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ON VECTOR VALUED ORLICZ SPACES

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Abstract

The theory of vector valued Orlicz spaces generated by generalized N-functions was introduced by M. S. Skaff (see [1, 2]). But the proofs of two main theorems in [2] are incorrect. In this paper, the author exhibits an example to show the incorrectness, gives the correct proofs and improves one of those theorems.

1. In this section, we quote some definitions from [1, 2] in brief.

Definition 1. *Let T be a space of points with non-atomic, σ-finite measure and E^n n-dimensional Euclidean space. We call a real valued non-negative function $M(t, x)$ defined on $T\times E^n$ a GN-function if it satisfies*

(1) *$M(t, x)=0$ if and only if $x=0$ for all t in T and all x in E^n;*

(2) *$M(t, x)$ is a continuous convex function of x for each t and a measurable function of t for each x;*

(3) *For each t in T, $\lim\limits_{|x|\to\infty} M(t, x)/|x|=+\infty$;*

(4) *There are constants $K\geqslant 1$ and $d\geqslant 0$ satisfying $\int_T \overline{M}(t, d)dt<\infty$ such that*

$$M(t, x)\leqslant KM(t, y)$$

for all t in T and all x, y in E^n satisfying $d\leqslant|x|\leqslant|y|$ where

$$\overline{M}(t, d)=\sup_{|x|=d} M(t, x).$$

Definition 2. *Orlicz class*

$$L_M=\{x(t)\in X \mid \text{modular } R_M(x)=\int_T M(t, x(t))dt<\infty\};$$

Orlicz space

$$L_M^*=\{y(t)\in X \mid \text{there is a constant } a>0 \text{ such that } a\cdot x(t)\in L_M\},$$

where

$$X=\{x(t)=(x_1(t), x_2(t), \cdots, x_n(t)) \mid x_k(t) \text{ is a measurable real function on T}, k=1, 2, \cdots, n\}.$$

Definition 3. *The norm of $x(t)$ in L_M^* is defined as*

$$\|x\|=\max\{\|x\|^+, \|-x\|^+\},$$

where

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$$\|x\|^+=\inf\{k>0 \mid R_M(x/k)\leqslant 1\}.$$

Definition 4. *We say a GN-function $M(t, x)$ satisfies a Δ-condition if there exist a constant $K\geqslant 2$ and a non-negative measurable function $q(t)$ such that the function $\overline{M}(t, 2q(t))$ is integrable over the domain T and for almost all t in T, we have*

$$M(t, 2x)\leqslant KM(t, x)$$

for all x satisfying $|x|\geqslant q(t)$.

2. In this section, we give an example to point out the mistake, the common part of the proofs of Theorems 2.2 and 3.3 in [2]:

(*) "if $M(t, x)$ does not satisfy a Δ-condition, there exists a sequence of points $\{x_k\}$ in E^n tending to infinity and a set T_0 of finite positive measure such that

$$M(t, 2x_k)>2^k M(t, x_k)$$

for all t in T_0 and all $k=1, 2, \cdots$."

Example Set $T=[0, 1]$, $E=R$ (the real line). Define

$$M(t, x)=\begin{cases} e^{2^k/k}[x^2-(2^k/k)^2+1]-1, \text{ when } t\in[1-1/2^{k-1}, 1-1/2^k) \\ \quad \text{and } |x|>2^k/k,\ k=1, 2, \cdots, \\ e^{|x|}-1, \text{ otherwise} \end{cases}$$

and

$$q_0(t)=\begin{cases} 2^k/k, \text{ when } t\in[1-1/2^{k-1}, 1-1/2^k),\ k=1, 2, \cdots, \\ +\infty, \text{ when } t=1. \end{cases}$$

