(x, \tau d\psi+\eta)=0$

$$\tau\in R^1,\ \eta\in R^n,$$

which means that P is of principal type, then the sign of (1.1) need not be taken into account and both (1.2) and (1.3) can be formulated.

Consider the following boundary value problem

$$P(x, D)u=f,\ \text{in } \Omega;\ B_{J_a}(x, D)u=g_{J_a},\ \text{on } S_a,\ J_a\in\mathscr{F}_a. \tag{1.4}$$

Choose suitable differential operators on S_a, $\{C_{J'_a}\}$ with $J'_a\in\mathscr{F}'_a=\{0, 1, \cdots, m-1\}\backslash\mathscr{F}_a$, in such a way that $\{B_{J_a}, C_{J'_a}\}$ forms a Dirichlet system. If $\{C'_{J_a}, B'_{J'_a}\}$ is the corresponding dual Dirichlet system, then

$$\int(v\overline{Pu}-P^*v\bar{u})dx=\sum_{J_a\in\mathscr{F}_a}\langle C'_{J_a}v, B_{J_a}u\rangle_{\partial\Omega}+\sum_{J'_a\in\mathscr{F}'_a}\langle B'_{J'_a}v, C_{J'_a}u\rangle_{\partial\Omega}. \tag{1.5}$$

The adjoint problem of (1.4), denoted by $T^*((T_+)^*\text{or}(T_-)^*)$, may be written as follows

$$P^*(x, D)v=f',\ \text{in } \Omega;\ B'_{J'_a}v=g'_{J'_a},\ \text{on } S_a,\ J'_a\in\mathscr{F}'_a. \tag{1.6}$$

In section 2, we shall prove that Problem T satisfies (1.2) if and only if Problem T^* does (1.3). In section 4, the well-posedness of problem T for differential operators of real principal type will be obtained with the aid of studying on propagation and reflection on boundary $S_a(a=1, -1)$ of singularities, which is based on several fundamental lemmas in section 3. Perhaps, such a method is more powerful than the usual way carried out in base space. Section 5 will be devoted to the BVPs of Lopatinski's type for differential operators with characteristic degenerate surfaces. The behavior of finite index will be found and the regularity of solutions relies heavily on the sign function. The major techniques are micro-local analysis near the multiple characteristic surface S_0.

The author would like to express his thanks to Professor Gu for his guidance.

§ 2. B. v. p. of Lopatinski's type

By Lop condition we mean (1.2) or (1.3). Since Lop condition is invariant under non-sigular transformation of coordinates, we can pass to study the case of homogeneous differential operators with constant coefficients after boundary flattened and coefficients frozen. For the details of techniques, see [5, P 133]. In this section, we shall restrict ourselves to study the case that P is defined in R^n_+ and $\{P, B_j\}$ are homogeneous operators with constant coefficients. The methods in [5] with slight modification may be applied to the case, including real roots. Set

$$\mathscr{H}^m_s=\{u\in\mathscr{D}'(R^1_+)\,|\,e^{st}D^k u\in H(R^1_+),\ k\leqslant m\}$$

and

$$\mathscr{L}_0=\bigcap_{s>0}\mathscr{H}^m_{-s}.$$

If $P(\eta, \tau)$ is a homogeneous polynomial of degree m and $P(0, 1)=1$, then, like the factorization of $p_m(x, \tau d\psi+\eta)$ mentioned above, we have

$$P(\eta, \tau)=P^+(\tau)P^-(\tau)P^0(\tau).$$

The polynomials $B_j(\eta, \tau)$ corresponding to boundary differential operators can be written as follows

$$B_j(\eta, \tau)=\sum_{k=1}^{m_e+m_h} b_{jk}(\eta)\tau^{k-1} \bmod P^+(\tau)P^0(\tau).$$

Condition (1.3) on S_{-1}, saying condition L_{+0}, is equivalent to

$$j=1, \cdots, m_h+m_e \text{ and } \det(b_{jk}(\eta))\neq 0,\ \forall\eta\in S^{n-2}.$$

Consider the boundary value problems for ordinary differential equation

$$P(\eta, D_t)u=0,\ t>0;\ B_j(\eta, D_t)u=g_j,\ t=0. \tag{2.1}$$

Lemma 2.1. *Lop condition L_{+0} is equivalent to each of the following ones*

(1) *for any given $\eta\in S^{n-2}$, the solution to (2.1) is unique in $\mathscr{L}_0$;*

(2) *for any given $\eta\in S^{n-2}$ and $\{g_j\}\in\mathbf{C}^{m_h+m_e}$ there is a solution to (2.1) in $\mathscr{L}_0$.*

Lemma 2.2. *$\{P, B_j\}$ satisfies Lop condition L_{+0} if and only if the problem*

$$P(\eta, D_t)u=f,\ t>0;\ B_j(\eta, D_t)=g_j,\ t=0, \tag{2.2}$$

where $f\in\mathscr{L}_0$ and $\{g_j\}\in\mathbf{C}^{m_h+m_e}$ has a solution in $\mathscr{L}_0$.

From the same argument as in [5, proposition 4.2] immediately follows Lemmas 2.1 and 2.2. Here we omit the details.

Analogously

$$B_j(\eta, \tau)=\sum_{k=1}^{m_e} b^+_{jk}(\eta)\tau^{k-1} \bmod P^+(\tau).$$

Condition (1.2) on S_{-1}, saying conditon L_+, holds if and only if

$$j=1, \cdots, m_e \text{ and } \det(b^+_{jk}(\eta))\neq 0,\ \forall\eta\in S^{n-2}.$$

Lemma 2.3. *Lop conditionL_+ is equivalent to each of the following ones*

(1) *the solution to (2.2) is unique in $\mathscr{H}^m_s$,*

(2) *for any given $f\in\mathscr{H}^0_s$ and $\{g_j\}\in\mathbf{C}^{m_e}$, (2.2) is solvable in $\mathscr{H}^m_s$, here $s\in(0, s_0]$ with $s_0=\frac{1}{2}\min\{I_m\lambda^k_+(\eta)\,|\,\eta\in S^{n-2}\}$.*

Theorem 2.1. *$\{P, B_j\}$ satisfies Lop condition L_{+0} if and only if $\{P^*, B'_{j'}\}$, its adjoint problem, satisfies Lop condition L_+.*

Proof Let $\{P^*, B'_{j'}\}$ satisfy Lop condition L_+ and let $u\in\mathscr{L}_0$ satisfy $Pu=0$, $B_ju|_{t=0}=0$. Then for any $\varphi\in C_C^\infty(\bar{R}^1_+)$ we can find a solution v_φ to Problem $P^*v=\varphi u$, $t>0$, $B'_{j'}v=0$, $t=0$, since $\{P^*, B'_{j'}\}$ satisfies Lop condition L_+ and $\varphi u\in\mathscr{H}^m_s(\forall s>0)$. By means of Green formula (1.2) we can derive

$$\int_0^\infty \varphi u^2\,dt=0$$

which gives $u\equiv0$. The proof of the converse part is the same as before. Details need not be repeated.

