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ON THE EXISTENCE OF PERIODIC SOLUTIONS OF NONLINEAR OSCILLATION EQUATIONS

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Abstract

This paper deals with the existence of periodic solutions of the nonlinear oscillation equation

$$\ddot{x}+f(x)\varphi(x)+\psi(x)\eta(x)=0. \tag{3}$$

The author offers a method which can reduce (3) into the system

$$\dot{x}=h(y)-e(y)F(x),\ \dot{y}=-g(x). \tag{9}$$

Some sufficient conditions for the existence of the limit cycles of (9) are obtained. These results generalize the results in [1,2, 3, 4, 5, 6]

It is well-know that we can reduce Liénard equation

$$\ddot{x}+f(x)\dot{x}+g(x)=0 \tag{1}$$

into system

$$\dot{x}=y-F(x),\ \dot{y}=-g(x) \tag{2}$$

with transformation

$$\dot{x}=y-F(x);\ F(x)=\int_0^x f(\tau)d\tau,$$

and it is much easier for us to discuss the behaviours of the solutions of (2) than those of (1). But, till now, we have never seen any method which can reduce the more complicated nonlinear oscillation equation

$$\ddot{x}+f(x)\varphi(\dot{x})+\psi(\dot{x})\eta(x)=0 \tag{3}$$

into some system like (2). In this paper, we offer such a method. Furthermore, by the convenient form of the new system and Liapunov functions, we obtain some sufficient conditions for the existence of periodic solutions of (3).

We always assume that $f(x)$, $\varphi(\dot{x})$, $\psi(\dot{x})$ and $\eta(x)$ are continuous on $(-\infty, +\infty)$ with respect to their own arguments, and the conditions for the existence and uniqueness of initial-value problem are satisfied.

At first, we reduce (3) into its equivalent system

$$\dot{x}=y,\ \dot{y}=-f(x)\varphi(y)-\psi(y)\eta(x). \tag{4}$$

Assuming $\psi(y)>0$ for all $y\in(-\infty, +\infty)$ and writing

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$$\int_0^{+\infty}\frac{dy}{\psi(y)}=u_+,\quad \int_0^{-\infty}\frac{dy}{\psi(y)}=u_-,$$

it follows $-\infty\leqslant u_-<0<u_+\leqslant+\infty$.

We rewrite the second equation of (4) as

$$\frac{\dot y}{\psi(y)}=-f(x)\frac{\varphi(y)}{\psi(y)}-\eta(x),$$

and use the transformation

$$u=\xi(y)=\int_0^y\frac{dy}{\psi(y)}. \tag{5}$$

It is clear that $\xi(y)$ increases strictly on $(-\infty,\ +\infty)$ and satisfies $y\xi(y)>0$ for all $y\neq0$. Let $\xi(+\infty)=u_+$, $\xi(-\infty)=u_-$. In addition, $\xi(y)$ has an inverse function

$$y=g(u)$$

which is strictly increasing on $(u_-,\ u_+)$ and satifies $ug(u)>0$ for $u\neq0$ with $g(u_+)=+\infty$, $g(u_-)=-\infty$. Thus, (4) becomes

$$\dot x=g(u),\ \dot u=-f(x)\frac{\varphi(g(u))}{\psi(g(u))}-\eta(x), \tag{6}$$

$$(u_-<u<u_+;\ \ -\infty<x<+\infty).$$

Exchanging x by $-x$ in (6), we obtain

$$\dot x=-g(u),\ \dot u=-f(-x)\frac{\varphi(g(u))}{\psi(g(u))}-\eta(-x) \tag{7}$$

$$(u_-<u<u_+;\ -\infty<x<+\infty).$$

In (7), exchanging x by y and u by x, it becomes

$$\dot x=-\eta(-y)-f(-y)\frac{\varphi(g(x))}{\psi(g(x))},\ \dot y=-g(x). \tag{8}$$

$$(u_-<x<u_+,\ -\infty<y<+\infty)$$

Denoting

$$h(y)=-\eta(-y),\ e(y)=f(-y),\ F(x)=\frac{\varphi(g(x))}{\psi(g(x))},$$

we obtain at last the system

$$\dot x=h(y)-e(y)F(x),\ \dot y=-g(x), \tag{9}$$

here $(x,\ y)\in D$: $u_-<x<u_+$; $-\infty<y<+\infty$. (9) is very similar to (2). Since all of the above transformations are topological, the trajectories of (4) in $(x,\ y)$-plane are mapped into the trajectories of (9) in D, and therefore they have the same topological structure.

