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UNIFORM STRONG CONVERGENCE RATE OF NEAREST NEIGHBOR DENSITY ESTIMATION

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Abstract

Based on [3] and [4], the authors study strong convergence rate of the k_n-NN density estimate $\hat{f}_n(x)$ of the population density $f(x)$, proposed in [1]. $f(x)>0$ and f satisfies λ-condition at x $(0<\lambda\leqq 2)$, then for properly chosen k_n

$$\varlimsup_{n\to\infty}\left(\frac{n}{\log n}\right)^{\lambda/(1+2\lambda)}|\hat{f}_n(x)-f(x)|\leqq C \quad a.s.$$

If f satisfies λ-condition, then for propeoly chosen k_n

$$\varlimsup_{n\to\infty}\left(\frac{n}{\log n}\right)^{\lambda/(1+3\lambda)}\sup_x|\hat{f}_n(x)-f(x)|\leqq C \quad a.s.,$$

where C is a constant. An order to which the convergence rate of $|\hat{f}_n(x)-f(x)|$ and $\sup_x|\hat{f}_n(x)-f(x)|$ cannot reach is also proposed.

§1. Introduction

Let $X_1, \cdots, X_n$ be i.i.d. samples taking values in R and having distribution function F and unknown density function f. A class of estimators of f proposed by Loftsgarden and Quesenberry[1] has the form

$$\hat{f}_n(x)=k/[2na_n(x)], \tag{1}$$

where $k=k_n$ is a sequence of positive integer chosen in advance and $a_n(x)$ equal to the distance from x to the k_nth nearest of $X_1, \cdots, X_n$. Call $\hat{f}_n(x)$ the nearest neighbor estimator of f. Since then, this estimate has been widely studied. Devroye and Wagner[2] showed the uniform strong consistency of f when f is uniformly continuous under the conditions $k/n\to 0$ and $\log n/k\to 0$. Concerning the uniform strong consistency and strong convergence rates of $\hat{f}_n$, the best result so far as we know was given by Chen ([3] & [4]):

1) It is impossible to establish any convergence rate of $\sup_x|\hat{f}_n(x)-f(x)|$ without

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some further conditions imposed on f besides the uniform continuity.

2) If f satisfies δ–Lipschitz condition, $0<\delta\leqq 1$, and we choose

$$k=[n^{2\delta/(1+3\delta)}], \tag{2}$$

then

$$\sup_x |\hat{f}_n(x)-f(x)|=O(n^{-\delta/(1+3\delta)}\sqrt{\log n}) \quad a.s. \tag{3}$$

3) For any $\delta\in(0, 1]$, there exists a density function f satisfying δ–Lipschitz condition such that

$$\sup_x |\hat{f}_n(x)-f(x)|=o(n^{-\delta/(1+3\delta)}) \quad a.s. \tag{4}$$

does not hold for any possible choice of k.

Later, Yang Zhenhai[5] proved that if $\delta=1$, on choosing k suitably, the right hand side of (3) can be improved to $O((\log n/n)^{1/4})$.

This paper is devoted to further study of this problem.

We call f satisfies λ–condition at x, $\lambda\in(0, 2]$, if there exists a $\lambda\in(0, 1]$ such that

$$|f(x)-f(y)|\leqq C|y-x|^{\lambda}, \ |y-x|\leqq h \tag{5}$$

for h is small enough, or there exists a $\lambda\in(1, 2]$ such that

$$|f'(x)-f'(y)|\leqq C|y-x|^{\lambda-1}, \ |y-x|\leqq h \tag{6}$$

for h is sufficiently small, where $C=C(x)$ is a constant depending on x.

We say that f satisfies λ–condition, $\lambda\in(0, 2]$, if $\lambda\in(0, 1]$ and f satisfies λ–Lipschitz condition, or $\lambda\in(1, 2]$ and $f'(x)$ is bounded and satisfies $(\lambda-1)$–Lipschitz condition.

