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THE TRIANGULAR MODEL OF SEMI-SIMPLE L^*-ALGEBRA OF H-S OPERATORS

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Abstract

In this paper, some properties of semi-simple L^*-algebras are considered.

At first, applying Cartan decomposition, the author constructs a family of nilpotent subalgebras in a semi-simple L^*-algebra and proves that whole algebra can be spanned by these subalgebras, their conjugations and Cartan subalgebras.

Then, the author proves that every nonzero root vector of semi-simple L^*-algebra of H-S operators is a finite rank operator and presents the triangular model of the algebra.

Finally, non-Voltera property of the algebra is shown.

In this paper we will follow the notations in [1, 2], The basic results we will use are as follows.

Theorem *If L is a semi-simple L^*-algebra with Cartan subalgebra H, L has a Cartan decomposition with respect to H.* ([2] p 348)

Suppose A is a bounded self-adjoint operator on L. For real λ and $\varepsilon>0$, let

$$V(\lambda, \varepsilon)=\{x;\ \|(A-\lambda)^n x\|\leqslant\varepsilon^n\|x\|,\ n=1, 2, \cdots\}.$$

For a Borel set M of the real numbers, let

$$V(M, \varepsilon)=SP\{V(\lambda, \varepsilon);\lambda\in M\}$$

and

$$V(M)=\bigcap_{\varepsilon>0}V(M, \varepsilon).$$

Furthermore, if $E(\lambda)$ is the real spectral measure of A such that

$$A=\int\lambda\, dE,$$

then the range of $E(M)$ is equal to $V(M)$ for M compact. For any Borel set M the range of $E(M)$ will be denoted by $S(M)$. Finally, for Borel sets M and N let

$$M+N=\{m+n;\ m\in M,\ n\in N\}\text{ and }-M=\{-m;\ m\in M\}.$$

The $M+N$ and $-M$ are alse Borel sets.

Proposition 1 *Suppose A is a bounded self-adjoint derivation on L and M. N are Borel sets of the real line. Then* $[S(M), S(N)]\subset S(M+N)$ *and* $S(M)^*=S(-M)$. ([1] p 335)

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§ 1. Nilpotent subalgebra and nilpotent root vector

In this section we will construct nilpotent subalgebras spanned by the root vectors.

Let L be a semi-simple L^*-algebra and h be a selfadjoint element of L. Then there is a Cartan subalgebra containing h. For every $x \in L$, $D_h x = [h, x]$ is a bounded selfadjoint derivation on L. Without loss of generality, we can suppose that the spectrum of D_h is in the interval $[-1, +1]$ including the end points. We take D_h as A in the proposition 1.

Proposition 2 *If* $|\lambda| > 1$, *then* $V(\lambda, \varepsilon) = \{0\}$ *when* ε *is small enough.*

Proof By hypothesis, we can suppose $\lambda > 1$. Therefore, there exists $(A-\lambda)^{-1}$ which is a bounded operator, $\|(A-\lambda)^{-1}\| \leqslant K$, where K is a constant. If $x \in V(\lambda, \varepsilon)$, $\|x\| = \|(A-\lambda)^{-n}(A-\lambda)^n x\| \leqslant (K\varepsilon)^n \|x\|$, $n = 1, 2, \cdots$, we take $\varepsilon < \frac{1}{2K}$ so that $\|x\| \leqslant \frac{1}{2}\|x\|$, therefore $x = 0$.

Q. E. D.

Theorem 1 *If L is a semi-simple L^*-algebra, then there is a decomposition of L*

$$L = S[-1, 0) \oplus H \oplus S(0, +1].$$

For every $\lambda > 0$, $S[\lambda, 1]$ *is a nilpotent subalgebra.*

Proof We take D_h as A in proposition 1. In views of the previous discussion, we can get $S[-1, 0)$ and $S(0, +1]$. $S[\lambda, 1]$ $(\lambda > 0)$ is a nilpotent subalgebra.

If α is a nonzero positive root of L, it is easy to see that a positive root vector e_α belongs to $V(\alpha(h), \varepsilon)$, that is, $V_\alpha \subset V(\alpha(h), \varepsilon) \subset S(0, +1]$

Therefore, $H \oplus V_\alpha \subseteq S[-1, 0) \oplus H \oplus S(0, +1] \subseteq L$, in which summation runs over all nonzero roots. By means of the existence of Cartan decomposition of semi-simple L^*-algebra, we have $L = H \oplus V_\alpha$. Consequently, we get the decomposition

$$L = S[-1, 0) \oplus H \oplus S(0, +1].$$

Q. E. D.

§ 2. Finite rank property of the root vectors

In this section we will consider a concrete semi-simple L^*-algebra. Then we can get the particular properties of root vectors.