It is easily verified that $M(t, x)$ is a GN-function and we show that it does not satisfy a Δ-condition first. For any constant $K\geqslant 2$ and any non-negative measurable function $q(t)$ satisfying $\int_T M(t, 2q(t))dt<\infty$, we choose $x_0<0$ such that

$$e^{2|x|}-1>K(e^{|x|}-1)$$

whenever $|x|\geqslant x_0$, an integer m such that $2^m/m>x_0$, and an integer $K_0\geqslant 1$ such that $\frac{1}{2}q_0(t)\geqslant 2^m/m$ for all t in $[1-1/2^{K_0}, 1]$. This K_0 exists because $q_0(t)\to+\infty$ as $t\to 1$. It follows from

$$\int_{1-1/2^{K_0}}^1 M(t, q_0(t))\,dt=\sum_{k=K_0+1}^{\infty}(e^{2^k/k}-1)1/2^k>\sum_{k=K_0+1}^{\infty}1/k-1=+\infty$$

and

$$\int_{1-1/2^{K_0}}^1 M(t, 2q(t))dt<+\infty$$

that the set

$$T^*=\{t\in[1-1/2^{K_0}, 1] \mid q_0(t)>2q(t)\}$$

is nonempty. For any t in T^*, defining

$$x_t=\frac{1}{2}q_0(t)>q(t)$$

and observing

$$q_0(t) > x_t = \frac{1}{2} q_0(t) \geqslant 2^m/m > x_0,$$

we have

$$M(t, 2x_t) = e^{2x_t} - 1 > K(e^{x_t} - 1) = KM(t, x_t).$$

Therefore, the GN-function $M(t, x)$ does not satisfy a Δ-condition

Next, we prove that for any non-negative constant $c<1$, there exists a constant $x_0>0$ such that

$$M(t, 2x) \leqslant 5M(t, x)$$

holds for all t in $[0, c]$ and all x in R satisfying $|x| \geqslant x_0$. In fact, let N be a positive integer such that $1/2^N < 1-c$, then for each $k=1, 2, \cdots, N$,

$$M(t, x) = e^{\frac{2^k}{k}}[x^2 - (2^k/k)^2 + 1] - 1 \tag{1}$$

for all t in $[1-1/2^{k-1}, 1-1/2^k)$ and $|x| > 2^k/k$. Since the expression (1) is a quadratic function of x, there is a constant $x'_k > 2^k/k$ such that

$$e^{\frac{2^k}{k}}[(2x)^2 - (2^k/k) + 1] - 1 \leqslant 5\{e^{\frac{2^k}{k}}[x^2 - (2^k/k)^2 + 1] - 1\}$$

for all x satisfying $|x| > x'_k$. Denote

$$x_0 = \max_{1 \leqslant k \leqslant N} \{x'_k\},$$

then

$$M(t, 2x) < 5M(t, x)$$

for all t in $[0, c] \subset \bigcup_{k=1}^{N} [1-1/2^{k-1}, 1-1/2^k)$ and all x in R satisfying $|x| > x_0$.

Now, for any set $T_0 \subset [0, 1]$ with positive measure and any sequence $\{x_k\}$ tending to infinity, we select constants $c>0$ and $x_0 \geqslant 0$ such that $0 < 1-c < |T_0|$ (where $|T|$ expresses the measure of T), and

$$M(t, 2x) \leqslant 5M(t, x)$$

for all t in $[0, c]$ and all x satisfying $|x| \geqslant x_0$. Since $1-c < |T_0|$, the set $[0, c] \cap T_0$ is nonempty and for all t in $[0, c] \cap T_0$, we have

$$M(t, 2x_k) \leqslant 5M(t, x_k)$$

for all x_k satisfying $|x_k| \geqslant x_0$, which means that the *affirmation* (*) is untrue and so the proofs of Theorem 2.2 and 3.3 in [2] are untrue.

3. We have shown in section 2 that the proof of Theorem 2.2 in [2] is incorrect. However, it is rather difficult to prove this theorem, which relies on the following lemmas

Lemma 1. *Suppose that A is a non-atomic measurable set with σ-finite measure, $a \geqslant 0$ a constant and $f(t)$ a finite, non-negative measurable function on A such that*

$$\int_A f(t)dt > a.$$

Then for any constant $e>0$, there exists a measurable set $A_0 \subset A$ such that