We shall prove the following theorems:

Theorem 1. *Suppose that $f(x)>0$ and f satisfies λ–condition at x for $\lambda\in(0,2]$. If we choose*

$$k=[n^{2\lambda/(1+2\lambda)}(\log n)^{1/(1+2\lambda)}],$$

then

$$\limsup_{n\to\infty}\left(\frac{n}{\log n}\right)^{\lambda/(1+2\lambda)}|\hat{f}_n(x)-f(x)|\leqq C \quad a.s. \tag{7}$$

where C is a constant depending on x. If

$$\lim_{h\to 0}\frac{1}{h^{1+\lambda}}\int_{x-h}^{x+h}(f(t)-f(x))dt=e_1\neq 0, \tag{8}$$

then

$$|\hat{f}_n(x)-f(x)|=o(n^{-\lambda/(1+2\lambda)}) \quad a.s. \tag{9}$$

does not hold for any possible choice of $k=o(n)$.

Theorem 2. *Suppose that f satisfies λ–condition for $\lambda\in(0, 2]$. If we choose*

$$k=[n^{2\lambda/(1+3\lambda)}(\log n)^{(1+\lambda)/(1+3\lambda)}], \tag{10}$$

then

$$\limsup_{n\to\infty}(n/\log n)^{\lambda/(1+3\lambda)}\sup_x|\hat{f}_n(x)-f(x)|\leqq C \quad a.s., \tag{11}$$

where C is a constant not depending on n and x.

Theorem 3. *For any $\lambda\in(0, 2]$, there exists a density f satisfying λ-condition such that*

$$\sup_x|\hat{f}_n(x)-f(x)|=o(n^{-\lambda(1+3\lambda)}) \quad a.s. \tag{12}$$

does not hold for any possible choice of k, and there exists a density function with bounded derivatives of any order such that for any choice of k, it is impossible that

$$\sup_x|\hat{f}_n(x)-f(x)|=o(n^{-2/7}) \quad a.s. \tag{13}$$

§ 2. Strong Convergence Rate of $f_n(x)$

In this section, x is fixed and C, C_1, C_2, $\cdots$ are all constants not depending on n (possibly depending on x).

Lemma 1. *Suppose that the random variable $Y\sim B(n, p)$, then for any $\varepsilon>0$, we have*

$$P(|Y/n-p|\geqq\varepsilon)\leqq 2\exp\{-n\varepsilon^2/(2p+\varepsilon)\}.$$

Refer to [6], Theoren 3.

Lemma 2. *Suppose $f(x)>0$ and f satisfies λ-condition at x, $\lambda\in(0, 2]$, we have*

1) *if $k\to\infty$ and $k=o(n^{2\lambda/(1+2\lambda)})$ as $n\to\infty$, then*

$$\sqrt{k}\,(\hat{f}_n(x)-f(x))/f(x)\xrightarrow{L}N(0, 1). \tag{14}$$

2) *if (8) holds and $\lim\limits_{n\to\infty}C_1n^{2\lambda/(1+2\lambda)}/k=1$, then*

$$\sqrt{k}\,(\hat{f}_n(x)-f(x))/f(x)-C_1^{\lambda+\frac{1}{2}}e_1/(2^{\lambda+1}f^{\lambda+1}(x))\xrightarrow{L}N(0, 1). \tag{15}$$

3) *if the conditions in 2) hold and $k=o(n)$, then*

$$Z_n\triangleq n^{\lambda/(1+2\lambda)}|\hat{f}_n(x)-f(x)|\xrightarrow{P.}0$$

is not true.

The argument is similar to that of [3], so we omit the proof.

Proof of Theorem 1 Choose $k=[n^{\frac{2\lambda}{1+2\lambda}}(\log n)^{\frac{1}{1+2\lambda}}]$, without loss of generality, we can suppose

$$k=n^{2\lambda/(1+2\lambda)}(\log n)^{1/(1+2\lambda)}, \tag{16}$$
$$q_n=(k/n)^{-\lambda}=(n/\log n)^{\lambda/(1+2\lambda)}.$$

Hence

$$P\{q_n(\hat{f}_n(x)-f(x))/f(x)\geqq 2C_2\}=P\{a_n(x)\leqq d_n\}, \tag{17}$$

where

$$d_n=d_n(x)=k/(2nf(x)(1+2C_2q_n^{-1}))=O(k/n). \tag{18}$$

Since f satisfies λ-condition at x, it is not difficult to get

$$p_n\triangleq\int_{x-d_n}^{x+d_n}f(t)dt=2d_nf(x)+\theta_nC_3d_n^{1+\lambda},\ |\theta_n|\leqq 1 \tag{19}$$

for n large enough.

