Chin. Ann. of Math.
5B (3) 1984

ON COMPLEX QUARTIC INTERPOLATING SPLINES WITH HIGHER DEFICIENCY

LU JIANKE (CHIEN-KE LU) (路见可)
(Wuhan University)

Abstract

In this paper, the error analysis between the complex quartic interpolating splines of deficiency$\geqslant 2$ and the interpolated functions on complex plane as well as their derivatives is studied. When the kernel density of a Cauchy principal integral is approximated by such splines, the order of the error produced between the integrals themselves is also estimated. The curve in consideration may be open or closed.

In [1], we analysed the errors of complex cubic (and modified cubic) interpolating splines. Here we shall discuss the similar problems for complex quartic splines. As in [2], we shall also estimate the errors of the Cauchy principal integral when its kernel density is approximated by such splines.

Many authors discussed real quartic splines, e. g., in [3, 4], but seldom about complex ones.

§ 1. Quartic interpolating splines with deficiency 2

Let $L=\widehat{ab}$ be an open smooth (or arcwise smooth) arc in the complex plane and $f(t)\in C^1$ be given on L (we exclude the tri-vial case that $f(t)$ is a polynomial of degree $\leqslant 4$). We shall use the notations in [1] and [2] without explanation.

By taking $\{t_j\}$ $(j=0, 1, \cdots, N)$ both as knots and nodes, the interpolating quartic spline of $f(t)$ may be written as

$$Q(t)\equiv Q_j(t)=S_j(t)+A_jR_j(t),\ t\in L_j,\ j=0,\ 1,\ \cdots,\ N-1, \tag{1.1}$$

where

$$S_j(t)=y_j+\frac{\Delta y_j}{\Delta t_j}(t-t_j)+\left(y'_j-\frac{\Delta y_j}{\Delta t_j}\right)\frac{(t-t_j)(t-t_{j+1})^2}{\Delta t_j^2} +\left(y'_{j+1}-\frac{\Delta y_j}{\Delta t_j}\right)\frac{(t-t_j)^2(t-t_{j+1})}{\Delta t_j^2}, \tag{1.2}$$

$$R_j(t)=\frac{(t-t_j)^2(t-t_{j+1})^2}{\Delta t_j^2}, \tag{1.3}$$

and $\{A_j\}$ are undetermined constants. In order that $Q(t)\in C^2$, we must require

Manuscript received September 27, 1982.

$$A_{j+1}-A_j=J_j, \tag{1.4}$$

where J_j is the jump of $S''(t)$ at t_j:

$$J_j=S_j''(t_j)-S_{j-1}''(t_j). \tag{1.5}$$

Hence, we may take, for instance, A_0 arbitrarily, then

$$A_j=A_0-\sum_{k=1}^{j} J_k=A_0+S_0''(t_1)-S_j''(t_j)+\sum_{k=1}^{j-1}[S_k''(t_k)-S_k''(t_{k+1})],\quad j=1,\cdots,N-1. \tag{1.6}$$

For brevity, we take $A_0=-S_0''(a)$, say. It may be easily verified that ($\delta=\max \Delta S_j$)

$$|S_j''(t_j)|\leqslant 6C_\delta\omega_1(\delta)/|\Delta t_j|,$$

$$|S_k''(t_k)-S_k''(t_{k+1})|\leqslant 12C_\delta\omega_1(\delta)/|\Delta t_k|.$$

Therefore

$$|A_j|\leqslant\frac{6C_\delta}{|\Delta t_j|}\omega_1(\delta)+12C_\delta^2\omega_1(\delta)\sum_{k=0}^{j-1}\frac{1}{\Delta s_k}. \tag{1.7}$$