$$a < \int_{A_0} f(t)dt < a + e.$$

Proof Set

$$B_n=\{t\in A\,|\,f(t)\leqslant n\},$$

$n=1,\ 2,\ \cdots$, then

$$\lim_{n\to\infty}\int_{B_n}f(t)\,dt=\int_A f(t)dt>a.$$

Therefore, there is an integer $n_0\geqslant 1$ such that

$$\int_{B_{n_0}}f(t)dt>a.$$

Since A is a non-atomic measurable set with σ-finete measure, there is a sequence $\{E_k\}$ of disjoint subsets of B_{n_0} satisfying $|E_k|<e/n_0$, $k=1,\ 2,\ \cdots$, such that

$$B_{n_0}=\bigcup_{k=1}^{\infty}E_k.$$

Moreover, the inequality

$$\sum_{k=1}^{\infty}\int_{E_k}f(t)dt=\int_{B_{n_0}}f(t)dt>a$$

implies that there is an integer K_0 such that

$$\sum_{k=1}^{k_0}\int_{E_k}f(t)dt>a$$

and

$$\sum_{k=1}^{k_0-1}\int_{E_k}f(t)dt\leqslant a.$$

When the last two inequalities are combined with the inequality

$$\int_{E_{k_0}}f(t)dt\leqslant n_0|E_{k_0}|<n_0\cdot e/n_0=e,$$

we obtain the expected result

$$a<\int_{\bigcup_{k=1}^{k_0}E_k}f(t)dt=\int_{\bigcup_{k=1}^{k_0-1}E_k}f(t)dt+\int_{E_{k_0}}f(t)dt<a+e.$$

Lemma 2. *Under the same conditions as in Lemma* 1, *there is a measurable set* B_0 *in* A *such that*

$$\int_{B_0}f(t)dt=a.$$

Proof Let $\{e_k\}$ be a non-increasing positive sequence tending to zero. By Lemma 1, there is a measurable set $B_1\subset A$ such that

$$a<\int_{B_1}f(t)dt<a+e_1,$$

and for the same reason, there is a measurable set B_2 in B_1 such that

$$a<\int_{B_2}f(t)dt<a+e_2,$$

$\cdots\cdots$ By induction, we obtain sets $B_1\supset B_2\supset\cdots$ such that

$$a<\int_{B_n}f(t)<a+e_n$$

$n=1,\ 2,\ \cdots$ and complete the proof by setting

$$B_0=\bigcap_{n=1}^{\infty} B_n.$$

Lemma 3. *Suppose that $M(t, x)$ is a GN-function on $T\times E^n$, I a measurable set in T and $\{x_k(t)\}$ a sequence in X tending to $x(t)$ (finite or infinite). Then for any non-negative constant c satisfying*

$$\int_I M(t, x(t))dt>c$$

(where $M(t, \infty)=+\infty$), there exists an integer $k_0\geqslant 1$ and a measurable set I_0 in I such that

$$\int_{I_0} M(t, x_{k_0}(t))dt=c.$$

Proof Since $M(t, x)$ is a continuous function of x for each t in T,

$$\lim_{k\to\infty} M(t, x_k(t))=M(t, x(t))$$

for all t in T, it follows by Fatou's Lemma that

$$\sup_{k\geqslant 1}\int_I M(t, x_k(t))dt\geqslant\int_I M(t, x(t))dt>c.$$

Thus there is an integer $k_0\geqslant 1$ such that

$$\int_I M(t, x_{k_0}(t))dt>c.$$

The desired conclusion follows immediately from Lemma 2.

Let $M(t, x)$ denote a GN-function on $T\times E^n$, $P_k\geqslant 2$ a constant and

$$R_1=\{r_1, r_2, \cdots\}$$

the set of all points in E^n with all coordinates being rationals. Moreover, we write

$$G_i=\{t\in T\,|\,M(t, 2r_i)>P_k M(t, r_i)\},$$

$$\bar{r}_i(t)=\begin{cases} r_i, & \text{when } t\in G_i,\\ 0, & \text{otherwise,}\end{cases}$$