From (16), we can choose sufficiently large C_2 such that

$$p_n=\frac{k}{n}[1-(2C_2q_n^{-1}-C_3\theta_n(k/n)^\lambda(1+o(1))]\leqq\frac{k}{n}(1-C_2q_n^{-1})\leqq k/n, \tag{20}$$

$$k/n-p_n\geqq C_2q_n^{-1}k/n=C_2(k/n)^{1+\lambda}. \tag{21}$$

Let $\mu_n(x, d_n)$ be the empirical measure of $[x-d_n, x+d_n]$.

We choose $C_2^2\geqq 12$ and fix it. From (20), (16) and Lemma 1,

$$\begin{aligned} P\{a_n(x)\leqq d_n\}&=P\{\mu_n(x, d_n)-p_n\geqq k/n-p_n\}\\ &\leqq P\{\mu_n(x, d_n)-p_n\\ &\geqq C_2(k/n)^{1+\lambda}\}\\ &\leqq 2\exp\{-nC_2^2(k/n)^{2+2\lambda}/[2k/n+C_2(k/n)^{1+\lambda}]\}\\ &\leqq 2\exp(-C_2^2\log n/3)<2n^{-2}. \end{aligned} \tag{22}$$

From (17) and (22)

$$\sum_n P\{q_n(\hat{f}_n(x)-f(x))\geqq 2C_2f(x)\}<\infty. \tag{23}$$

Hence, from Borel-Cantelli's lemma, $\limsup_{n\to\infty} q_n(\hat{f}_n(x)-f(x))$ is bounded above a. s..

In the same way, we can also prove $\underline{\lim}_{n\to\infty}\sup q_n(\hat{f}_n(x)-f(x))$ is bounded bellow a. s..

Hence we have proved the first part of the theorem. The rest follows from 3) of Lemma 2.

§ 3. Strong Convergence Rate of $\sup_x|f_n(x)-f(x)|$

The constants $C, C_1, C_2, \cdots, N, N_1, \cdots$ in this section are all independent of x and n.

Lemma 3. *Let $X_1, \cdots, X_n$ be i. i. d. samples taken from a one-dimensional population, $\mu(A)$ and $\mu_n(A)$ be it's probability distribution and empirical measure respectively. Suppose $T\subset R$, $\mathscr{A}_l=\{[x-l', x+l']: x\in T, l'\leqq l\}$ and*

$$\sup_{\mathscr{A}_{2l}}\mu(A)\leqq b\leqq 1/4, \tag{24}$$

then, for $\varepsilon>0$ and $n\geqq\max(1/b, 8b/\varepsilon^2)$, we have

$$P\{\sup_{\mathscr{A}_l}|\mu_n(A)-\mu(A)|\geqq\varepsilon\}\leqq 16n^2\exp\{-n\varepsilon^2/(64b+4\varepsilon)\}+8n\exp\{-nb/10\}. \tag{25}$$

Proof See [2].

