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CHARACTERISTIC FORMS OF TRANSFORMATIONS

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Abstract

In this paper, the author defines boundary preserving transformations and proves that they are homeomorphisms; defines interior preserving transformations and proves that they usually are open imbedding; and defines the co-continuous transformations, which have not been discussed in continuous, closed and open transformations. The characteristic forms of transformations are most important in the discussion, and there are 17 cases for homeomorphism.

All spaces considered are connected T_1.

In a topological space X, let A be a subset in X, the boundary of A is denoted by ∂A, with boundary operator ∂. The interior of A is the set

$$\mathrm{Int}A=(1-\partial)A(=A-\partial A).$$

The closure of A is the set

$$\bar{A}=(1+\partial)A(=A\cup\partial A).$$

Definition 1. *Let f: $X\to Y$ be a transformation (not necessarily continuous). Then f is boundary preserving iff it satisfies the relation $\partial fA=f\partial A$ for any subset $A\subset X$, and is denoted by the characteristic form*

$$\partial f=f\partial. \tag{I}$$

Lemma 1. *A boundary preserving transformation f: $X\to Y$ is a bijection.*

Proof To prove onto, suppose f is not onto, let $fX=U\neq Y$. Denote $Y-U=V\neq\varnothing$. We have $\partial fX=\partial U=\partial V\neq\varnothing$, for Y is connected. But $f\partial X=f\varnothing=\varnothing$, and $\partial fX\neq f\partial X$. This is a contradiction.

To prove one-to-one, suppose $x_1\neq x_2$ in X, and $fx_1=fx_2=y\in Y$. Put $A=X-x_2$, then $\partial A=x_2$, since X is connected T_1. Then $fA=Y$, $\partial fA=\partial Y=\varnothing$. But $f\partial A=fx_2=y\neq\varnothing$, and $\partial fA\neq f\partial A$. This is a contradiction.

Lemma 2. *The inverse transformation f^{-1} of a boundary preserving transformation f:$X\to Y$, is also a boundary preserving.*

Proof Let B be any subset in Y, and $f^{-1}B=A$, $fA=B$. We have $\partial fA=f\partial A$, or $\partial B=f\partial f^{-1}B$. Then

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$$f^{-1}\partial B=f^{-1}f\partial f^{-1}B=\partial f^{-1}B.$$

So f^{-1} is also boundary preserving.

Theorem 3. *A boundary preserving transformation f: $X\to Y$ is a homeomorphism.**

Proof We have proved that f is bijective and f^{-1} is also boundary preserving in lemmas 1 and 2. It will be known below that f is bicontinuous, and hence a homeomorphism.

Example 1 The inverse of lemma 2 is not valid in general, though f^{-1} satisfies $\partial f^{-1}B=f^{-1}\partial B$, yet f^{-1} and f may not be one-to-one. Let $X=S^1$: $x^2+y^2=1$, $Y=[-1, 1]$ or $-1\leqslant x\leqslant 1$, $y=0$; p: $S^1\to Y$ be the projection $p(x, y)=(x, 0)$. p is not boundary preserving, but p^{-1} is.

Definition 2. *f is an interior [closure] preserving iff it satisfies the relation*

$$(1-\partial)fA=f(1-\partial)A,\ [(1+\partial)fA=f(1+\partial)A]$$

for any subset $A\subset X$, and is denoted by the characteristic form

$$(1-\partial)f=f(1-\partial), \qquad \text{(II)}$$

$$[(1+\partial)f=f(1+\partial). \qquad \text{(III)}]$$

Definition 3. *f has following property iff it satisfies the corresponding relation for any subset $A\subset X$:*

continuous: $(1+\partial)fA\supset f(1+\partial)A,$ (1)

closed: $(1+\partial)fA\subset f(1+\partial)A,$

open: $(1-\partial)fA\supset f(1-\partial)A,$

co-continuous: $(1-\partial)fA\subset f(1-\partial)A;$

and is denoted by the characteristic form:

$$(1+\partial)f\supset f(1+\partial), \qquad \text{(IV)}$$

$$(1+\partial)f\subset f(1+\partial), \qquad \text{(V)}$$

$$(1-\partial)f\supset f(1-\partial), \qquad \text{(VI)}$$

$$(1-\partial)f\subset f(1-\partial). \qquad \text{(VII)}$$

Definition 3′. *f has following property iff it satisfies the corresponding relation for any subset $A\subset X$:*

continuous: $(1+\partial)fA=(1+\partial)f(1+\partial)A,$ (2)

closed: $f(1+\partial)A=(1+\partial)f(1+\partial)A,$

open: $f(1-\partial)A=(1-\partial)f(1-\partial)A,$

co-continuous: $(1-\partial)fA=(1-\partial)f(1-\partial)A.$

Theorem 4. *The definitions 3 and 3′ is equivalent for each case.*

Proof Let f: $X\to Y$ be any transformation, and A be any subset in X. Consider the following inclusion diagram:

* Wallace, A. D. mentioned the concept of boundary preserving transformation in his paper "Some characterizations of interior transformations", Amer. Jour. of Math., 61 (1939), 757—763.

$$(1+\partial)fA \subset (1+\partial)f(1+\partial)A \supset f(1+\partial)A, \quad (3)$$
$$(1-\partial)fA \supset (1-\partial)f(1-\partial)A \subset f(1-\partial)A.$$

It is clear that definition 3′ implies definition 3. Conversely, let $(1+\partial)$ be acted on (1), we have

$$(1+\partial)^2 fA \supset (1+\partial)f(1+\partial)A,$$

or

$$(1+\partial)fA \supset (1+\partial)f(1+\partial)A. \quad (4)$$

Comparing (4) with (3), (2) is correct. All other cases can be proved similarily.

Proposition 5. *Interior preserving is equivalent to both open and co-continuous.*

Proposition 5C. *Closure preserving is equivalent to both continuous and closed.*

Example 2. Let $X = I = [0, 1]$, $Y = I$, and

$$f(x) = \begin{cases} x, & \text{when } x \text{ is a rational in } I, \\ \frac{1}{2}, & \text{when } x \text{ is an irrational in } I. \end{cases}$$

f is not continuous, not closed, not open but co-continuous.

Example 3. Let $X = I = Y$, and

$$f(x) = \begin{cases} x, & \text{when } x \text{ is a rational in } I, \\ x+\frac{1}{2}, & \text{when } x \text{ is an irrational in } \left[0; \frac{1}{2}\right] \\ x-\frac{1}{2}, & \text{when } x \text{ is an irrational in } \left[\frac{1}{2}, 1\right]. \end{cases}$$

f dose not satisfy characteristic forms (IV, V, VI, VII). Let A be a set of all rationals in $\left[0, \frac{1}{4}\right]$ and of all irrationals in $\left[\frac{1}{2}, \frac{3}{4}\right]$. We have $(1-\partial)fA = \left[0, \frac{1}{4}\right)$, and

$$f(1-\partial)A = \varnothing.$$

Definition 4. *The space X is called substitution topology if for any point $x_1 \in X$ with neighborhood U and $x_2 \notin U$, $(U - x_1) + x_2$ is an neighborhood of x_2; X is called non-substitution if $(U - x_1) + x_2$ is not an neighborhood of x_2.*

Lemma 6. *If X is non-substitution, then one-to-one is the necessary condition of interior preserving transformation.*

Proof Suppose that the inverse image $\{f^{-1}y\}$ of $y \in Y$ is more than one point. Let x_1 and x_2 be two points in $\{f^{-1}y\}$ and U be an neighborhood of x_1 not containing x_2, and $(U - x_1) + x_2$ is not an neighborhood of x_2. Put $A = (U - \{f^{-1}y\}) + x_2$. Then

$$(1-\partial)fA = (1-\partial)f[(U - \{f^{-1}y\}) + x_2] = (1-\partial)fU = f(1-\partial)U,$$

and

$$f(1-\partial)A = f(1-\partial)[(U - \{f^{-1}y\}) + x_2] = f(1-\partial)(U - \{f^{-1}y\}).$$

Hence

$$(1-\partial)fA \neq f(1-\partial)A,$$

and f is not interior preserving.

Let f: $X \to Y$ be a transformation, and $fX = Y_1 \subset Y$. Denote f_1: $X \to Y_1 \subset Y$ by $f_1 x = fx$ for $x \in X$. If f is an interior preserving, then so is f_1.

Theorem 7. *If X is non-substitution, and f: $X \to Y$ is an interior preserving* *transformation, then so is* $f_1^{-1}: Y_1 \to X$.

Proof Let B be any subset in Y_1. Putting $f_1^{-1}B = A$ and $B = f_1 A$. we have

$$(1-\partial)B = (1-\partial)f_1 A = f_1(1-\partial)A = f_1(1-\partial)f_1^{-1}B,$$

$$f_1^{-1}(1-\partial)B = (1-\partial)f_1^{-1}B.$$

f_1^{-1} satisfies the characteristic form

$$(1-\partial)f_1^{-1} = f_1^{-1}(1-\partial), \tag{II'}$$

and f_1^{-1} is also interior preserving.

Proposition 8. *Let $f: X \to Y$ be a transformation. It is well known that the following statements are equivalent:*

1° *f is continuous.*

2° $(1+\partial)fA \supset f(1+\partial)A$, *for any subset* $A \subset X$.