$i=1, 2, \cdots$. Since both $M(t, 2r_i)$ and $M(t, r_i)$ are measurable functions of t, the set G_i is measurable, and so, $|\bar{r}_i(t)|$ a simple function on T. Moreover

$$r_k(t)=\sup_{1\leqslant i}|\bar{r}_i(t)|=\lim_{j\to\infty}\max_{1\leqslant i\leqslant j}|\bar{r}_i(t)|$$

is a measurable function on T. For each $j=1, 2, \cdots$, set $|r_{i_1}|\geqslant|r_{i_2}|\geqslant\cdots\geqslant|r_{i_j}|$, (where $\{i_1, i_2, \cdots, i_j\}=\{1, 2, \cdots, j\}$), and denote

$$x_j^{(k)}(t)=\begin{cases} r_{i_k}, & \text{when } t\in G_{i_k}-\bigcup_{\nu=1}^{i_k-1} G_{i_\nu},\ k=2, \cdots, j,\\ \bar{r}_{i_1}(t), & \text{otherwise,}\end{cases} \tag{a}$$

$$g_j^{(k)}(t)=(|x_j^{(k)}(t)|, 0, \cdots, 0)\in X \tag{b}$$

and

$$g_k(t)=(r_k(t), \overbrace{0, \cdots, 0}^{n-1}). \tag{c}$$

We have

$$M(t,\ 2x_j^{(k)}(t))>P_kM(t,\ x_j^{(k)}(t))$$

whenever $x_j^{(k)}(t)\neq 0$,

$$r_k(t)=\lim_{j\to\infty}\operatorname{Max}_{1\leq i\leq j}|\bar{r}_i(t)|=\lim_{j\to\infty}|x_j^{(k)}(t)| \tag{d}$$

and

$$g_k(t)=\lim_{j\to\infty} g_j^{(k)}(t). \tag{e}$$

We will always employ the notations (a)—(e) throughout this section without any further explanation.

Lemma 4. *The inequality*

$$M(t,\ 2x)\leqslant P_kM(t,\ x)$$

holds for all t in T and all x in E^n satisfying $|x|>r_k(t)$.

Proof Otherwise, there is some t in T and x_0 in E^n satisfying $|x_0|>r_k(t)$ such that

$$M(t,\ 2x_0)>P_kM(t,\ x_0);$$

then by the continuity of $M(t,\ x)$, there is a positive constant $c<|x_0|-r_k(t)$ such that

$$M(t,\ 2x)>P_kM(t,\ x)$$

for all x in E^n satisfying $|x_0-x|<c$. Select a point r in R_1 such that $|x_0-r|<c$. Then we have

$$M(t,\ 2r)>P_kM(t,\ r)$$

and

$$r_k(t)<|x_0|-c<|x_0|-|x_0-r|\leqslant|x_0|-(|x_0|-r)=r=|\bar{r}(t)|.$$

This contradicts the definition of $r_k(t)$.

Lemma 5. *If $P_1<P_2<\cdots$ and*

$$\int_T M(t,\ g_k(t))dt=\infty$$

$k=1,\ 2,\ \cdots$, then there is a sequence $\{T_j\}$ of disjoint subsets of T and a sequence $\{x_j(t)\}$ of points in X such that

$$\int_{T_j} M(t,\ x_j(t))dt=1/P_{k_j}$$

and

$$M(t,\ 2x_j(t))>P_{k_j}M(t,\ x_j(t)),\ t\in T_j,$$

$j=1,\ 2,\ \cdots$, where $\{P_{k_j}\}$ is a subsequence of $\{P_k\}$.

Proof We know that the sequence $\{r_k(t)\}$ is non-increase since $P_1<P_2<\cdots$. Let

$$I_k=\{t\in T\,|\,r_k(t)=\infty\}$$

and $I_0=\bigcap_{k=1}^{\infty} I_k$. Clearly, $I_1\supset I_2\supset\cdots$.

1° If $|I_0|>0$, we can choose a non-overlapping sequence $\{T'_k\}$ of sets in I_0 with positive measure (see [3], § 41, Exercise (2)). For each $k=1,\ 2,\ \cdots$, by (d):

$$\lim_{j\to\infty}|x_j^{(k)}(t)|=r_k(t)=\infty,\ t\in T_k'$$

and Lemma 3, there is a point $x_{jk}^{(k)}(t)\equiv x_k(t)$ in X and a set T_k in $T_k'(x\neq 0)$ such that