Remark Consider the case that there occurs the multiple characteristic surface S_0. Note that $p^*_m=p_m$ and for P^*

$$C^*_{m-1}(x, \xi) = \text{principal part of } \frac{\sqrt{-1}}{2}(P^*-(P^*)^*) = -C_{m-1}(x, \xi).$$

Thus

$$\begin{aligned}\operatorname{sgn}(S_0, P^*) &= \operatorname{Re} C^*_{m-1}(x, d\psi)+\frac{m-1}{2}d\psi\cdot d(p_m(x, d\psi))/|d\psi|^2\\ &= -\operatorname{sgn}(S_0, P)+(m-1)d\psi\cdot d(p_m(x, d\psi))/|d\psi|^2,\end{aligned}$$

which implies that the adjoint problem of T_+, $(T_+)^*$, satisfies (1.3), namely, $(T_+)^*=T_-$ when sgn $(S_0, P)>(m-1)d\psi\cdot d(p_m(x, d\psi))/|d\psi|^2$ and the adjoint problem of T_-, $(T_-)^*$, always satisfies (1.2), namely, $(T_-)^*=T_+$.

§ 3. Several lemmas

In this section we present several fundamental lemmas for needs in section 4, 5.

Let

$$E^m=\{\text{the space of all polynomials of degree}\leqslant m\}.$$

Lemma 3.1. *If $Q_i(\tau)\in E^{m_i}$ $(i=1, 2)$ have no common root and both of coefficients in leading term are equal to 1, then for any given $R(\tau)\in E^{m_1+m_2-1}$, there exist only two polynomials $P_i(\tau)\in E^{m_i-1}(i=1, 2)$, satisfying*

$$Q_1(\tau)P_2(\tau)+Q_2(\tau)P_1(\tau)=R(\tau). \tag{3.1}$$

Proof Obviously, it is sufficient to verify that the map

$$E^{m_1-1}\times E^{m_2-1}\ni(P_1, P_2)\to\varphi(P_1, P_2)=Q_1P_2+Q_2P_1\in E^{m_1+m_2-1} \tag{3.2}$$

is injective. Assume $Q_1P_2+Q_2P_1=0$. The fact that Q_i have no common root and $P_i\in E^{m_i-1}(i=1, 2)$ yields $P_i\equiv0$, which implies Lemma 3.1.

Corollary 3.1. *Let*

$$Q_1(\tau)=\tau^{m_1}+\sum_{l=1}^{m_1}a_l\tau^{m_1-l},\quad Q_2(\tau)=\tau^{m_2}+\sum_{l=1}^{m_2}b_l\tau^{m_2-l}$$

and

$$R(\tau)=\sum_{l=0}^{m_1+m_2-1} c_l\tau^{m_1+m_2-1-l}.$$

Then the coefficients of solution (P_1, P_2) to (3.1) are smooth functions of $\{a_l\}$, $\{b_l\}$ and $\{c_l\}$.

Proof Suppose that

$$P_i=\sum_{l=0}^{m_i-1} d_l^i\tau^{m_i-1-l}\quad (i=1, 2).$$

Indeed, the map φ (3.2) may be regarded as that of

$$\mathbf{C}^{m_1}\times\mathbf{C}^{m_2}\ni(\{d_l^1\}, \{d_l^2\})\mapsto\{c_l\}\in\mathbf{C}^{m_1+m_2}.$$

Its injection yields $d\varphi$ to be non-singular for given $\{a_l\}\in\mathbf{C}^{m_1}$, $\{b_l\}\in\mathbf{C}^{m_2}$. So the theorem of implicit functions presents the expected assertion.

Set

$$P(x, t, D_x, D_t)=\sum_{j=0}^{m} A^j(t)\Lambda^j D_t^{m-j},\ \text{in } R^n=R^{n-1}\times R^1, \tag{3.3}$$

where $A^0(t)\equiv I$ and $A^j(t)$ $(j\neq 0)$ are some smooth one parameter-t families of operators in OPS^0 with asymptotic expansions

$$\sigma(A^j)\sim a_0^j+a_{-1}^j+\cdots$$

and $\mathrm{Supp}_x\,\sigma(A^j)\subset K$ for some compact set $K\subset R^{n-1}$, independent of t. In the sequal, unless stated otherwise, capital letters A, B, C, $\cdots$ stand for such pseudodifferential operators and small letters $\{a_{-\nu}\}$, $\{b_{-\nu}\}$, $\{c_{-\nu}\}$ stand for ν-th terms of corresponding expansions. Denote naturally the principal part of P by

$$p_m(x, t, \xi, \tau)=\sum_{j=0}^{m} a_0^j(x, t, \xi)|\xi|^j\tau^{m-j}.$$

Lemma 3.2. *If, as polynomials of τ, $p_m(x, t, \xi, \tau)=p_{m_1}(x, t, \xi, \tau)\cdot p_{m-m_1}(x, t, \xi, \tau)$ in a conic neighbourhood of (x_0, t_0, ξ_0), $\Gamma(x_0, t_0, \xi_0)$, and $p_{m_1}(\tau)$, $p_{m-m_1}(\tau)$ satisfy the conditions in Lemma 3.1, then there exist two differential operators P_{m_1}, P_{m-m_1} like (3.3) with principal parts equal to $p_{m_1}(x, t, \xi, \tau)$ and $p_{m-m_1}(x, t, \xi, \tau)$ such that*

$$P=P_{m_1}(x, t, D_x, D_t)P_{m-m_1}(x, t, D_x, D_t),\ \text{near } (x_0, t_0, \xi_0).$$

Proof Suppose

$$P_{m_1}=\sum_{j=0}^{m_1} B^j\Lambda^j D_t^{m_1-j},\quad P_{m-m_1}=\sum_{j=0}^{m-m_1} C^j\Lambda^j D_t^{m-m_1-j}.$$