Lemma 4. *Use the notations of Lemma 3. Suppose that $k=o(n)$ and*

$$\sup_{\mathscr{A}_{2l}}\mu(A)\leqq 10k/n\triangleq b_n\leqq 1/4. \tag{26}$$

Then, for $r>0$ and $0\leqq s\leqq 1/2$, there exists $C_1>0$ such that

$$\limsup_{n\to\infty}\left\{\left(\frac{n}{k}\right)^{1+r}(\log n)^s\sup_{\mathscr{A}_l}|\mu_n(A)-\mu(A)|\right\}\leqq C_1 \ a.s. \tag{27}$$

whenever

$$k\geqq\beta n^{2r/(1+2r)}(\log n)^{(1-2s)/(1+2r)},\ (\beta>0 \text{ is a const.}) \tag{28}$$

$n>b_n^{-1}$ and $n>8b_n/\varepsilon_n^2$ hold for large n, from Lemma 3 we have

$$P\{\sup_{\mathscr{A}_1}|\mu_n(A)-\mu(A)|\geqq\varepsilon_n\}$$
$$\leqq 16n^2\exp\{-nC_1^2(k/n)^{2+2r}(\log n)^{2s}/[640k/n+4\varepsilon_n]\}+8ne^{-k}$$
$$\leqq 16n^2\exp\{-C_1^2\beta^{1+2r}\log n/650\}+8ne^{-k}\leqq 16n^{-2}+8ne^{-k}. \tag{29}$$

Hence

$$\sum_n P\{(n/k)^{1+r}(\log n)^{-s}\sup_x|\mu_n(A)-\mu(A)|\geqq C_1\}<\infty, \tag{30}$$

and from Borel-Cantelli's lemma, (27) is concluded.

Lemma 5. *Suppose that the density function f satisfies λ-Lipschitz condition, $\lambda\in(0, 1]$. Then for any $N_1>0$ there exists N depending only upon N_1, such that for any $v_n\geqq n^{-N_1}$, the set $B_n=\{x: f(x)\geqq 2v_n\}$ is compact and there exist n^N closed intervals $B_{ni}\subset\{f(x)\geqq v_n\}$ such that $L(B_{ni})\leqq n^{-N_1}$, $\omega_f(B_{ni})<n^{-N_1}$ and $\bigcup_{i=1}^{n^N}B_{ni}\supset B_n$,* where $\omega_f(B)$ $=\sup_{x\in B}f(x)-\inf_{x\in B}f(x)$ *and L the Lebesgue measure.*

Proof It is obvious that B_n is compact. From the fact that f satisfies λ-Lipschitz condition, for any $x\in B_n$ there exists an open interval I_x containing x such that $L(I_x)=C_2n^{-N_1/\lambda}$ and $\omega_f(I_x)\leqq v_n$. By Heine-Borel's theorem, we can choose a finite number of closed $\bar{I}_x$'s such that their union $A\supset B_n$ and $f(x)\geqq v_n$ on A, where $\bar{I}_x$ denotes the closure of I_x.

The set A is a union of l_n closed intervals without common points, whose lengehs are not less than $C_2n^{-N_1/\lambda}$.Hence

$$C_2n^{-N_1/\lambda}l_n\leqq L(A)\leqq L\{f(x)\geqq n^{-N_1}\}\leqq n^{N_1}\int_{\{f(x)\geqq n^{-N_1}\}}f(x)dx\leqq n^{N_1}.$$

Therefore $l_n\leqq n^{N_2}$. Also, the length of every interval not exceed n^{N_1}. Since f satisfies λ-Lipschitz condition, it is easy to express those closed intervals by a union of some B_{ni} satisfying the requirements of Lemma 5.

Lemma 6. *Let $X_1, \cdots, X_n$ be i.i.d. variables with continuous distribution function F, and denote their empirical distribution funotion by F_n, then*

$$\limsup_{n\to\infty}[\sup_x\sqrt{n}\,|F_n(x)-F(x)|/\sqrt{2\log\log n}]=1, \quad a.s. \tag{31}$$

Proof See [7].