Now let L_H be a semi-simple L^*-algebra composed by Hilbert-Schmidt operators. The Lie product is given in usual way, $[A, B] = AB - BA$ for $A, B \in L_H$, and inner product is defined by $(A, B) = \text{trace}(B^* A)$. We will prove that all root vectors corresponding to nonzero roots are finite rank operators.

Lemma 1 *If A is a fixed nonzero bounded operator on a Hilbert space such that $[[A, A^*], A]=\lambda A$ for some $\lambda\neq 0$, and n is the greatest integer such that $A^n\neq 0$. Then A^*A has finite spectra contained in the set $\{K(\lambda/2)$: $K=0, 1, \cdots, n(n+1)\}$.* ([2] p 342)

Proposition 3. *If e_α is a nonzero root vector of L_H, then e_α is a finite rank operator.*

Proof If e_α is a nonzero root vector, without loss of generality, we can suppose e_α corresponds to a positive root. By means of Theorem 1, e_α belongs to some nilpotent subalgebra $S[\lambda, 1]$, where $\lambda>0$.

Thus there exists n which is the largest integer such that $e_\alpha^n\neq 0$. Evidently,

$$[[e_\alpha, e_\alpha^*], e_\alpha]=\alpha(h_\alpha)e_\alpha.$$

Accordinf to Lemma 1, the spectrum of operator $e_\alpha^*e_\alpha$ is contained in the set $\{\alpha(h_\alpha)K/2; K=0, 1, \cdots, n(n+1)\}$. Since $e_\alpha^*e_\alpha$ is a completely continuous operator, if $\alpha(h_\alpha)k/2$ is the nonzero characteristic value of $e_\alpha^*e_\alpha$, the corresponding characteristic subspace is finite dimension. Because of the spectral theorem of selfadjoint completely continuous operator, $e_\alpha^*e_\alpha$ is a finite rank operator: Since the null space of $e_\alpha^*e_\alpha$ contains in the null space of e_α, thorefore, e_α is a finite rank operator.

§ 3. The triangular model of L

In this section, we prove, with the help of [4], that every nonzero root vector can be expressed in triangular model. It is somewhat like uppertriangular form.

Chain A set $\mathscr{B}$ of orthoprojectors is called a chain, if for any pair $p_1, p_2\in\mathscr{B}$ either $p_1<p_2$ or $p_2<p_1$.

Eigenchain We shall say that a cfain $\mathscr{B}$ is an eigenchain of the operator A, if each of the subspaces $R(p)(p\in\mathscr{B})$ is invariant with respect to A, in other words, if $pAp=Ap(p\in\mathscr{B})$.

Lemma 2. *Every completely continous linear operator has a maximal eigenchain.* ([4] p 15)

The rank of a chain A system of vectors $\{\chi_j\}_1^r$ $(1\leqslant r\leqslant+\infty)$ is called a reproducing system for the chain $\mathscr{B}$, if the closed linear hull of the set of vectors $p\chi_j(j=1, 2, \cdots, r, p\in\mathscr{B})$ coincides with the entire space H_1. The smallest of the cardinalities of all possible reproducing system for the chain $\mathscr{B}$ is called the rank of the chain, and is denoted by $r(\mathscr{B})$.

Triangular Model. For brevity, we denote by $L_2^{(r)}(1\leqslant r\leqslant+\infty)$ the Hilbert space $L_2^r(Q)$, where $Q=[0, 1]$. Thus, an element $f\in L_2^{(r)}$ is an r-dimensional vector function $f=\{f_\nu(t)\}_1^r$ with measurable components $f_\nu(t)$ $(0\leqslant t\leqslant 1)$ such that

$$|f|^2=\int_0^1 \sum_{\nu=1}^{r} |f_\nu(t)|^2 dt<\infty.$$

For the scalar product of the elements $f,\ g\in L_2^{(r)}$, we have

$$(f,\ g)=\int_0^1 g^*(t)f(t)\,dt=\int_0^1 \sum_{\nu=1}^{r} f_\nu(t)\overline{g_\nu(t)}\,dt.$$

Let $\tilde{p}(t)$ $(0\leqslant t\leqslant 1)$ be the truncation projector-function defined by the condition $\hat{p}(0)=0$, $\hat{p}(1)=I$ and

$$(\hat{p}(s)f)(t)=\begin{cases} f(t), & 0\leqslant t<s,\\ 0, & s<t\leqslant 1 \end{cases} \quad (0<t<1).$$

Let A be some (abstract) Voltera operator acting in a Hilbert space H_1. A (concrete) Voltera operator $\mathscr{A}$, acting on $L_2^{(r)}$ and having $\hat{p}(t)$ as an eigen-projector-function, is called a triangular model of A, if $\mathscr{A}$ is unitary equivalent to A or to an inessential extension of A.