2.1° $(1+\partial)fA \supset f\partial A$, *for any subset* $A \subset X$.

3° $(1-\partial)f^{-1}B \supset f^{-1}(1-\partial)B$, *for any subset* $B \subset Y$.

4° $(1+\partial)f^{-1}B \subset f^{-1}(1+\partial)B$, *for a ny subset* $B \subset Y$.

4.1° $\partial f^{-1}B \subset f^{-1}(1+\partial)B$, *for any subset* $B \subset Y$.

Corollary 9. *If $f: X \to Y$ is an one-to-one interior preserving transformation, then f and $f_1: X \to Y_1 \subset Y$ are continuous.*

Proof Since f_1^{-1} satisfies (II′) in Theorem 7, thus by 3° in proposition 8, f_1 and f are continuous.

Corollary 10. *If $f: X \to Y$ is an one-to-one interior preserving transformation, then $f_1^{-1}: Y_1 \to X$ is continuous.*

Proof We have

$$(1-\partial)f_1 A \supset f_1(1-\partial)A, \tag{5}$$

for any subset $A \subset X$. Since f_1 is an interior preserving transformation and bijective, hence $(f_1^{-1})^{-1} = f_1$, and (5) is the continuity of f_1^{-1} by 3° in proposition 8.

Theorem 11. *If $f_1: X \to Y_1 \subset Y$ is an interior preserving transformation, and X is a non-substitution space, then f_1 is a homeomorphism, and $f: X \to Y$ is an open imbedding.*

Proof By Theorem 7, Corollarys 9 and 10, $f_1: X \to Y_1 \subset Y$ is bijective and bicontnuous, so f_1 is a homeomorphism, and $f: X \to Y$ is an open imbedding.

Theorem 7C. *If $f: X \to Y$ is an one-to-one closure preserving transformation, then so is $f_1^{-1}: Y_1 \to X$.*

Example 4 The inverses of Theorems 7 and 7C are not valid in general. As in example 1, p is not interior preserving, but p^{-1} is . Again, let S^1: $x^2+y^2=1$, $X = S^1 - (x_0, y_0)$, where $(x_0, y_0) = (0, 1)$, $Y = [-1, +1]$, or $-1 \leqslant x \leqslant 1$, $y=0$. $p: X \to Y$ as before, $p(x, y) = (x, 0)$. p is not closure preserving, but p^{-1} is.

Corollary 9C. *If $f: X \to Y$ is an one-to-one closure preserving transformation, then f and $f_1: X \to Y_1 \subset Y$ are co-continuous.*

Corollary 10C. *If f: $X \to Y$ is an one-to-one closure preserving transformation, then f_1^{-1}: $Y_1 \to X$ is continuous.*

Theorem 11C. *If f_1: $X \to Y_1 \subset Y$ is an one-to-one closure preserving transformation, then f_1 is a homeomorphism, and f_1 is a closed imbedding.*

Lemma 12. *Onto is the necessary condition of both closed and open transformation.*

Proof Let f: $X \to Y$ be both closed and open. Since X and Y are all connected, except the empty set $\varnothing$, the both closed and open sets are X and Y only, and $fX \subset Y$ is both closed and open. Hence it must be $fX = Y$, so f is onto.

Theorem 13. *If f: $X \to Y$ is a transformation for non-substitution space X, then the following 6 statements are equivalent:*