From the fundamental calculus of pseudodifferential operators, it follows that

$$\sum_{j=0}^{m_1} b_0^j|\xi|^j\tau^{m_1-j}=p_{m_1}(x, t, \xi, \tau), \tag{3.5}$$

$$\sum_{j=0}^{m-m_1} c_0^j|\xi|^j\tau^{m-m_1-j}=p_{m-m_1}(x, t, \xi, \tau), \tag{3.6}$$

and transport equations

$$\begin{aligned}&\Big(\sum_{j=1}^{m_1} b_{-1}^j|\xi|^j\tau^{m_1-j}\Big)p_{m-m_1}(x, t, \xi, \tau)+p_{m_1}(x, t, \xi, \tau)\Big(\sum_{j=1}^{m-m_1} c_{-1}^j|\xi|^j\tau^{m-m_1-j}\Big)\\&=-\sum_{l=1}^{n}\frac{1}{\sqrt{-1}}\,p_{m_1}^{(l)}\cdot p_{m-m_1,(l)}+p_{m-1}(x, t, \xi, \tau)\end{aligned} \tag{3.7}$$

$$\left(\sum_{j=1}^{m_1} b^j_{-\nu}|\xi|^j\tau^{m_1-j}\right)p_{m-m_1}+p_{m_1}\left(\sum_{j=1}^{m-m_1} c^j_{-\nu}|\xi|^j\tau^{m-m_1-j}\right)=\cdots \tag{3.8}$$

The terms omitted in (3.8) depend only on $b^j_{-\nu+k}$, $c^j_{-\nu+k}$, $k\geqslant 1$, and P. Evidently, $b^j_0(x, t, \xi)$, $c^j_0(x, t, \xi)$ obtained from (3.5) (3.6) are in S^0 and smooth for t in $\Gamma(x_0, t_0, \xi_0)$. By means of Lemma 3.1 and (3.7) we can also obtain $b^j_{-1}, c^j_{-1}\in S^{-1}$ which have the same smoothness as b^j_0, c^j_0. In fact, Corollary 3.1 shows their smoothness, and the uniqueness of solution to (3.1) gives their homogeneity. Similarly, other $b^j_{-\nu}$, $c^j_{-\nu}$ can be found recursively. Lemma is proved.

We shall next derive a theorem of existence and regularity for Cauchy problem of some kind of elliptic operators. The technique we now use is due to Treves in [8, p 140] and Taylor in [10, p192]. Suppose the principal part of (3.3),

$$p_m(x, t, \xi, \tau) \text{ only has roots with positive imaginary parts.} \tag{3.9}$$

Consider the problem

$$Pu=f,\ \text{in } \Omega=R^{n-1}\times(0, T),\ D_t^j u=g_j,\ t=0,\ j=0, \cdots, m-1. \tag{3.10}$$

We try to construct a parametrix $S(t)$ for (3.10), which is of the form

$$S(t)(g_0, \cdots, g_{m-1}, f)=\sum_{l=1}^{m} B^l(t)g_{l-1}+\int_0^t B^m(t, t_1)f(t_1)dt_1 \tag{3.11}$$

where $B^l(t)$ $(l=1, \cdots, m)$ and $B^m(t, t_1)$ will next be determined by asymptotic expansions. Applying P to (3.11) we have

$$0=PS(t)(g_0, \cdots, g_{m-1}, f)-f=\sum_{l=1}^{m}\bar{B}^l(t)g_{l-1}+\int_0^t \bar{B}^m(t, t_1)f(t_1)dt,$$

where

$$\bar{b}^l(x, t, \xi)=e^{-ix\xi}(PB^l)e^{ix\xi},$$
$$\bar{b}^m(x, t, t_1, \xi)=e^{-ix\xi}(PB^m(t_1))e^{ix\xi}.$$

From the lemma of the fundamental asymptotic expansion, $\bar{b}^l\sim\sum p^{(\alpha)}(x, t, \xi, D_t)\times b^l_{(\alpha)}(x, t, \xi)/\alpha!$ it follows immediately that

$$p_m(x, t, \xi, D_t)b^l_0=0,\ \text{with } D_t^k b^l_0|_{t=0}=\delta_l^{k+1} \tag{3.12}$$

and transport equations

$$p_m(x, t, \xi, D_t)b^l_{-1}=-p_{m-1}(x, t, \xi, D_t)b^l_0-\sum_{|\alpha|=1}p_m^{(\alpha)}(x, t, \xi, D_t)b^l_{0,(\alpha)} \tag{3.13}$$

$$p_m(x, t, \xi, D_t)b^l_{-\nu}=\cdots \tag{3.14}$$

$$D_t^k b^l_{-\nu}|_{t=0}=0,\ \nu=1, \cdots,\ k=0, 1, \cdots, m-1.$$

The terms omitted in (3.14) depend only on $b^l_{-\nu'}(\nu'<\nu)$, P and are homogeneous functions of degree $m-\nu$ in (ξ, D_t). We have

Lemma 3.3. *Let (3.9) be fulfilled. Then there exists a positive constant $C_1>0$, independent of $l, k, x, t, N, \alpha, \nu, \beta$ such that when $|\xi|\geqslant 1$, $(x, t_1)\in\Omega$*

(1) $|D_\xi^\beta D_t^k D_x^\alpha b^l_{-\nu}(x, t, \xi)|\leqslant C_{\beta,k,l,\alpha}(1+|\xi|)^{-(l+\nu+|\beta|)+k+1}e^{-C_1|\xi|t}$,

(2) $|\widehat{D_t^k D_x^\alpha b^l_{-\nu}}(\eta, t, \xi)|\leqslant C_{k,l,\alpha,N}(1+|\eta|)^{-N}(1+|\xi|)^{-(l+\nu)+k+1}\cdot e^{-C_1|\xi|t}$.

Moreover, estimates (1) and (2) are still true if $b^l_{-\nu}(x, t, \xi)$ is replaced by $b^m_{-\nu}(x, t, t_1, \xi)$ in the left-hand side of (1), (2) and t by $(t-t_1)$ in the right-hand side.