We have the following results.

Theorem 1. *If*

1) $\psi(y)>0$ *for all* $y\in(-\infty,\ +\infty)$;
2) $y\varphi(y)\leqslant0$ (or$\geqslant0$) *for all* y, *but* $\varphi(y)\not\equiv0$ *for sufficiently small* y;
3) $f(x)>0$ *for all* $x\in(-\infty,\ +\infty)$;
4) $x\eta(x)>0$ *for all* $x\neq0$,

then (4) *has no closed trajectory* (*hence* (3) *has no periodic solutions*).

The proof is similar to Theorem 1 of [1].

Theorem 2. *If*

1) $yh(y)>0$ *for* $y\neq 0$, $h(+\infty)=+\infty$, $h(-\infty)=-\infty$;

2) $xg(x)>0$ *for* $x\neq 0$, $G(\pm\infty)=+\infty$;

3) $e(y)>0$ *for all* $|y|<+\infty$, and

$$\lim_{y\to\pm\infty}\frac{e(y)}{h(y)}=0;$$

4) *there exists* $\delta>0$ *such that* $xF(x)\leqslant 0$ *but* $F(x)\not\equiv 0$ *for* $|x|\leqslant\delta$;

5) *there exists* $N>0$ *and constants* $K>K'$ *such that*

$$F(x)\geqslant\frac{h(y)-h(y-K)}{e(y)} \text{ for } x\geqslant N \text{ and } |y|<+\infty;$$

$$F(x)\leqslant\frac{h(y)-h(y-K')}{e(y)} \text{ for } x\leqslant -N \text{ and } |y|<+\infty,$$

then (9) *has at least one limit cycle* (*hence* (3) *has at least an isolated periodic solution*).

Proof The method of proof is to construct Poincaré-Bendixson annular region. Consider

$$V(x, y)=H(y)+G(x),\quad H(y)=\int_0^y h(y)dy.$$

It is obvious that $V(x, y)$ is definite positive in a sufficiently small neighbourhood of (0, 0), and we have

$$\dot V_{(9)}=-g(x)F(x)e(y)\leqslant 0.$$

Thus, for sufficiently amall c, the trajectories of (9) starting from the points on closed curve Γ_0: $V(x, y)=c$ go out of the interior region of Γ_0 as t increases. So we can take Γ_0 as the interior boundary.

Next, let's construct the exterior boundary. Denote

$$V_1(x, y)=H(y-K)+G(x),$$

$$V_2(x, y)=H(y-K')+G(x).$$

By assumption 5), it follows that

$$\dot V_{1(9)}=g(x)[h(y)-h(y-K)-e(y)F(x)]\leqslant 0 \text{ for } x\geqslant N,\ |y|<+\infty,$$

$$\dot V_{2(9)}=g(x)[h(y)-h(y-K')-e(y)F(x)]\leqslant 0 \text{ for } x\leqslant -N,\ |y|<+\infty.$$

By assumption 3), we can prove that there exists a $N_1>0$ such that

$$\dot x=h(y)-e(y)F(x)>0,$$

and

$$\frac{dy}{dx}=\frac{-g(x)}{h(y)-e(y)F(x)}<\frac{K-K'}{2N},$$

for $|x|\leqslant N$, $y\geqslant N_1$; and

$$\dot x=h(y)-e(y)F(x)<0,$$

and

$$\frac{dy}{dx}=\frac{-g(x)}{h(y)-e(y)F(x)}<\frac{K-K'}{2N}$$

for $|x|\leqslant N$, $y\leqslant -N_1$.