$$\int_{T_k} M(t,\ x_k(t))dt=1/P_k.$$

It is obvious from the definition of $x_k(t)$ that

$$M(t,\ 2x_k(t))>P_kM(t,\ x_k(t)),\ t\in T_k.$$

2° If $|I_0|=0$ and every $|I_k|>0$, then there are integers $1\leqslant k_1<k_2<\cdots$ such that $|I_{k_j}-I_{k_{j+1}}|>0$, $j=1,\ 2,\ \cdots$. For each $j=1,\ 2,\ \cdots$, as in case 1°, we can select a set T_j in $I_{k_j}-I_{k_{j+1}}$ and a point $x_t(t)$ in X such that

$$\int_{T_j} M(t,\ x_j(t))dt=1/P_{k_j}$$

and

$$M(t,\ 2x_j(t))>P_{k_j}M(t,\ x_j(t)),\ t\in T_j.$$

Clearly, $\{T_j\}$ is disjoint.

3° Suppose that neither case 1° nor 2° is satisfied, then there is some $|I_{k_0}|=0$. For convenience, we say $I_1=\emptyset$, therefore $\emptyset=I_1=I_2=\cdots$. By Definition 1, there are constants $K\geqslant 1$ and $d\geqslant 0$ satisfying

$$\int_T \overline{M}(t,\ d)dt<\infty$$

such that

$$M(t,\ x)\leqslant KM(t,\ y)$$

for all t in T and all $x,\ y$ in E^n satisfying $d\leqslant|x|\leqslant|y|$. Since

$$\int_T M(t,\ g_1(t))dt=\infty$$

by Lemma 2, there exists a set E_1 in T such that

$$\int_{E_1} M(t,\ g_1(t))dt=3K/P_1+\int_T \overline{M}(t,\ d)dt.$$

Thus, by (e) and Lemma 3, there is a set T_1' in E_1 and a point $g_{m_1}^{(1)}(t)$ in X such that

$$\int_{T_1'} M(t,\ g_{m_1}^{(1)}(t)\ dt=2K/P_1+\int_T \overline{M}(t,\ d)dt.$$

Clearly.

$$\int_{T_1'(|g_{m_1}^{(1)}|>d)} M(t,\ 2g_{m_1}^{(1)}(t))dt\geqslant 2K/P_1$$

and by (c)

$$\int_{T_1'(|x_1|>d)} M(t,\ x_1(t))dt\geqslant 1/K\int_{T_1'(|g_{m_1}^{(1)}|>d)} M(t,\ g_{m_1}^{(1)}(t))dt\geqslant 2/P_1,$$

where $x_1(t)\equiv x_{m_1}^{(1)}(t)$. It follows by Lemma 2 that there is a set T_1 in $T_1'(|x_1|>d)$ such that

$$\int_{T_1} M(t,\ x_1(t))dt=1/P_1.$$

And it is easy to see that

$$M(t,\ 2x_1(t))>P_1M(t,\ x_1(t)),\quad t\in T_1.$$

Hence

$$\int_{T-T_1}M(t,\ g_2(t))dt\geqslant\int_T M(t,\ g_2(t))dt-\int_{T_1}M(t,\ g_1(t))dt=+\infty.$$

Similarly, we can obtain a point $x_2(t)\equiv x^{(2)}_{m_2}(t)$ and a set T_2 in $T-T_1$ such that

$$\int_{T_2}M(t,\ x_2(t))dt=1/P_2$$

and

$$M(t,\ 2x_2(t))>P_2M(t,\ x_2(t)),t\in T_2.$$

And so on, by induction, we obtain a sequence $\{x_k(t)\}$ in X and a disjoint sequence $\left\{T_k\subset T-\bigcup_{i=1}^{k-1}T_i\right\}$ of sets such that

$$\int_{T_k}M(t,\ x_k(t))dt=1/P_k$$

and

$$M(t,\ 2x_k(t))>P_kM(t,\ x_k(t)),\ t\in T_k,$$

$k=1,\ 2,\ \cdots$, completing the proof.