Noting that

$$|R_j(t)|\leqslant\frac{1}{32}C_\delta^2\Delta s_j^2,\quad t\in L_j, \tag{1.8}$$

we then have

$$|A_jR_j(t)|\leqslant\frac{3}{16}C_\delta^4\omega_1(\delta)(\delta+2K_\Delta^2L),\quad j=0,1,\cdots,N-1, \tag{1.9}$$

where

$$K_\Delta=\max \Delta s_j/\min \Delta s_j.$$

Applying the estimate of $|f(t)-S(t)|$ in [1], we obtain by (1.9)

$$|f(t)-Q(t)|\leqslant\left(\frac{1}{2}C_\delta+\frac{1}{4}C_\delta^3+\frac{3}{16}C_\delta^4\right)\omega_1(\delta)\delta+\frac{3}{8}K_\Delta^2LC_\delta^4\omega_1(\delta), \tag{1.10}$$

or, when we keep K_Δ bounded

$$|f(t)-Q(t)|=O(\omega_1(\delta)). \tag{1.10$'$}$$

If we take $A_0=0$, it may be shown that the above estimate remains valid. Moreover, if we take

$$A_0=S_0''(t_1)+\sum_{k=1}^{[\frac{N}{2}]}[S_k''(t_k)-S_k''(t_{k+1})],$$

then the last term in the right-hand member of (1.10) may be multiplied by a factor $\frac{1}{2}$ when N is even or $\frac{1}{2}+\frac{1}{2N}$ when N is odd.

We can get a better estimation if we take $A_0=\alpha$ as the "pseudo-centre" of the set of points

$$z_0=0,\ z_j=\sum_{k=1}^{j} J_k\qquad (j=1,\cdots,N-1),$$

i. e.,

$$\alpha=\min_{z}\max_{0\leqslant j\leqslant N-1}|z-z_j|,$$

whose unique existence may be easily proved. In case K_Δ is bounded, (1.10)$'$ is valid, provided

$$A_0=O\left(\frac{\omega_1(\delta)}{\delta^2}\right).$$

Now we assume $f(t)\in C^2$. If we take $A_0=0$, then

$$|A_j|\leqslant 6jC_\delta^3\omega_2(\delta);$$

and hence, by the result in [1], we may get

$$|f(t)-Q(t)|\leqslant\left(\frac{1}{8}+\frac{1}{16}C_\delta+\frac{1}{4}C_\delta^3\right)\omega_2(\delta)\delta^2+\frac{3}{16}K_\Delta LC_\delta^5\omega_2(\delta)\delta,$$

which certainly remains true if $A_0=\alpha$. In general, if we take

$$A_0=O\left(\frac{\omega_2(\delta)}{\delta}\right),$$

then we have

$$|f(t)-Q(t)|=O(\omega_2(\delta)\delta).$$

In an analogous way, we may obtain

$$|f'(t)-Q'(t)|\leqslant\left(\frac{1}{2}+\frac{3}{8}C_\delta+C_\delta^3\right)\omega_2(\delta)\delta+\frac{3}{2}K_\Delta LC_\delta^5\omega_2(\delta).$$

Similarly, if A_0 satisfies the above equality, then

$$|f'(t)-Q'(t)|=O(\omega_2(\delta)),$$

provided K_Δ is bounded.

For $f(t)\in C^3$, we may obtain in a similar way if we take $A_0=0$ or α:

$$|f(t)-Q(t)|\leqslant\left(\frac{1}{24}+\frac{1}{16}C_\delta+\frac{1}{12}C_\delta^3\right)\omega_3(\delta)\delta^3+\left(\frac{13}{64}C_\delta^3+\frac{1}{16}C_\delta^5\right)K_\Delta L\omega_3(\delta)\delta^2,$$

$$|f'(t)-Q'(t)|\leqslant\left(\frac{1}{4}+\frac{3}{8}C_\delta+\frac{1}{3}C_\delta^3\right)\omega_3(\delta)\delta^2+\left(\frac{13}{8}C_\delta^3+\frac{1}{2}C_\delta^5\right)K_\Delta L\omega_3(\delta)\delta,$$

$$|f''(t)-Q''(t)|\leqslant\left(\frac{13}{4}C_\delta+C_\delta^3\right)\omega_3(\delta)\delta+\left(\frac{39}{4}C_\delta^3+3C_\delta^5\right)K_\Delta L\omega_3(\delta).$$