Denote $l=\max[H(N_1-K'), H(-N_1-K)]$. For definition, we may as well assume $l=H(-N_1-K)\geqslant H(N_1-K')$. At first, consider curve Γ_1(Fig. 1)

$$V_1(x, y)=l+G(N).$$

Let $x=N$, we have

$$H(y_C-K)=H(-N_1-K),\ H(y_B-K)\geqslant H(N_1-K'),$$

hence

$$y_C=-N_1,\ y_B\geqslant N_1+(K-K').$$

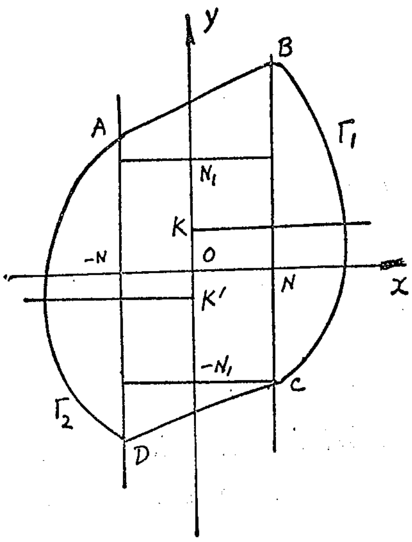
Then, we consider curve Γ_2

$$V_2(x, y)=l+G(-N).$$

Let $x=-N$, we have

$$H(y-K')=H(-N_1-K)\geqslant H(N_1-K').$$

It is easy to prove $|O_2D|=|O_1C|$, $|O_1B|=|O_2A|$.



(Fig. 1)

Hence, the trajectiries of (9) starting from the points on closed curve Γ; $\widehat{ABCDA}$ enter the interior region of Γ. Therefore, we obtain an annular region bounded by Γ_0 and Γ. Theorem 2 is proved.

Theorm 2 generalizes the main theorem of [2] and [3].

Corollary 1. *If*

1) $\psi(y)>0$ *for all* $y(-\infty, +\infty)$, *and*

$$\xi(+\infty)=\int_0^{+\infty}\frac{dy}{\psi(y)}=+\infty;\ \xi(-\infty)=\int_0^{+\infty}\frac{dy}{\psi(y)}=-\infty;$$

2) $x\eta(x)>0$ *for* $x\neq 0$, $\eta(+\infty)=+\infty$, $\eta(-\infty)=-\infty$;

3) $f(x)>0$ *for all* $x\in(-\infty, +\infty)$, *and*

$$\lim_{x\to\pm\infty}\frac{f(x)}{\eta(x)}=0;$$

4) $\varphi(0)=0$, $\varphi'(0)<0$;

5) *there exist a* $M>0$ *and constants* $K>K'$, *such that*

$$\frac{\varphi(y)}{\psi(y)}\geqslant\frac{\eta(x+K)-\eta(x)}{f(x)},\ \textit{for } y\geqslant M,\ |x|<+\infty,$$

$$\frac{\varphi(y)}{\psi(y)}\leqslant\frac{\eta(x+K')-\eta(x)}{f(x)},\ \textit{for } y\leqslant -M,\ |x|<+\infty,$$

then (4) *has at least one limit cycle.*

Corollary 2. *If the assumptions* 1)—4) *of Corollary* 1 *are satisfied, and we have*

5) *for any constant* K, *there exists an* $L(K)$ *such that*

$$\left|\frac{\eta(x+K)-\eta(x)}{f(x)}\right|\leqslant L \text{ for all } x;$$

6) $\lim\limits_{y\to+\infty}\dfrac{\varphi(y)}{\psi(y)}=+\infty,\quad \lim\limits_{y\to+\infty}\dfrac{\varphi(y)}{\psi(y)}=-\infty,$

then (4) *has at least one limit cycle.*

Next, we are going to do more investigation in the existence of the limit cycles of (9). This is of course very helpful for us to find the conditions for the existence of periodic solutions of (3).

We assume $xg(x)>0$ for $x\neq0$ and $G(\pm\infty)=+\infty$. Following Filippov [4], let

$$z=z_i(x)=\int_0^x g(x)dx,\ (-1)^{i+1}x\geqslant0,\ i=1,\ 2, \tag{10}$$

and denote their inverse functions as

$$x=x_i(z),\ i=1,\ 2,\ z\geqslant0.$$

Substituting them into (9), we get two new systems

$$\dot{z}=g_1(z)\,(h(y)-e(y)F_1(x)),\ \dot{y}=-g_1(z), \tag{11}$$

$$\dot{z}=g_2(z)\,(h(y)-e(y)F_2(x)),\ \dot{y}=-g_2(z), \tag{12}$$

where

$$g_i(z)=g(x_i(z)),\ F_i(z)=F(x_i(z)),\ i=1,\ 2.$$

Also, we denote

$$H_1(y)=\frac{h(y)}{e(y)}.$$

Theorem 3. *If*

1) $xg(x)>0$ *for* $x\neq0$, $G(\pm\infty)=+\infty$;