1° *f is boundary-preserving or homeomorphic.*

2° *f is interior-preserving and closed.*

3° *f is closure-preserving, open and one-to-one.*

4° *f is bijectively closure-preserving.*

5° *f is both interior and closure-preserving.*

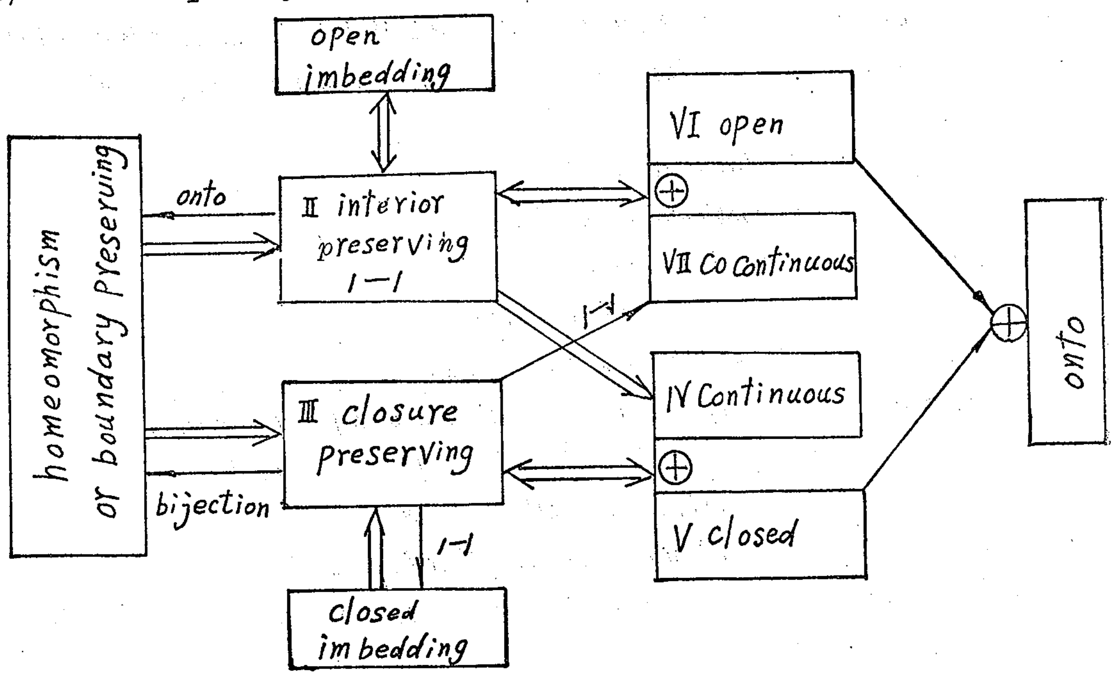
6° *f is closed, open and co-continuous.*

Proof In all statements (except 4°), by consequence of Lemmas 6 and 12, f is a bijection.

The equvalence is a consequence of Theorem 3, Propositions 5 and 5C, and Theorems 11 and 11C.

Thus we have the diagram, where the spaces are all connected T_1 non-substitutive.

Definition 5. *Let f: $X \to Y$ be a transformation. f has following property iff it satisfies the corresponding relation for any subset $A \subset X$:*



boundary continuous $\partial fA \supset f\partial A$, *boundary closed* $\partial fA \subset f\partial A$;

and is denoted by the corresponding characteristic form:

$$\partial f \supset f\partial, \tag{VIII}$$

$$\partial f \subset f\partial. \tag{IX}$$

Example 5 Jordan curve is an imbedding $f: S^1 \to S^2$, it's boundary continuous, but not boundary closed. In example 1, the projection $p: S^1 \to [-1, 1]$, p is boundary closed, not boundary continuous.

Proposition 14. *Boundary preserving transformations are equivalent to both boundary continuous and boundary closed.*

Theorem 15. *If $f: X \to Y$ is bijective, then the following* 6 *statements are equivalent*:

1° *f is cocontinuous,* $(1-\partial)f \subset f(1-\partial)$. (VII)

2° *f is continuous,* $(1+\partial)f \supset f(1+\partial)$. (IV)

3° *f is boundary continuous,* $\partial f \supset f\partial$. (VIII)

4° f^{-1} *is boundary closed,* $\partial f^{-1} \subset f^{-1}\partial$. (IX)

5° f^{-1} *is closed,* $(1+\partial)f^{-1} \subset f^{-1}(1+\partial)$. (V)

6° f^{-1} *is open,* $(1-\partial)f^{-1} \supset f^{-1}(1-\partial)$. (VI)

Theorem 15C. *f may be replaced by f^{-1}, and vice versa.*

Theorem 16. *Let $f: X \to Y$ be a transformation for non-substitution space X, and X, Y be all connected T_1. Then the following* 17 *statements are all equivalent to a homeomorphism.*

1° *f is boundary-preserving.*

2° *f is both boundary continuous and boundary closed.*

3° *f is both interior-and closure-preserving.*

4° *f is closed, open and cocontinuous.* 5° *f is both interior-preserving and closed.*

6° *f is one-to-one and both closure-preserving and open. For the following cases, f is a bijection.*

7° *Either f or f^{-1} is closure-preserving.*

8° *Either f or f^{-1} is continuous and closed.*

9° *Either f or f^{-1} is continuous and open.*

10° *Either f or f^{-1} is open and co-continuous.*

11° *Either f or f^{-1} is closed and co-continuous*

12° *Both f and f^{-1} are continuous.* 13° *Both f and f^{-1} are closed.*

14° *Both f and f^{-1} are open.* 15° *Both f and f^{-1} are cocontinuous.*

16° *Either f or f^{-1} is closed, and the other is open.*

17° *Either f or f^{-1} is continuous, and the other is cocontinuous.*

Proof Some had been shown in Theorem 13, and others can be shown in Theorem 15 and Proposition 14.