Lemma 6. *Assume that $M(t,\ x)$ does not satisfy a Δ-condition. Then for any $P_k\geqslant 2$,*

$$\int_T M(t,\ 2g_k(t))dt=\infty.$$

Proof Suppose that there exists some $P_k\geqslant 2$ such that

$$\int_T M(t,\ 2g_k(t))dt<\infty.$$

Let

$$q(t)=\begin{cases}|g_k(t)|, & \text{when}\ |g_k(t)|>d/2,\\ d/2, & \text{otherwise}\end{cases}$$

By Definition 1, we have

$$\int_T\overline{M}(t,\ 2q(t))dt\leqslant\int_T\overline{M}(t,\ d)dt+\int_{T(2|g_k|>d)}\overline{M}(t,\ 2|g_k(t)|)dt$$

$$\leqslant\int_T\overline{M}(t,\ d)dt+K\int_{T(2|g_k|>d)}M(t,\ 2g_k(t))dt<\infty,$$

and for any $t\in T$ and $x\in E^n$ satisfying $|x|>q(t)\geqslant|g_k(t)|=r_k(t)$, Lemma 4 asserts that

$$M(t,\ 2x)\leqslant P_kM(t,\ x).$$

This contradicts the assumption that $M(t,\ x)$ does not satisfy a Δ-condition.

Now, we can prove Theorem 2.2 appeared in [2].

Theorem 1. *Orlicz class L_M is a vector space if and only if $M(t,\ x)$ satisfies a Δ-condition.*

Proof We only show the necessity, since the sufficiency is obvious. If $M(t,\ x)$

does not satisfy a Δ-condition, then for each $k=1, 2, \cdots$, setting $P_k=2^k$, Lemmm 6 shows

$$\int_T M(t, 2g_k(t))dt=\infty$$

and the condition that L_M is a vector space yields

$$\int_T M(t, g_k(t))dt=\infty.$$

It follows from Lemma 5 that there exists a non-overlapping sequence $\{T_j\}$ of subsets of T and a sequence $\{x_j(t)\}$ of points in X such that

$$\int_{T_j} M(t, x_j(t))dt=1/2^{k_j}$$

and

$$M(t, 2x_j(t))>2^{k_j}M(t, x_j(t)), \ t\in T_j.$$

Let us define

$$x(t)=\begin{cases} x_j(t), & \text{when } t\in T_j, \ j=1, 2, \cdots, \\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\int_T M(t, x(t))dt=\sum_{j=1}^{\infty}\int_{T_j} M(t, x_j(t))dt=\sum_{j=1}^{\infty}1/2^{k_j}\leqslant 1$$

and

$$\int_T M(t, 2x(t))dt=\sum_{j=1}^{\infty}\int_{T_j} M(t, 2x_j(t))dt\geqslant\sum_{j=1}^{\infty}2^{k_j}\int_{T_j} M(, x_j(t))dt=\sum_{j=1}^{\infty}1=\infty.$$

This means $x(t)\in L_M$ and $2x(t)\,\overline{\in}\, L_M$, contradicting the hypothesis that L_M is a vector space.

4. The purpose of this section is to prove and improve Theorems 3.2 and 3.3 in [2].

Lemma 7. *Let $M(t, x)$ be a GN-function on $T\times E^n$. Then*

$$\underline{M}(t, c)=\inf_{|x|=c} M(t, x)>0$$

for all constant $c>0$ and all t in T.

Proof If there exists some $c>0$ and t in T such that

$$\underline{M}(t, c)=\inf_{|x|=c} M(t, x)=0,$$

there is a sequence of points $\{x_k\}$ in E^n satisfying $|x_k|=c$, $k=1, 2, \cdots$ such that

$$\lim_{k\to\infty} M(t, x_k)=0.$$

By the compactness of E^n, there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and $x\in E^n$ such that $x_{k_j}\to x$ as $j\to\infty$. This implies, from the continuity of $M(t, x)$, that

$$M(t, x)=\lim_{j\to\infty} M(t, x_{k_j})=0,$$

contradicting the definition of $M(t, x)$. Hence $|x|=c>0$.

Lemma 8. *Assume that $M(t, x)$ is a GN-function on $T\times E^n$, $|T|<\infty$, and*

$\{x_k(t)\}\subset X_0$. *Then* $M(t, x_k(t))\Rightarrow 0$ *(convergence in measure) as* $k\to\infty$ *if and only if* $x_k(t)\Rightarrow 0$ *as* $k\to\infty$.