Moreover, in case K_Δ is bounded, we have

$$|f^{(p)}(t)-Q^{(p)}(t)|=O(\omega_3(\delta)\delta^{2-p}),\quad p=0,\ 1,\ 2,$$

provided

$$A_0=O(\omega_3(\delta)).$$

Thus, we have

Theorem 1. *Assume $f(t)\in C^r(1\leqslant r\leqslant 3)$ and K_Δ is bounded, then*

$$|f^{(p)}(t)-Q^{(p)}(t)|=O^r(\omega_r(\delta)\delta^{r-p-1}),\quad p=0,\ \cdots,\ r-1, \tag{1.11}$$

provided

$$A_0=O(\omega_r(\delta)/\delta^{3-r}). \tag{1.12}$$

Corollary. *Assume $f^{(r)}(t)\in H^\mu(0<\mu\leqslant 1)$ and K_Δ is bounded, then*

$$|f^{(p)}(t)-Q^{(p)}(t)|=O(\delta^{\mu+r-p-1}),\quad p=0,\ \cdots,\ r-1, \tag{1.13}$$

provided

$$A_0=O(\delta^{\mu+r-3}). \tag{1.14}$$

§ 2. Quartic interpolating splines with deficiency 3

The splines discussed in § 1 are not available for closed L. In order to get a kind of quartic interpolating splines available both for closed and open L, we may require $Q(t)\in C^1$ only and add a new node t_j^* on each L_j, for example, the mid-point of L_j. That is, we require additionally

$$Q(t_j^*)=f(t_j^*)(=y_j^*),\quad j=0,1,\cdots,N-1.$$

Then, we should take

$$A_j=[y_j^*-S_j(t_j^*)]/R_j(t_j^*). \tag{2.1}$$

If $f(t)\in C^1$, by using the results in [1], we may get

$$\left.\begin{aligned}|f(t)-Q(t)|&\leqslant(1+C_\delta^4)\Big(\frac{1}{2}C_\delta+\frac{1}{4}C_\delta^3\Big)\omega_1(\delta)\delta,\\ |f'(t)-Q'(t)|&\leqslant\Big(C_\delta+\frac{3}{2}C_\delta^3+4C_\delta^5+2C_\delta^7\Big)\omega_1(\delta).\end{aligned}\right\} \tag{2.2}$$

If $f(t)\in C^2$, similarly we may get

$$\left.\begin{aligned}|f(t)-Q(t)|&\leqslant(1+C_\delta^4)\Big(\frac{1}{8}+\frac{1}{16}C_\delta+\frac{1}{4}C_\delta^3\Big)\omega_2(\delta)\delta^2,\\ |f'(t)-Q'(t)|&\leqslant\Big(\frac{1}{2}+\frac{3}{8}C_\delta+C_\delta^3+C_\delta^4+\frac{1}{2}C_\delta^5+2C_\delta^7\Big)\omega_2(\delta)\delta,\\ |f''(t)-Q''(t)|&\leqslant 3C_\delta^3\Big(1+C_\delta+\frac{1}{2}C_\delta^2+2C_\delta^4\Big)\omega_2(\delta);\end{aligned}\right\} \tag{2.3}$$

if $f(t)\in C^3$, we may get

$$\left.\begin{aligned}|f(t)-Q(t)|&\leqslant(1+C_\delta^4)\Big(\frac{1}{24}+\frac{1}{16}C_\delta+\frac{1}{12}C_\delta^3\Big)\omega_3(\delta)\delta^3,\\ |f'(t)-Q'(t)|&\leqslant\Big(\frac{1}{4}+\frac{3}{8}C_\delta+\frac{1}{3}C_\delta^3+\frac{1}{3}C_\delta^4+\frac{1}{2}C_\delta^5+\frac{2}{3}C_\delta^7\Big)\omega_3(\delta)\delta^2,\\ |f''(t)-Q''(t)|&\leqslant\Big(\frac{13}{4}C_\delta+C_\delta^3+C_\delta^4+\frac{3}{2}C_\delta^5+2C_\delta^7\Big)\omega_3(\delta)\delta,\\ |f'''(t)-Q'''(t)|&\leqslant(3C_\delta+2C_\delta^3+4C_\delta^4+6C_\delta^5+8C_\delta^7)\omega_3(\delta).\end{aligned}\right\} \tag{2.4}$$