Proof of Theorem 2 We can choose

$$k=n^{2\lambda/(1+3\lambda)}(\log n)^{(1+\lambda)/(1+3\lambda)}, \tag{32}$$

$$\theta v_n=(k/n)^{\lambda/(1+\lambda)},\ q_n=\rho v_n^{-1}=\rho\theta(n/k)^{\lambda/(1+\lambda)}, \tag{33}$$

where $0<\theta<1$ and $\rho>1$ will be chosen later. We can take N_1 so large that $v_n\geqq n^{-N_1}$, then $B_n=\{f(x)\geqq 2v_n\}$ is covered by the union of n^N B_{ni}'s satisfying the conditions of Lemma 5 and $L(B_{ni})\leqq n^{-N_1}$, $\omega_f(B_{ni})\leqq n^{-N_1}$ for each B_{ni}. Denote

Let $\mu(x, d)$ and $\mu_n(x, d)$ be the probability distribution and empirical measure of $[x-d, x+d]$ respectively. Taking $s=0$ and $r=\lambda/(1+\lambda)$ in Lemma 4, we can assert with probability one that, for n large enough, the inequality

$$\mu(x,\ d)-A_n\leqq\mu_n(x,\ d)\leqq\mu(x,\ d)+A_n \tag{34}$$

holds uniformly for all x, d satisfying

$$\mu(x,\ 2d)\leqq 10\,k/n, \tag{35}$$

where

$$A_n=2C_1(\log n/n)^{(1+2\lambda)/(1+3\lambda)}. \tag{36}$$

Now suppose $x\in B_n^c=\{f(x)<2v_n\}$. Since f satisfies λ-condition, it is easy to get

$$\mu(x,\ d)=\int_{x-d}^{x+d}f(t)\,dt\leqq 2df(x)+C_3d^{1+\lambda}\leqq 4d\theta^{-1}(k/n)^{\lambda/(1+\lambda)}+C_3d^{1+\lambda}.$$

Noticing $k/n=(\log n/n)^{(1+\lambda)/(1+3\lambda)}$, $k/n\geqq 2A_n$ for large n, we can take $d_n=C_4(k/n)^{1/(1+3\lambda)}$ with suitably chosen C_4 such that

$$\mu(x,\ d_n)\leqq k/2n\leqq k/n-A_n, \tag{37}$$

$$\mu(x,\ 2d_n)\leqq 10k/n. \tag{38}$$

From (34)—(38), we have

$$\mu_n(x,\ d_n)\leqq\mu(x,\ d_n)+A_n\leqq k/n,$$

$$a_n(x)\geqq d_n=C_4(k/n)^{1/(1+\lambda)},$$

$$\hat f_n(x)=\frac{k}{2n}C_4^{-1}(k/n)^{-1/(1+\lambda)}\leqq C_5(\log n/n)^{\lambda/(1+3\lambda)}, \tag{39}$$

and it follows that

$$\lim_{n\to\infty}\sup\{(n/\log n)^{\lambda/(1+3\lambda)}\sup_{x\in B_n^c}|\hat f_n(x)-f(x)|\}\leqq C_6,\quad a.\ s. \tag{40}$$

On the other hand, for fixed $C_7>C$.

$$P\{\sup_{x\in B_n}|\hat f_n(x)-f(x)|\geqq C_7q_n^{-1}\}\leqq\sum_{i=1}^{n^N}P\{\sup_{x\in B_{ni}}|\hat f_n(x)-f(x)|\geqq C_7q_n^{-1}\}. \tag{41}$$

Denote $m_i=\min_{B_{ni}}f(x)$ and $M=\sup_x f(x)$, we have $v_n\leqq m_i\leqq M$. It is easy to see, $|a_n(x)-a_n(y)|\leqq|x-y|$ for any x,y. Therefore

$$\inf_{B_{ni}}a_n(x)\geqq\sup_{B_{ni}}a_n(x)-n^{-N_1}. \tag{42}$$

Obviously

$$P\{\sup_{B_{ni}}|\hat f_n(x)-f(x)|\geqq C_7q_n^{-1}\}\leqq P\{\sup_{x\in B_{ni}}\hat f_n(x)\geqq m_i+C_7q_n^{-1}\}$$

$$+P\{\inf_{x\in B_{ni}}\hat f_n(x)\leqq m_i+n^{-N_1}-C_7q_n^{-1}\}\triangleq I_{ni}+J_{ni}. \tag{43}$$