2) $e(y)>0$ *for all* y;

3) $yh(y)>0$ *for all* $y\neq0$, $h(+\infty)=+\infty$, $h(-\infty)=-\infty$, $H_1(y)$ *is strictly increasing in* $(-\infty,\ +\infty)$, $H_1(+\infty)=+\infty$, $H_1(-\infty)=-\infty$;

4) *there exists a* $\delta>0$ *such that* $xF(x)\leqslant0$ *for* $|x|\leqslant\delta$, *but* $F(x)\not\equiv0$ *for sufficiently small* $|x|$;

5) *there exists a* $\Delta>0$ *and a constant* K, *such that*

$$F_2(z)\leqslant\frac{h(y)-h(y-K)}{e(y)}\leqslant F_1(z)$$

$$\text{for } z\geqslant\Delta,\ -\infty<y<+\infty;$$

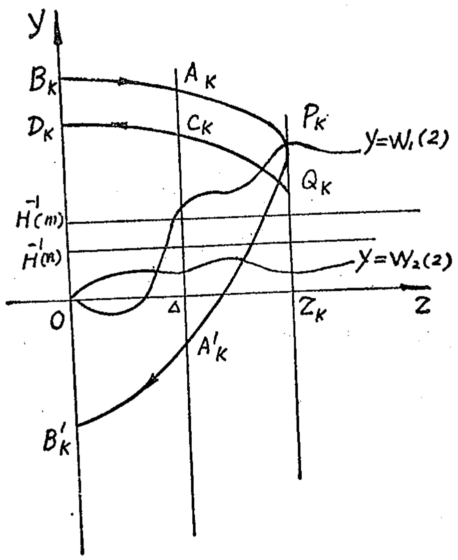
6) *there exists a* $\sigma>0$ *and a sequence* $z_k\to+\infty$ *for* $k\to+\infty$ *such that* $F_1(z_k)\geqslant m+\sigma$ (or $F_2(z_k)\leqslant n-\sigma$),

where

$$m=\sup_{(-\infty,+\infty)}\frac{h(y)-h(y-K)}{e(y)},\ n=\inf_{(-\infty,+\infty)}\frac{h(y)-h(y-K)}{e(y)},$$

then (9) *has at least one limit cycle.*

Proof We still construct Poincaré-Bendixson annular region.



(Fig. 2)

The construction of the interior boundary is similar to that of Theorem 2, so we omit it and only discuss the exterior boundary as follows.

By assumption 3), solving y from equations

$$H_1(y)-F_i(z)=0 \quad (i=1,\ 2),$$

we obtain

$$y=H_1^{-1}(F_i(z))=w_i(z),\ i=1,\ 2,\ z\geqslant\Delta,$$

which satisfy

$$w_2(z)\leqslant H_1^{-1}(n)\leqslant H_1^{-1}(m)\leqslant w_1(z),\ z\geqslant\Delta.$$

Let Y^+, Y^- denote the positive half y-axis and negative half y-axis, $\Gamma_i(P)$, $\Gamma_i^+(P)$, $\Gamma_i^-(P)$ $(i=1,\ 2)$ denote the trajectory, positive half-trajectory and negative half-trajectory of (11), (12) passing through point P respectively.

Denoting $P_k=P_k(z_k,\ w_1(z_k))$, we have $\dot{z}>0$ along $\Gamma_1^-(P_k)$ which is above $y=w_1(z)$ and $\dot{z}<0$ along $\Gamma_1^+(P_k)$ which is below $y=w_1(z)$.

By 3), it follows

$$\lim_{y\to+\infty}\frac{e(y)}{h(y)}=0,$$

and $\Gamma_1^-(P_k)$ must cross $z=\Delta$ and Y^+ at A_k and B_k, $\Gamma_1^+(P_k)$ must cross $z=\Delta$ and Y^- at A'_k and B'_k. Thus we have $y_{B_k}\geqslant 0$, $y_{B'_k}\leqslant 0$.