Proof It only needs to discuss the case of $l=m$. (3.12) may be rewritten as follows

$$p_m(x,\ 0,\ \xi,\ D_t)b_0^m=-t\sum_{j=1}^{m}\left[\frac{\partial a^j}{\partial t}\right](x,\ t,\ \xi)\,|\xi|^jD_tb_0^m,\quad D_t^kb_0^m\Big|_{t=0}=\delta_m^{k+1}\qquad(3.12')$$

where $\left[\frac{\partial a_0^j}{\partial t}\right](x,\ t,\ \xi)=\int_0^1\frac{\partial a_0^j}{\partial t}(x,\ \lambda t,\ \xi)d\lambda$. If $\Gamma_{\xi x}$ is some simple curve in upper half plane, surrounding all roots of $p_m(x,\ t,\ \xi,\ \tau)=0$, then

$$\tilde{b}_0^m=\frac{1}{2\pi i}\int_{\Gamma_{\xi x}}(e^{i\tau t}/p_m(x,\ 0,\ \xi,\ \tau))d\tau.\qquad(3.15)$$

is the solution to homogeneous equation of (3.12′). Changing variables $\tau=|\xi|\tau_1$, one can find a simple curve Γ in upper half plane, independent of x, ξ, t, such that

$$\tilde{b}_0^m=\frac{|\xi|^{-(m-1)}}{2\pi i}\int_{\Gamma}\left(e^{i\tau|\xi|t}\Big/\sum_{j=0}^{m}a_0^j(x,\ 0,\ \xi)\,\tau^{m-j}\right)d\tau.\qquad(3.15')$$

Evidently, estimates (1) and (2) are valid for $\tilde{b}_0^m$, when

$$C_1=\frac{3}{4}\min_{|\xi|=1,(x,t)\in\bar{\Omega}}\{\text{imaginary parts of roots for } p_m(\tau)=0\}=\frac{3}{4}C^*.$$

As is well known, the solution of (3.12′)

$$b_0^m(t)=\tilde{b}_0^m(t)-\int_0^t\left[\tilde{b}_0^m(t-t_1)t_1\sum_{j=1}^{m}\left[\frac{\partial a_0^j}{\partial t}\right](t_1)\,|\xi|^jD_{t_1}^{m-j}b_0^m(t_1)\right]dt_1.$$

By induction one can obtain estimates for iterated sequence of this integral equation, which yields that (1) and (2) hold for b_0^m, provided $C_1=\frac{1}{2}C^*$. Set

$$b_0^m(t,\ t_1)=\tilde{b}_0^m(t-t_1)-\int_{t_1}^t\left[\tilde{b}_0^m(t-t_2)t_2\sum_{j=1}^{m}\left[\frac{\partial a_0^j}{\partial t}\right](t_2)\,|\xi|^jD_{t_2}^{m-j}b_0^m(t_2,\ t_1)\right]dt_2.$$

Without difficulties, we can get

$$|D_t^kD_x^\alpha b_0^m(x,\ t,\ t_1,\ \xi)|\leqslant C_{k,\alpha}(1+|\xi|)^{-m+k+1}e^{-C_1|\xi|(t-t_1)}\qquad(3.16)$$

and for any $N\in\mathbf{Z}$

$$|\widehat{D_t^kD_x^\alpha b_0^m}(\eta,\ t,\ t_1,\ \xi)|\leqslant C_{k,\alpha,N}(1+|\eta|)^{-N}(1+|\xi|)^{-m+k+1}\cdot e^{-C_1|\xi|(t-t_1)}.\qquad(3.17)$$

Similarly, $b_{-\nu}^m(x,\ t,\ \xi)$ and $b_{-\nu}^m(x,\ t,\ t_1,\ \xi)$ also satisfy (1) and (2) when $C_1=\frac{1}{4}C^*$ because the right side of (3.14) are homogeneous in ξ and D_t. Lemma 3.3 is proved.

So far we have constructed the operators $B^l(t)\in OPS^{-l+1}$ and $B^m(t,\ t_1)\in OPS^{-m+1}$, smooth for t and $(t,\ t_1)$ with symbols as asymptotic to sums

$$\sum_{\nu=0}^{\infty}b_{-\nu}^l(x,\ t,\ \xi)\text{ and }\sum_{\nu=0}^{\infty}b_{-\nu}^l(x,\ t,\ t_1,\ \xi).\qquad(3.18)$$

Lemms 3.4. *Let* (3.9) *be fulfilled. Then for given* $(g_0,\ \cdots,\ g_{m-1})\in\mathscr{E}'(R^{n-1})$, $f\in\mathscr{E}'(\bar{R}_+^n)$,

$$u(x,\ t)=S(t)(g_1,\ \cdots,\ g_{m-1},\ f)=\sum_{l=1}^{m}B^l(t)g_{l-1}+\int_0^tB^m(t,\ t_1)f(t_1)dt_1\qquad(3.19)$$

satisfies

(1) $Pu-f\in C^\infty(R^{n-1}\times[0,\ T))$; $D_t^ku-g_k\in C^\infty(R^{n-1})$.

(2) $S(t)$ *is a linear bounded operator of* $H_{s-m}(R^{n-1}\times(0,T))\times\prod_{0\leqslant k\leqslant m-1}H_{s-k-\frac{1}{2}}(R^{n-1})$

into $H_s(R^{n-1}\times(0, T))$ *with* $m\leqslant s\in\mathbf{Z}$.

(3) $S(t)$ *is regularizing for* $t>0$.

Proof (1) (3) is the trivial consequence of Lemma 3.3 (1). So it only remains to verify the assertion (2). Let u be defined as (3.19) and let

$$g_{l-1}\in H_{s-l+\frac{1}{2}}(R^{n-1}),\ f\in H_{s-m}(R^{n-1}\times(0, T)).$$

From the continuity of pseudodifferential operators and (3.19) it follows only that

$$D_t^k u(t)\in H_{s-k-\frac{1}{2}}(R^{n-1})$$

for $t\in[0, T]$ when $k\leqslant s-1$. Therefore we must more closely study

$$D_t^k u(x, t)\quad (k=0, 1, \cdots, m-1)$$

$$\begin{aligned}\widehat{D_t^k u}(\eta, t)&=\sum_{l=1}^{m}\int_{|\xi|\geqslant 1}+\int_{|\xi|\leqslant 1}\widehat{D_t^k b^l}(\eta-\xi, t, \xi)\hat{g}_{l-1}(\xi)d\xi\\&\quad+\int_0^t\int_{|\xi|\geqslant 1}+\int_{|\xi|\leqslant 1}\widehat{D_t^k b^m}(\eta-\xi, t, t_1, \xi)\hat{f}(\xi, t_1)d\xi\, dt_1\\&=I_1(\eta, t)+I_2(\eta, t)+I_3(\eta, t)+I_4(\eta, t).\end{aligned}\tag{3.20}$$