Proof Necessity. Otherwise, there are positive constants a, e and a subsequence $\{x_{k_j}(t)\}$ of $\{x_k(t)\}$ such that

$$|T_{k_j}|\equiv \text{mes}\{t\in T\,|\,|x_{k_j}(t)|>a\}\geqslant e,$$

$j=1, 2, \cdots$. It follows from Lemma 7 and the convexity of $M(t, x)$ that

$$M(t, x_{k_j}(t))\geqslant M(t, (a/|x_{k_j}(t)|)x_{k_j}(t))\geqslant \underline{M}(t, a)>0,\ t\in T_{k_j} \tag{f}$$

for all $j=1, 2, \cdots$. Since $\underline{M}(t, a)>0$ and T is a set with non-atomic, finite measure, there is a constant $c>0$ and a set T' in T such that $|T'|<e/2$, and $\underline{M}(t, a)<c$ for all t in $T-T'$. Combining (f), we have

$$\text{mes}\{t\in T\,|\,M(t, x_{k_j}(t))>c\}\geqslant \text{mes}\{t\in T_{k_j}\,|\,\underline{M}(t, a)>c\}\geqslant|T_{k_j}-T'|>e/2$$

for all $j=1, 2, \cdots$. This contradicts that $M(t, x_k(t))\Rightarrow 0$ as $k\to\infty$.

Sufficiency. For any positive constants a and e, since $\overline{M}(t, c)\to 0$ as $c\to 0$ for all t in T, and T is a set with non-atomic finite measure, there is a constant $c_0>0$ and a set T_0 in T such that $|T_0|<e/2$, and $\overline{M}(t, c_0)<a$ for all t in $T-T_0$. By hypothesis, there is an integer $N\geqslant 1$ such that

$$|E_k|\equiv \text{mes}\{t\in T\,|\,|x_k(t)|>c_0\}<e/2$$

for all $k\geqslant N$. It follows that

$$\text{mes}\{t\in T\,|\,M(t, x_k(t))>a\}\leqslant \text{mes}\{t\in E_k\,|\,M(t, x_k(t))>a\}$$
$$+\text{mes}\{t\in T-E_k\,|\,\overline{M}(t, c_0)>a\}<e/2+|T_0|<e$$

for all $k\geqslant N$. That is, $M(t, x(t))\Rightarrow 0$ as $k\to\infty$.

The following lemma is Theorem 3.2 in [2] without proof.

Lemma 9. *The convergences in norm and in modular are equivalent in* L_M^* *if* $|T|<\infty$ *and* $M(t, x)$ *satissfies a* Δ*-condition.*

Proof If $R_M(x_k)\to 0$ as $k\to\infty$, then $M(t, x_k(t))\Rightarrow 0$ as $k\to\infty$, and by Lemma 8, so does $M(t, 2x_k(t))$. Hence Lemma 8 implies that $\{x_k(t)\}$, and so $\{2x_k(t)\}$, converges to zero in measure. For each $k=1, 2, \cdots$, set

$$x'_k(t)=\begin{cases} x_k(t), & \text{when } |x_k(t)|\leqslant q(t),\\ 0, & \text{otherwise,}\end{cases}$$

($q(t)$ and the following K are defined as in Definition 4), then by Lebesgue's Dominated Convergence Theorem and the hypothesis, we have

$$\int_T M(t, 2x_k(t))dt\leqslant K\int_T M(t, x_k(t))dt+\int_T M(t, 2x'_k(t))dt\to 0$$

as $k\to\infty$, and finish the proof by applying (3.1.1) and (3.1.2) in [2].

Theorem 2. *The convergences in norm and in modular are equivalent in* L_M^* *if and only if* $M(t, x)$ *satisfies a* Δ*-condition.*

Proof sufficiency. By hypothesis, there is a constant $K\geqslant 2$ and a nonnegative measurable function $q(t)$ satisfying $\int_T \overline{M}(t, 2q(t))dt<\infty$ such that for almost all t

in T

$$M(t, 2x) \leqslant KM(t, x)$$

for all x in E^n satisfying $|x| \geqslant q(t)$. Since T is a set with σ-finite measure, there is a non-overlapping sequence $\{T_k\}$ of sets with finite measure in T such that

$$T = \bigcup_{k=1}^{\infty} T_k,$$

and so

$$\sum_{k=1}^{\infty} \int_{T_k} \overline{M}(t, 2q(t))dt = \int_T \overline{M}(t, 2q(t))dt < \infty.$$

Therefore, for arbitrary $e > 0$, there exists an integer N such that

$$\int_{T_0} \overline{M}(t, 2q(t))dt < e/3,$$

where $T_0 = \bigcup_{k=N}^{\infty} T_k$.