Consider function

$$u(x,\ y)=H(y-K)+z,\ z\geqslant 0,\ |y|<+\infty,$$

we have

$$\dot{u}_{(11)}=g_1(z)\,[h(y)-h(y-K)-e(y)F_1(z)]\leqslant 0 \text{ for } z\geqslant\Delta.$$

Therefore, $u(A_k)\geqslant u(P_k)$. Since $\lim\limits_{k\to+\infty}u(P_k)=+\infty$, it follows $\lim\limits_{k\to+\infty}u(A_k)=+\infty$. Hence we have $\lim\limits_{k\to+\infty}y_{A_k}=+\infty$.

Since $\dot{y}=-g_1(z)\leqslant 0$ along $\Gamma_1(P_k)$, it follows $\lim\limits_{k\to+\infty}y_{B_k}=+\infty$. By 5), we have

$$w_1(z_k)\geqslant H_1^{-1}(m+\sigma)\geqslant H_1^{-1}(m)+\varepsilon,$$

where $\varepsilon>0$. Denoting $H_1^{-1}(m)=\overline{m}$, we have

$$w_1(z_k)\geqslant\overline{m}+\varepsilon,\quad (k=1,\ 2,\ \cdots\cdots).$$

Next, we consider $\Gamma_2^+(Q_k)$, $Q_k=Q_k\left(z_k,\overline{m}+\frac{\varepsilon}{2}\right)$. It is easy to know that $\Gamma_2^+(Q_k)$ must intersect $z=\Delta$ and Y^+ at C_k and D_k. Besides, y_{C_k} and y_{D_k} increase strictly as $k\to+\infty$.

If $\lim\limits_{k\to+\infty}y_{C_k}<+\infty$, we can prove that $\lim\limits_{k\to+\infty}y_{D_k}<+\infty$, hence we have $y_{B_k}>y_{D_k}$ for sufficiently large k.

If $\lim_{k\to+\infty} y_{C_k}=+\infty$, we have $\lim_{k\to+\infty} y_{D_k}=+\infty$, This time, we also have

$$u(A_k)-u(C_k)\geqslant u(P_k)-u(Q_k)=\int_0^{w_1(z_k)-K} h(y)dY-\int_0^{y_{Q_k}-K} h(y)dy$$

$$\geqslant\int_{\overline{m}+\frac{\varepsilon}{2}-K}^{\overline{m}+\varepsilon-K} h(y)dy=l>0 \quad (l\text{—constant}).$$

But

$$u(B_k)-u(D_k)=u(A_k)-u(C_k)+\int_0^{\Delta}\frac{F_1(z)e(y_1)-h(y_1)+h(y_1-K)}{h(y_1)-e(y_1)F_1(z)}dz$$

$$-\int_0^{\Delta}\frac{F_2(z)e(y_2)-h(y_2)+h(y_2-K)}{h(y_2)-e(y_2)F_2(z)}dz,$$

where y_1, y_2 denote the expressions $y=y_{ik}(z)$ $(i=1, 2)$ of $\widehat{A_kB_k}$ and $\widehat{C_kD_k}$ respectively. Let $k\to+\infty$, $y_{ik}(z)\to0$ $(i=1, 2)$ uniformly on $[0, \Delta]$.

By assumptions 3), 5), the absolute values of the above integrals may be arbitrarily small. Thus, for sufficiently largs k, we have

$$u(B_k)-u(D_k)\geqslant\frac{l}{2}>0,$$

hence

$$\int_0^{y_{B_k}-K} h(y)dy>\int_0^{y_{D_k}-K} h(y)dy$$

and

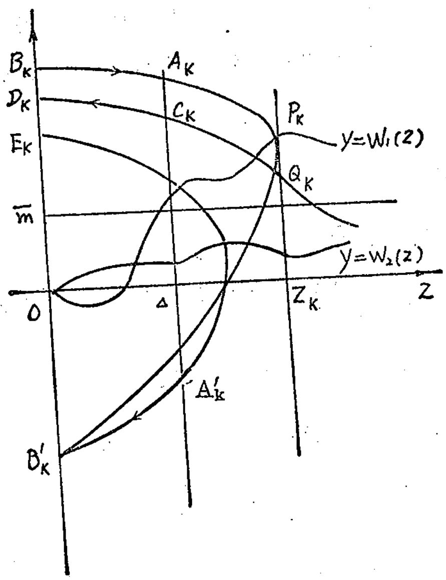
$$y_{B_k}>y_{D_k}.$$

Next, let us consider $\Gamma_2^-(Q_k)$. There are two cases:

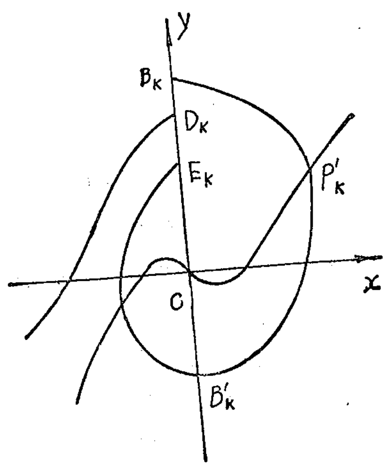
1°. If there exists a $\bar{k}$ such that $\Gamma_2^-(Q_{\bar{k}})$ does not intersect $y=w_2(z)$ (Fig. 3), then all $\Gamma_2^-(Q_k)$ will never intersect $y=w_2(z)$ for every $k>\bar{k}$. Then, we consider $\Gamma_2^+(B'_k)$ $(k\geqslant\bar{k})$. Since $\lim_{z\to+\infty} u(z, y)=+\infty$ and

$$\dot{u}_{(12)}=g_2(z)\,[h(y)-h(y-K)-e(y)F_2(z)]\leqslant0 \text{ for } z\geqslant\Delta,$$

$\Gamma_2^+(B'_k)$ can not extend to infinity in the right half (z, y)-plane, but comes back



(Fig. 3)

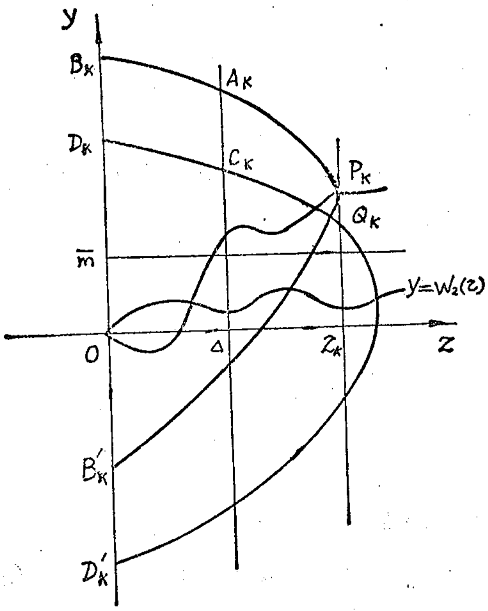


(Fig. 4)

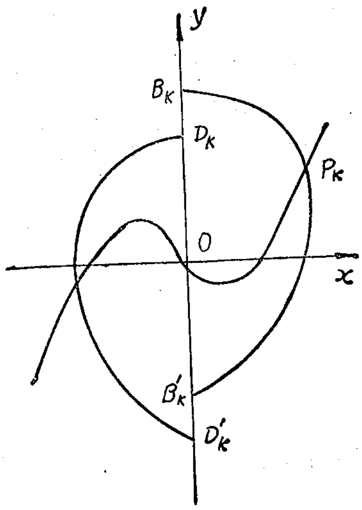
crossing Y^+ at some point E_k, where $y_{E_k}>y_{D_k}$. Hence, for sufficiently large k, we have $y_{E_k}<y_{B_k}$. Returning to (x, y)-plane, we obtain a closed curve $\widehat{B_kP'_kB'_kE_kB_k}$ (Fig. 4) which can be regarded as the desired exterior boundary.

2°. If all $\Gamma_2^-(Q_k)$ intersect $y=w_2(z)$, then they must also intersect $z=\Delta$ and Y^- at points C'_k, D'_k. By a similar argument for $\Gamma_2^+(Q_k)$, we can prove that, for sufficiently large k, we have $y_{B'_k}>y_{D'_k}$ (Fig. 5).

Returning back to (x, y)-plane, we get a closed curve $\widehat{B_kP'_kB'_kD'_kD_kB_k}$ which can be



(Fig. 5)



(Fig. 6)

regarded as the desired exterior boundary (Fig. 6).

Summing up the above arguments, we can coostruct the desired annular region and Theorem 3 is proved. This Theorem generalizes the main theorem of [3], and it is parallel to the famous result of [4] and the more recent result of [5, 6] Moreover, Theorem 3 provides new possibilities for nonlinear oscillation to have periodic solutions.

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