Compute

$$\int_0^T\int|\widehat{D_t^k u}(\eta, t)|^2(1+|\eta|^2)^{s-k}d\eta\, dt.$$

Certainly, the integrations with respect to I_2, I_4 are bounded. Using Lemma 3.3 (2) and Cauchy inequality, we have, for $k=0, 1, \cdots, m-1$,

$$\begin{aligned}&\int_0^T\int|I_1(\eta, t)|^2(1+|\eta|^2)^{s-k}d\eta\, dt\\&\leqslant C\sum_{l=1}^{m}\int_0^T\int_{|\xi|\geqslant 1}(1+|\xi|^2)^{s-k}(1+|\xi|^2)^{k+1-l}|\hat{g}_{l-1}(\xi)|^2e^{-2C_1|\xi|t}d\xi\, dt\\&\leqslant C'\sum_{l=1}^{m}\|g_{l-1}\|^2_{s-l+\frac{1}{2}}.\end{aligned}$$

Similarly, with $k=0, 1, \cdots, m-1$,

$$\int_0^T\int|I_3(\eta, t)|^2(1+|\eta|^2)^{s-k}d\eta\, dt\leqslant C''\|f\|^2_{s-m}.$$

Application of original equation in (3.10) gives the estimates for $D_t^k u(x, t)$, $k=m, \cdots, s$, which completes the proof.

Let differential operator

$$R=D_t^m+\sum_{|\alpha|\leqslant m,\alpha_n\leqslant m-1}a^\alpha D^\alpha$$

be defined in an open subset Ω of $\overline{R}^n_+$. Set

$$\mathscr{H}=\{u\in H^{\text{loc}}(\Omega)\,|\,Ru\in H^{\text{loc}}_{s-m}(\Omega),\ m\leqslant s\in\mathbf{Z}\}.$$

By the theorem on partial hypoellipticity given by Hörmander [7, p. 107] we have, for $k=0, 1, \cdots, s-1$,

$$\gamma_k u=\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0}\in H^{\text{loc}}_{-k-\frac{1}{2}}(\partial\Omega\cap\{t=0\}),\ \text{if } u\in\mathscr{H}.$$

Indeed, this famous theorem implies

Lemma 3.5. *γ_k is a continuous map of $\mathscr{H}$ into $H^{loc}_{-k-\frac{1}{2}}(\partial\Omega\cap t=0)$ if $\mathscr{H}$ is equipped with topology of usual countable norms.*

The following lemma is an important result about regularity up to boundary of solutions for elliptic operators, which is due to Trèves and its proof may be refered to [8, p. 140].

Lemma 3.6. *If (3.9) is fulfilled and $u\in H^{loc}(R^n_+)$, $Pu|_{O(x_0)\times[0,T)}\in C^\infty$, $D^k_t u|_{O(x_0)\times\{0\}}\in C^\infty$, $k=0, \cdots, m-1$, then $u|_{O(x_0)\times[0,T)}\in C^\infty$.*

§ 4. B. v. p. for differential operators of real principal type

In this section we are concerned with the well-posedness of boundary value problems for differential operators of real principal type. When there is no multiple characteristic surface in $\overline{\Omega}$, we prove that Problem T is of the behavior of finite index and $\mathrm{Ker}(T)\subset C^\infty(\overline{\Omega})$. The plan is, firstly, to discuss the regularity of solutions to problem T, which, indeed, is of behavior with loss of 1-derivative for f, and secondly, to obtain a priori estimates for Problem T by the theorem of closed graph, which, immediately, gives the existence.

If $u\in H(\Omega)$, $Pu=f\in H(\Omega)$, then it follows that $r_k u$ $(k=0, \cdots, m-1)$ make sense from the fact that $\partial\Omega$ is non-characteristic and the theorem on partial hypoellipticity. In the sequal, we often use this assertion without statement.

Theorem 4.1. *If (1.4) satisfies (H_1) $(H_2)'$ and (1.2) or (1.3) and $u\in H(\Omega)$, $\{P, B_{J_a}\}u\in H_{s-m}(\Omega)\times H_{s-1-r_{J_a}}(S_a)$ with $m\leqslant s\in\mathbf{Z}$, then*

$$u\in H_{s-1}(\Omega),\ r_k u\in H_{s-1-k}(\partial\Omega),\ 0\leqslant k\leqslant m-1. \tag{4.1}$$

Furthermore, the inequality

$$\|u\|^2_{s-1}+\sum_{k=0}^{m-1}\|r_k u\|^2_{s-1-k}\leqslant C_s(\|Pu\|^2_{s-m}+\sum_{J_a}\|B_{J_a}u\|^2_{s-1-r_{J_a}}+\|u\|^2) \tag{4.2}$$

is valid, where C_s is independent of u.

Proof First of all, it only needs to study the case of (1.3), since the verification for (1.2) is the same as that for (1.3). Secondly, we can prove that (4.2) is the consequence of (4.1). Set

$$\mathscr{H}^1=\{u\in H(\Omega)\,|\,Pu\in H_{s-m}(\Omega),\ B_{J_a}u\in H_{s-r_{J_a}-1}(S_a),\ J_a\in\mathscr{F}_a\}$$

equipped with the norm $\|Pu\|^2_{s-m}+\sum_{J_a}\|B_{J_a}u\|^2_{s-r_{J_a}-1}+\|u\|^2$ and set

$$\mathscr{H}^2=\{u\in H_{s-1}(\Omega)\,|\,r_k u\in H_{s-1-k}(\partial\Omega),\ k=0, \cdots, m-1\}.$$

equipped with the norm

$$\|u\|^2_{s-1}+\sum_{k=0}^{m-1}\|r_k u\|^2_{s-1-k}$$

Obviously, $\mathscr{H}^2$ is a Hilbert space. In fact, $\mathscr{H}^1$ is also a Hilbert space. Suppose that $\{u_n\}$ is a Cauchy sequence in $\mathscr{H}^1$. From the definition of $\mathscr{H}^1$ it is easily seen that there exist $u_0 \in H(\Omega)$, $f \in H_{s-m}(\Omega)$, $g_{J_a} \in H_{s-1-r_{J_a}}(\partial\Omega)$ such that $\lim u_n = u_0$, $\lim Pu_n = f$, $\lim B_{J_a} u_n = g_{J_a}$ in respective spaces. The continuity in $\mathscr{D}'(\Omega)$ of P gives $Pu_0 = f \in H_{s-m}(\Omega)$ and Lemma 3.5 yields that $B_{J_a} u_n \to B_{J_a} u_0$ in $H_{-r_{J_a}-\frac{1}{2}}(\partial\Omega)$. Thus, $B_{J_a} u_0 = g_{J_a} \in H_{s-1-r_{J_a}}(\partial\Omega)$ which means $u_0 \in \mathscr{H}^1$. The inclusion map i:

$$\mathscr{H}^1 \ni u \mapsto u \in \mathscr{H}^2$$

does make sense with the aid of (4.1) and is evidently closed. Therefore, i is continuous, namely, (4.2) is valid.