Now we estimate I_{ni}. By (42), we have

$$I_{ni}=P\left\{\inf_{B_{ni}}a_n(x)\leqq\frac{k}{2nm_i}\Big/(1+C_7q_n^{-1}m_i^{-1})\right\}\leqq P\left\{\sup_{B_{ni}}a_n(x)\leqq d_n\right\}, \tag{44}$$

where

$$d_n=\frac{k}{2nm_i}\Big/(1+C_7q_n^{-1}m_i^{-1})+n^{-N_1}. \tag{45}$$

Take ρ so large that $C_7/\rho<1/8$, then $C_7q_n^{-1}m_i^{-1}=C_7\rho^{-1}v_n/m_i<1/8$. Noticing $1/(1+x)<1-7x/8$, for $0\leqq x<1/8$, we have

$$d_n\leqq\frac{k}{2nm_i}\left(1-\frac{7}{8}C_7q_n^{-1}m_i^{-1}\right)+n^{-N_1}. \tag{46}$$

Fixed C_7 and ρ, one can choose θ so small that, for n large enough, the following inequalities hold uniformly for all i and $x\in B_{ni}$

$$\mu(x, d_n)=\int_{x-d_n}^{x+d_n} f(t)\,dt\leqq 2d_n f(x)+C_3 d_n^{\lambda+1}\leqq 2d_n\Big(m_i+n^{-N_1}+\frac{1}{2}C_3 d_n^{\lambda}\Big)$$
$$\leqq\frac{k}{n}\Big(1-\frac{3}{4}C_7 q_n^{-1} m_i^{-1}\Big)\Big(1+C_8\Big(\frac{k}{n}\Big)^{\lambda} m_i^{1-\lambda}\Big)$$
$$\leqq\frac{k}{n}\Big(1-\frac{3}{4}C_7 q_n^{-1} m_i^{-1}+C_8\rho\theta^{\lambda+1} q_n^{-1} m_i^{-1}\Big)\leqq\frac{k}{n}\Big(1-\frac{1}{2}C_7 q_n^{-1} m_i^{-1}\Big), \tag{48}$$
$$\mu(x, 2d_n)\leqq 10\,k/n\leqq 1/4, \tag{49}$$
$$k/n-\mu(x, d_n)\geqq\frac{k}{2n}C_7 q_n^{-1} M^{-1}, \tag{50}$$

$$I_{ni}\leqq P\Big\{\sup_{B_{ni}}(\mu_n(x, d_n)-\mu(x, d_n))\geqq\frac{k}{2n}C_7 q_n^{-1} M^{-1}\Big\}$$
$$\leqq 16n^2\exp\Big\{-nC_7^2\Big(\frac{k}{2n}\Big)^2 q_n^{-2} M^{-2}\Big/\Big(640k/n+\frac{2k}{n}C_7 q_n^{-1} M^{-1}\Big)\Big\}+8ne^{-k}$$
$$\leqq 16n^2\exp\Big\{-\frac{1}{2600}C_7^2\rho^{-1}\theta^{-1}M^{-2}\log n\Big\}+8ne^{-k}<\frac{1}{2}n^{-N-2}. \tag{51}$$

The last one is obtaind from Lemma 3. The estimate of J_{ni} is similar to this. Therefore, on account of (41) and (43)

$$\sum_n Pq_n\{\sup_{x\in B_n}|\hat{f}_n(x)-f(x)|\geqq C_7\}<\infty. \tag{52}$$

By Borel-Cantelli's lemma, we have

$$\lim_{n\to\infty}\sup\{q_n\sup_{x\in B_n}|\hat{f}_n(x)-f(x)|\}\leqq C_7 \quad a.s. \tag{53}$$

Consequently, (11) follows from (40) and (53).

Finally, we give the proof of Theorem 3. For $\lambda\in(0, 1]$, The proof is given in [3]; for $\lambda\in(1, 2]$, the proof can be given in a simillar way by using Lemmas 2 and 4. We omit it here.