Now, if $\{x_k(t)\} \subset X$ and $R_M(x_k) \to 0$ as $k \to \infty$, then there is an integer $N_1 \geqslant 1$ such that $R_M(x_k) < e/3K$ whenever $k > N_1$, and by Lemma 9, there is an integer $N_2 \geqslant 1$ such that

$$\int_{T-T_0} M(t, 2x_k(t))dt < e/3$$

for all $k > N_2$. It follows immediately that

$$R_M(2x_k) = \int_{T-T_0} M(t, 2x_k(t))dt + \int_{T_0} M(t, 2x_k(t))dt$$

$$\leqslant e/3 + \int_{T_0} \overline{M}(t, 2q(t))dt + K\int_{T_0} M(t, x_k(t))dt < e/3 + e/3 + Ke/3K = e$$

for all $k > \max\{N_1, N_2\}$. In other words, $R_M(2x_k) \to 0$ as $k \to \infty$. We finish the proof of the sufficiency by applying (3.1.1) and (3.1.2) in [2].

Necessity. If $M(t, x)$ does not satisfy a Δ-condition, by Lemma 5, there exists a disjoint sequence $\{T_j\}$ of subsets of T and a sequence $\{x_j(t)\}$ in X such that

$$\int_{T_j} M(t, x_j(t))dt = 1/2^{k_j}$$

and

$$M(t, 2x_j(t)) > 2^{k_j} M(t, x_j(t)), \ t \in T_j$$

for all $j = 1, 2, \cdots$, where $\{k_j\}$ are integers satisfying $1 \leqslant k_1 < k_2 < \cdots$. Defining

$$x'_j(t) = \begin{cases} x_j(t), & \text{when } t \in T_j, \\ 0, & \text{when } t \in T - T_j, \end{cases}$$

$j = 1, 2, \cdots$, we have

$$R_M(x'_j) = \int_T M(t, x'_j(t))dt = \int_{T_j} M(t, x_j(t))dt = 1/2^{k_j} \to 0$$

as $j \to \infty$, and

$$R_M(2x'_j) = \int_{T_j} M(t, 2x_j(t))dt \geqslant 2^{k_j} \int_{T_j} M(t, x_j(t))dt = 1.$$

This implies by [2], Theorem 3.1 that the convergences in norm and in modular are not equivalent.

[1)]Neither Theorem 2.2 nor 3.3 in [2] holds without this condition as T being nonatomic in [1]. For example, let T be a space of points with a point t_0 in it and with the measure m defined as follows: for any $A \subset T$, $mA=1$ if $t_0 \in A$, and $mA=0$ otherwise. Obviously, T is a σ-finite measurable space with an atom t_0. Now, define

$$M(t, x) = e^{|x|} - |x| - 1$$

for all t in T and x in E^n. We can show the incorrectness of the Theorems 2.2 and 3.3 in [2] without any difficulties. For instance, we verify the first one.

It is easy to verify that $M(t, x)$ is a GN-function not satisfying a Δ-condition. We only need to show that the Orlicz class L_M is a vector space which contradicting the Theorem 2.2 in [2]. For arbitrary $x(t)$ in X, we have

$$R_M(x) = \int_T M(t, x(t))dt = \int_{t_0} M(t, x(t))dt = M(t_0, x(t_0)) \cdot mt$$
$$= M(t_0, x(t_0)) < \infty.$$

That is to say, $L_M = X$ being a vector space.

Referenees

[1] Skaff, M. S., Vector valued Orlicz spaces, generalized N-function, I, *Pacific. J. Math.*, 28 (1969), 193—206.

[2] Skaff, M. S., ibit, 28 (1969), 413—430.

[3] Halmos, P. R., Measure Theory, New York, 1950.