We now proceed to prove (4.1). It is sufficient to prove (4.1) locally. Start with discussion near S_{-1}. After boundary flatten and solution u localized, we are faced to the following situation

$$Pu = f, \text{ in } R_+^n \tag{4.3}$$

with

$$B_{J_{-1}} u = g_{J_{-1}}, \text{ on } t=0, \; J_{-1} \in \mathscr{F}_{-1}, \tag{4.4}$$

where $u \in \mathscr{E}'(\bar{R}_+^n) \cap H(R_+^n)$, $f \in \mathscr{E}'(\bar{R}_+^n)$, $g_{J_{-1}} \in \mathscr{E}'(R^{n-1})$ and there exists a neighbourhood in $\bar{R}_+^n$ of $(x_0, t_0) = (0, 0)$, $\mathscr{N} = O(0) \times [0, T)$, such that

$$f|_{\mathscr{N}} \in H_{s-m}, \; g_{J_{-1}}|_{O(0)} \in H_{s-1-r_{J_a}}. \tag{4.5}$$

Of course, $\{P, B_{J_{-1}}\}$ in (4.3) (4.4) still satisfies $(H_1)(H_2)'$ and (1.3). By means of Lemma 3.2, one can find operators P^-, P^0, P^+ like (3.3) with principal parts

$$p_{m_e}^-(x, t, \xi, \tau) = p_m^-(\tau), \quad p_{m_0}^0(x, t, \xi, \tau) = p_m^0(\tau), \quad p_{m_e}^+(x, t, \xi, \tau) = p_m^+(\tau),$$

resp., satisfying

$$Pu = P^- P^0 P^+ u = f, \text{ mod } C^\infty(\mathscr{N}), \tag{4.6}$$

if necessary, shrinking $\mathscr{N}$. Applying Lemma 3.4 and Lemma 3.6 to operator P^- and solving a backward Cauchy problem, we can get

$$P^0 P^+ u|_{O(0)\times[0,T)} \in H_{s-m+m_e} \tag{4.7}$$

for some new $O(0)$ and T, which means

$$D_t^k P^0 P^+ u|_{O(0)\times(0)} \in H_{s-m+m_e-k-\frac{1}{2}}, \; 0 \leqslant k \leqslant m_e - 1. \tag{4.8}$$

(4.4) and (4.8) form a system of equations of pseudodifferential operators for $(\Lambda^{m-1}u, \Lambda^{m-2}D_t u, \cdots, D_t^{m-1}u) = (u_0, \cdots, u_{m-1})$. Write

$$\Lambda^{m_e-1-k} D_t^k P^0 P^+ u = \sum_{l=0}^{m-1} B^{kl}(x, 0, D_x) u_l$$

and

$$\Lambda_{m-1-r_{J_{-1}}} B_{J_{-1}} u = \sum_{l=0}^{m-1} B^{J_{-1}l}(x, D_x) u_l.$$

The system of equations of pseudodifferential operators mentioned above may be rewritten as follows

$$\begin{cases} \sum_{l=0}^{m-1} B^{kl} u_l|_{O(0)} \in H_{s-m+\frac{1}{2}}, \; k = 0, \cdots, m_e - 1, \\ \sum_{l=0}^{m-1} B^{J_{-1}l} u_l|_{O(0)} \in H_{s-m}, \; J_{-1} \in \mathscr{F}_{-1}. \end{cases} \tag{4.9}$$

Note that Hypothesis (1.3) implies that (4.9) is elliptic. So

$$\Lambda^{m-1-l}D_t^l u=u_l\in H_{s-m}(O'(0))\ (l=0,\ \cdots,\ m-1) \text{ for any } O'(0)\subset\subset O(0),$$

namely

$$r_l u|_{O'(0)}\in H_{s-1-l}. \tag{4.10}$$

Combining (4.7) with (4.10), solving a forward Cauchy problem for the strictly hyperbolic operator P^0, we get

$$P^+u|_{O'(0)\times[0,T)}\in H_{s-m_e-1}$$

for some new $O'(0)$ and T. Application of Lemma 3.4 and Lemma 3.6 to P^+ just gives

$$u|_{O'(0)\times[0,T)}\in H_{s-1}.$$

Let us return to original coordinates. One can find a neighbourhood in $\bar{\Omega}$ of S_{-1}, $\mathcal{N}(S_{-1})$, such that $u\in H_{s-1}(\mathcal{N}(S_{-1}))$, $r_k u\in H_{s-1-k}(S_{-1})$. Using the Hörmander's theorem on propagation of singularities, we can reach that

$$u\in H_{s-1} \text{ at any point } x\in\bar{\Omega}\backslash S_1. \tag{4.11}$$

It only remains to prove the regularity up to S_1. By the similar argument carried out near S_{-1} with slight modification, the assertion (4.1) near S_1 may be done. We leave it to readers.

Corollary 4.1. *If assumptions in Theorem 4.1 are fulfilled, then $Ker(T)\subset C^\infty(\bar{\Omega})$ is of finite dimension and Problem T has finite index.*

Corollary 4.1 is the consequence of (4.2) and Peetr's Theorem. On the other hand, by the compactness of $\bar{\Omega}$ and using a standard proceedure of functional analysis, we can get

Theorem 4.2. *If assumptions in Theorem 4.1 are fulfilled and*

$$\{f,\ g_{J_a}\}\in H_{s-m}(\Omega)\times\prod_{J_a} H_{s-1-r_{J_a}}(S_a)$$

with $m\leqslant s\in\mathbf{Z}$, then (1.4) is of solvability if and only if

$$(v,\ f)_\Omega-\sum_{J_a}\langle C'_{J_a}v,\ g_{J_a}\rangle_{\partial\Omega}=0,\ \forall v\in \mathrm{Ker}(T^*).$$

For details of such techniques, see [6].