To prove (13), construct a density function

$$f(x)=\begin{cases} C_9 x^2\exp\{1/(x^2-1)\}, & \text{if}\,|x|<1,\\ 0, & \text{if}\,|x|\geqq 1.\end{cases} \tag{54}$$

Obviously, it's derivatives of any order are bounded, and $f(0)=0$, $f''(0)=2C/e\triangleq b$. We can suppose R satisfies one of the following four assumptions (if necessary, we can choose a suitable subsequence instead of the original one).

A. $k\geqq\alpha n$, $0<\alpha<1$ is a constant;

B. $k=o(n)$ and there is a constant $\beta>0$ such that $k\geqq\beta n^{4/7}$;

C. $k\to\infty$ and $k=o(n^{4/7})$;

D. $k=k_0$, a constant.

Case A From (54), $a_n(0)\leqq 1$ and $\hat{f}_n(0)=k/(2na_n(0))\geqq\alpha/2$. In this case,

$$|\hat{f}_n(0)-f(0)|\overset{a.s.}{=\!=}O(n^{-2/7})$$

is not true.

Case B Denote $A_n=2(\log\log n/n)^{\frac{1}{2}}$. From B, $A_n\leqq k/n$ for large n. We have with probability one that

$$\mu(d_n)\triangleq\mu[-d_n,\ d_n]\leqq\mu_n[-d_n,\ d_n]+A_n\triangleq\mu_n(d_n)+A_n \tag{55}$$

for n large enough and for all d_n. Taking

$$d_n=\left\{\frac{4}{b}\left(\frac{k}{n}+A_n\right)\right\}^{1/3},$$

we obtain from $k=o(n)$ that

$$\mu(d_n)=\int_{-d_n}^{d_n}f(t)dt=\frac{f''(0)}{3}d_n^3(1+o(1))\geqq\frac{b}{4}d_n^2=k/n+A_n,$$

By (55), we have $\mu(d_n)\geqq k/n$. From the definition of $a_n(0)$, $a_n(0)\leqq d_n\leqq C_{10}(k/n)^{1/3}$. Therefore we have with probability one that

$$\hat{f}_n(0)\geqq\frac{k}{2n}C_{10}^{-1}(k/n)^{-1/3}=C_{11}(k/n)^{2/3}\geqq C_{12}n^{-2/7}, \tag{56}$$

so $|\hat{f}_n(0)-f(0)|\overset{a.s.}{=\!=}o(n^{-2/7})$ is not true in case B.

Case C We arbitrarily choose $x_0\in(0,\ 1)$, $f(x_0)\neq0$. By 1) of Lemma 2

$$\sqrt{k}\,(\hat{f}_n(x_0)-f(x_0))\xrightarrow{L.}N(0,\ 1). \tag{57}$$

From $k=O(n^{2/7})$, we see that $|\hat{f}_n(x_0)-f(x_0)|\overset{a.s.}{=\!=}O(n^{-2/7})$ is not true.

Case D For x_0 mentioned in case C

$$P\{n^{2/7}(\hat{f}_n(x_0)-f(x_0))\leqq y\}=P\{a_n(x_0)\geqq d_n\}, \tag{58}$$

where

$$d_n=k_0/\{2nf(x_0)(1+n^{-2/7}y/f(x_0)). \tag{59}$$

Denote $p_n=\int_{x-d_n}^{x+d_n}f(t)dt$, we see that $np_n\to k_0$ from the continuity of f at x_0. Let $\xi_{n1},\cdots,\xi_{nn}$ be i.i.d. variables with $P(\xi_{n1}=1)=p_n$. According to the Possion approximation of binomial distribution, we obtain

$$\lim_{n\to\infty}P\{n^{2/7}(\hat{f}_n(x_0)-f(x_0))\leqq y\}=\lim_{n\to\infty}P\left\{\sum_{i=1}^{n}\xi_{ni}\leqq k_0\right\}=\sum_{i=0}^{k_0}e^{-k_0}k_0^i/i!. \tag{60}$$

Note that the right hand side of (62) does not depend on y, so that,

$$|\hat{f}_n(x_0)-f(x_0)|\overset{a.s.}{=\!=}O(n^{-2/7})$$

is not true. The proof of Theorem 3 is concluded.

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