Lemma 3. *Every Hilbert-Schmidt Voltera operator $\mathscr{A}$ of rank r has as a triangular model an integral operator $\mathscr{A}$, which acts on the space $L_2^{(r)}$ according to the formula*

$$(\mathscr{A}f)(t)=\int_t^1 \mathscr{A}(t,\ s)f(s)ds,$$

where

$$\mathscr{A}(t,\ s)=\|a_{\mu\nu}(t,\ s)\|_1^r \quad (0\leqslant t\leqslant s\leqslant 1)$$

is a Hilbert-Schmidt matrix kernel i. e.

$$\int_0^1\int_t^1 \sum_{\mu,\nu=1}^{r} |a_{\mu\nu}(t,\ s)|^2 ds\,dt<\infty.$$

([4] p 221)

Lemma 4. *If A is a finite-rank operator, r is its rank of operator and $r(\mathscr{B})$ is the rank of its eigenchain, then there exists an inequality*

$$r(\mathscr{B})\leqslant r$$

Proof Let A be a finite rank operator on H_1. The domain of A is a r-dimensional space, and $x_1,\ x_2,\ \cdots,\ x_r$ is its orthonormal basis. Then we can extend this basis to the whole space H_1 and get the orthonormal basis of H_1 and denote it as $x_1,\ x_2,\ \cdots,\ x_r,\ e_{r+1},\ e_{r+2},\ \cdots$ If the chain of A is $\mathscr{B}'$, we can extend $\mathscr{B}'$ to $\mathscr{B}$ such that

$$E\{x_1,\ x_2,\ \cdots x_r,\ e_{r+1},\ \cdots,\ e_{r+i}\}\ (i=1,\ 2,\ \cdots)\in\mathscr{B}.$$

Therefore $\mathscr{B}$ is the chain of A.

Let

$$\chi_i=x_i+\sum_{k=1}^{\infty} e_{r+k}/2^k \quad (i=1,\ 2,\ \cdots,\ r.).$$

It is easy to see that $\{\chi_i\}_1^r$ is a reproducing system for A, consequently

$$r(\mathscr{B})\leqslant r.$$

Theorem 2. *Let L_H be a semi-simple L^*-algecra of H-S operators on H_1, α be a nonzero root, and e_α be a corresponding root vector. Then e_α has a triangular model*

$$(\mathscr{A}f)(t)=\int_t^1 \mathscr{A}(t,\ s)f(s)ds,$$

$$\mathscr{A}(t,\ s)=\|a_{\mu\nu}(t,\ s)\|_1^r,\ (0\leqslant t\leqslant s\leqslant 1)$$

$$\int_0^1\int_t^1 \sum_{\mu\nu=1}^r |a_{\mu\nu}(t,\ s)|^2 ds\,dt<\infty,$$

$$r\leqslant\left[\frac{1}{\alpha(h)}\right]+1.$$

Proof In view of proposition 3, for every nonzero root α, e_α is a finite rank operator. By means of Lemma 4, the Chain $\mathscr{B}$ corresponding to e_α is finite rank. Therefore, owing to Lemma 3, e_α has the form in the theorem.

By means of Theorem 1 $S[\alpha(h),\ 1]$ is a nilpotent subalgebra. Therefore

$$e_\alpha^{\left[\frac{1}{\alpha(h)}\right]+1}=0.$$

As e_α is a finite rank operator, applying Jordan canonical matrix to e_α, we can prove that if the rank of e_α is r, then

$$r\leqslant\left[\frac{1}{\alpha(h)}\right]+1.$$

Consequently

$$r(\mathscr{B})\leqslant r\leqslant\left[\frac{1}{\alpha(h)}\right]+1.$$

Q. E. D.

§ 4. L_H is not a Voltera algebra

Definition. *The algebra L consisting of bounded operators on a Hilbert space H_1 is called Voltera, if every operator in L is a Voltera operator.*

In this section, we will prove that L is not a Voltera algebra.

Theorem. *Let L be a separable Voltera algebra containing a finite rank operator. Then L has a proper closed ideal.* ([3] p 271)

Theorem 3. *If L_H is a semi-simple L^*-algebra, then L_H is not a Voltera algebra.*

Proof If L'_H is a semi-simple subalgebra of L_H, L'_H will be called regular (with respect to a Cartan subalgrbra H) if and only if L'_H is separ able and $H'=H\cap L'_H$ is a Cartan subalgebra of L'_H. In [2] p 344, the construction of regular semi-simple subalgebras of semi-simple L^*-algebra is given.

Now we assume L_H is a Voltera algebra. Therefore there exists a regular subalgebra L'_H of L, which is a Voltera algebra. Accoding to the definition, L'_H is separable.

Since a semi-simple L^*-algebra can be decomposed as direct sum of simple closed ideals, without loss of generality, we may suppose that L_H is a separable simple Voltera algebra.

Because of Proposition 3, L_H has finite operators. Owing to previous theorem, L_H contains a proper closed ideal. It contradicts the simple property of L_H. So L_H is not a Voltera algebra.

Q. E. D.

Reference

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