§ 5. B. v. p. for differential operators with characteristic degenerate surfaces

In this section we are concerned with the well-posedness of boundary value problems for differential operators with characteristic degenerate surfaces. The emergence of multiple characteristic surface causes some difficulties different from those in Problem T. The crucial point to attack Problems T_+ and T_- lies in study of the behavior near S_0. Let us first introduce a lemma about reduction of operators, which is necessary for microlocal analysis.

After surface S_0 flatten, operator P we consider is of the form

$$P=tD_t^m+\sum_{l=1}^{n-1}a_lD_{x_l}D_t^{m-1}+\sum_{l_1,l_2=1}^{n-1}a_{l_1l_2}D_{x_{l_1}}D_{x_{l_2}}D_t^{m-2}+\sum_{|\alpha|\geq 3}^{m}a_\alpha D_x^\alpha D_t^{m-|\alpha|}$$
$$+\frac{1}{\sqrt{-1}}bD_t^{m-1}+\sum_{k+|\alpha|<m,k\leq m-2}a_\alpha D_x^\alpha D_t^k. \tag{5.1}$$

Because $(H_1)(H_2)(H_3)$ are invariant under the transformation used above, we have

$$a_l(x, 0)=0 \quad (l=1, \cdots, n-1), \tag{5.2}$$

$$\operatorname{sgn}(S_0, P)=(d\psi\cdot d(p_m(x, d\psi))/|d\psi|^2)\cdot\left(\operatorname{Re}b-\frac{1}{2}\right). \tag{5.3}$$

Lemma 5.1. *Let (5.1) satisfy (5.2). Then one can find a pseudodifferential operator of order 2, $P_2(x, t, D_x, D_t)$ such that*

$$P=D_t^{m-2}P_2$$

mod $OPS^{-\infty}$, near n_0

$$n_0=(x_0, t_0, \xi_0, \tau_0)=(0, 0, 0, 1). \tag{5.4}$$

Proof We shall obtain P_2 as asymptotic to a sum

$$\sigma(P_2)\sim p_2+p_1+\cdots.$$

Evidently, by a standard proceedure of symbols calculus, we have

$$p_2=t\tau^2+e(x, t, \xi, \tau) \tag{5.6}$$

with

$$e=\sum_{l=1}^{n-1}a_l\xi_l\tau+\sum_{l_1,l_2=1}^{n-1}a_{l_1}a_{l_2}\xi_{l_1}\xi_{l_2}+\sum_{|\alpha|\geq 3}^{m}a_\alpha\xi^\alpha\tau^{-|\alpha|+2}, \tag{5.7}$$

and

$$p_1=\frac{1}{\sqrt{-1}}(b-m+2)\tau+\sum_{|\alpha|\geq 1}^{m-1}b_\alpha\xi^\alpha\tau^{1-|\alpha|}-\frac{1}{\sqrt{-1}}e_t\tau^{-1}. \tag{5.8}$$

The others may be obtained in a similar way.

Let $(x_0, \xi_0)\in T^*(\Omega)$. We say that $u\in H_k(x_0, \xi_0)$ if and only if there exist $\varphi(x)\in C_c^\infty(\Omega)$ with $\varphi(x_0)\neq 0$ and a conic neighbourhood of ξ_0, $\Gamma(\xi_0)$, such that

$$(1+|\xi|^2)^{\frac{k}{2}}\widehat{\varphi u}(\xi)\in L^2(\Gamma(\xi_0)).$$

It is easily seen that $u\in H(x_0)\Leftrightarrow u\in H_k(x_0, \xi)$, $\forall\xi\in S^{n-1}$.

Lemma 5.2. *If (5.1) satisfies (5.2) and $u\in H(x_0, t_0)$, $Pu\in H_k(n_0)$ with $k\in\mathbf{Z}$ or $k=0$, then (1) $u\in H_{k+(m-1)}(\Omega)$ when $\operatorname{Re}b-\frac{1}{2}>m-1$, (2) $u\in H_{k+(m-1)}(n_0)$, when $\operatorname{Re}b-\frac{1}{2}+k<0$ and there is a conic neighbourhood of n_0, $\Gamma(n_0)$, such that $u\in H_{k+(m-1)}$ (x, t, ξ, τ) with $\xi\neq 0$ and $(x, t, \xi, \tau)\in\Gamma(n_0)$.*

Proof Lemma 5.1 shows that $P_2=A_{-(m-2)}P$ mod $OPS^{-\infty}$ near (n_0). From now on, by A_k we mean some elliptic pseudodifferential operator of order k at the point in $T^*(\Omega)$ under consideration. Besides, the principal symbol of P_2, (5.6), satisfies

$$e(x, t, 0, \tau)\equiv 0,\ e_\xi(x, 0, 0, \tau)=0. \tag{5.9}$$

In the meantime, from a slight computation it will be seen that there is a conic neighbourhood of n_0, $\Gamma(n_0)$, such that

$$P_2 \text{ is non-radial on } p_2^{-1}(0)\cap\Gamma(n_0)\setminus(\Sigma=\{t=0,\ \xi=0\}). \tag{5.10}$$

According to Lemmas 2.3 and 2.4 in [12], one can fiind a Fourier integral operator F associated with a sympletic transformation χ:

$$T^*(R^n_{xt})\supset\Gamma(n_0)\to\Gamma'(n_0)\subset T^*(R^n_y)$$

and two elliptic pseudodifferential operators A_0, A_{-1} in $\Gamma'(n_0)$ such that

$$A_{-1}FP_2F^{-1}A_0=(D_n+q(y')),\ \text{near } n_0 \tag{5.11}$$

with

$$q(y')=\frac{1}{\sqrt{-1}}(b-m+1)\circ x^{-1}|_{y_n=0,\eta'=0}. \tag{5.12}$$

Let u and f be involved in this lemma. Thus with $F^{-1}A_0v=u$

$$A_{-1}FA_{-(m-2)}f=(D_n+q'(y'))v,\ \text{near } n_0. \tag{5.13}$$

Obviously, $v\in H^{\text{loc}}(R^n_y)$, $A_{-1}FA_{-(m-2)}f\in H_{k+(m-1)}(n_0)$. Therefore, by means of the hypotheses in this lemma, and Lemmas 3.3 and 3.4 in [12], we have

$$v\in H_{k+m-1}(n_0) \text{ if } \operatorname{Re}\sqrt{-1}\,q(y')>\frac{1}{2},\ \text{i. e., } \operatorname{Re}b-\frac{1}{2}>(m-1)$$

and

$$v\in H_{k+m-1}(n_0) \text{ if } \operatorname{Re}\sqrt{-1}\,q(y')-\frac{1}{2}+(m-1)+k<0 \text{ i. e., } \operatorname{Re}b-\frac{1}{2}+k<0,$$

which proves the results expected for u, since

$$u=F^{-1}A_0v.$$

Now we proceed to study Problems T_+ and T_-.

Theorem 5.1. *Let* (1.4) *satisfy* $(H_1)(H_2)$ (H_3) *and* (1.2) *or* (1.3). *Assume that*

$$u\in H(\Omega),\ \{P,\ B_{J_a}\}u=\{f,\ g_{J_a}\}\in H_{s-m}(\Omega)\times\prod_{J_a}H_{s-1-r_{J_a}}(S_a)$$

with $m\leqslant s\in\mathbf{Z}$. *Then*

$$u\in H_{s-1}(\Omega),\ r_ku\in H_{s-1-k}(\partial\Omega)$$

when

$$\operatorname{sgn}(S_0,\ P)>(m-1)d\psi\cdot d(p_m(x,\ d\psi))/|d\psi|^2$$

or

$$\operatorname{sgn}(S_0,\ P)+(s-m)d\psi\cdot d(p_m(x,\ d\psi))/|d\psi|^2<0, \tag{5.14}$$

and

$$\|u\|^2_{s-1}+\sum_{k=0}^{m-1}\|r_ku\|^2_{s-1-k}\leqslant C_s(\|Pu\|^2_{s-m}+\sum_{J_a}\|B_{J_a}u\|^2_{s-1-r_{J_a}}+\|u\|^2) \tag{5.15}$$

is valid.

Proof The proofs for T_+ and T_- are the same, so we only deal with Problem T_-. As is done in verification of Theorem 4.1 we can reach the conclusion

$$r_ku|_{s_{-1}}\in H_{s-k-1};\ u\in H_{s-1},$$

near and up to S_{-1}.

By Hörmander's theorem on propagation of singularites we can obtain that

$$u \in H_{s-1} \text{ at any point } x \in \overline{\Omega} \backslash S_1. \tag{5.16}$$

Let $\psi(x)<0$ at point x and $\xi \in R^n \backslash \{0\}$. If $p_m(x, \xi) \neq 0$ which means P is elliptic at this point, then the fact that $Pu \in H_{s-m}(\Omega)$ implies

$$u \in H_s(x, \xi).$$

If

$$p_m(x, \xi)=0,$$

then the projection on to the base $\overline{\Omega}$ of the bicharacteristic through (x, ξ) always meets S_{-1} at last. Hörmander's theorem shows

$$u \in H_{s-1}(x, \xi), \tag{5.16'}$$

which yields (5.16).

Let $0<\psi(x)<1$ at point x. Conditions (H_3) and (H_2) guarantee that the projection onto base $\overline{\Omega}$ of the bicharacteristic through any point (x, ξ) with $p_m(x, \xi)=0$, $\xi \neq 0$, is certainly transversal to S_0. Then along the familiar line used above, (5.16) can obtained. Let $x \in S_0$ and $\xi \not\parallel d\psi(x)$. Condition (H_2) implies that the projection onto base $\overline{\Omega}$ of the bicharacteristic through (x, ξ) is transversal to S_0 too. So (5.16') is true. It only remains to deal with the case of $x \in S_0$, $\xi \parallel d\psi(x)$. When S_0 is flatten, P is of the form (5.1) and (x, ξ) is mapped as $n_0=(x_0, t_0, \xi_0, \tau_0)=(0, 0, 0, \pm 1)$ and

$$u \in H(R^n_{xt}),\ Pu=f \in H_{s-m}(n_0), \tag{5.17}$$

$$u \in H_{s-1}(x, t, \xi, \tau)$$

with (x, t) near $(0, 0)$ and $\xi \neq 0$. Applying Lemma 5.2 to (5.17), we have

$$u \in H_{s-1}(n_0) \text{ if } \operatorname{Re} b-\frac{1}{2}+(s-m)<0, \text{ i. e.,}$$

$$0>\operatorname{sgn}(S_0, P)+(s-m)d\psi \cdot d(p_m(x, d\psi))/|d\psi|^2.$$

So the result like (5.16') follows at once. The proof of regularity up to S_1, as before, is left to readers. From (5.14) and the same argument as in Theorem 4.1, (5.15) follows immediately. This proves Theorem 5.1.

Theorem 5.1 shows the difference in regularity between Problems T_+ and T_-. It is worth noting that the regularity of solutions to Problem T_- can not be improved infinitely, even if

$$\{f, g_{J_a}\} \in C^\infty(\overline{\Omega}) \times \prod_{J_a} C^\infty(S_a).$$

Corollary 5.1. *If the assumptions in Theorem* 5.1 *are fulfilled, then* $T_\pm$ *is of finite index and* $\operatorname{Ker}(T_+) \subset C^\infty(\overline{\Omega})$ *when*

$$\operatorname{sgn}(S_0, P)-(m-1)d\psi \cdot d(p_m(x, d\psi))/|d\psi|^2>0,$$

whereas $\operatorname{Ker}(T_-) \subset H_{m-1}(\Omega)$ *when*

$$\operatorname{sgn}(S_0, P)<0.$$

From the Remark of Theorem 2.1, it is easily seen $(T_-)^*=T_+$. So (5.15) holds for the adjoint problem of (1.4), i. e., (1.6). Application of a standard proceedure of functional analysis gives

Theorem 5.2. *If the assumptions in Theorem* 5.1 *are fulfilled and*

$$\{f,\ g_{J_a}\}\in H_{s-m}(\Omega)\times\prod_{J_a}H_{s-1-r_{J_a}}(S_a)$$

with $m\leqslant s\in\mathbf{Z}$, *then* (1.4) *is solvability if and only of*

$$(v,\ f)_\Omega-\sum_{J_a}\langle C'_{J_a}v,\ g_{J_a}\rangle_{\partial\Omega}=0,\ \forall v\in \mathrm{Ker}(T^*).$$

Remark 1. In [4], there is an example to show $\mathrm{Ker}(T_\pm)\neq\{0\}$.

Remark 2. For the case of $m=2$, see [11].

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