Thus, we obtain

Theorem 2. *If $f(t)\in C^r$ $(r=1,2,3)$, for the above quartic spline $Q(t)$ with Deficiency 3, we have*

$$|f^{(p)}(t)-Q^{(p)}(t)|=O(\omega_r(\delta)\delta^{r-p}),\quad 0\leqslant p\leqslant r, \tag{2.5}$$

whether L is closed or not; in particular, if $f^{(r)}(t)\in H^\mu$ $(0<\mu\leqslant 1)$, we have

$$|f^{(p)}(t)-Q^{(p)}(t)|=O(\delta^{\mu+r-p}),\quad 0\leqslant p\leqslant r. \tag{2.6}$$

Remark 1. Theorem 2 remains true if we take t_j^* so that $\widehat{t_j^*t_{j+1}}/\widehat{t_jt_j^*}$ and its reciprocal are bounded.

Remark 2. K_Δ does not appear in all the estimates in this section, so there is no restriction on the partition Δ for the validity of (2.5).

§ 3. Application to the approximation of Cauchy principal integrals

Let L be a smooth curve (or an arcwise smooth curve without cusps). Denote

$$(T_L\varphi)(t) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau, \quad t\in L,$$

where $\varphi(t)\in H^\mu$ $(0<\mu<1)$ on L. It is well-known[5] that

$$\|T_L\varphi\|_\infty \leqslant B_\mu\{\|\varphi\|_\infty + M_\mu(\varphi)\}, \tag{3.1}$$

where

$$M_\mu(\varphi) = \sup_{t,t'\in L}\frac{|\varphi(t)-\varphi(t')|}{|t-t'|^\mu}$$

and B_μ is a constant depending on μ (and L). In case $L=\widehat{ab}$ is an open arc, (3.1) remains valid provided

$$\varphi(a) = \varphi(b) = 0^{[2]}.$$

Let $Q(t)$ be the quartic interpolating spline with Deficiency 2 introduced in § 1 (L is assumed to be open in this case). Let $f(t)\in C^1$. Denote

$$Q(t) - S(t) = R(t) = A_j R_j(t), \quad t\in L_j.$$

Note that

$$\|T_L f - T_L Q\|_\infty \leqslant \|T_L f - T_L S\|_\infty + \|T_L R\|_\infty.$$

For arbitrarily small $\varepsilon>0$, we have

$$\|T_L R\|_\infty \leqslant B_\varepsilon\{\|R\|_\infty + M_\varepsilon(R)\},$$

since $R(a)=R(b)=0$ and thereby (3.1) is applicable to $R(t)$, where B_ε is a constant depending on ε. If A_0 is chosen as in § 1, then by (1.9) we have

$$\|R\|_\infty \leqslant BK_\Delta^2\omega_1(\delta),$$

where B is a constant (in the sequel, B or B_ε may have different values in different cases). On the other hand, using the same method as in [2], we may obtain

$$|R(t)-R(t')| \leqslant BK_\Delta^2\frac{\omega_1(\delta)}{\delta}|t'-t| \leqslant BK_\Delta^2\frac{\omega_1(\delta)}{\delta^\varepsilon}|t'-t|^\varepsilon.$$

Thereby

$$M_\varepsilon(R) \leqslant BK_\Delta^2\omega_1(\delta)\delta^{-\varepsilon},$$

and so

$$\|T_L R\|_\infty \leqslant B_\varepsilon K_\Delta^2\omega_1(\delta)\delta^{-\varepsilon}.$$

By the results in [2], we may get

$$\|T_L(f-Q)\|_\infty \leqslant B_\varepsilon K_\Delta^2\omega_1(\delta)\delta^{-\varepsilon}. \tag{3.2}$$

Particularly, if $f'(t)\in H^\mu$, then

$$\|T_L(f-Q)\|_\infty \leqslant B_\varepsilon K_\Delta^2\delta^{\mu-\varepsilon}. \tag{3.2}$$

If $f(t)\in C^2$, in a similar way, we may get

$$\|T_L(f^{(p)}-Q^{(p)})\|_\infty \leqslant B_\varepsilon K_\Delta\omega_2(\delta)\delta^{1-p-\varepsilon}, \quad p=0,\ 1. \tag{3.3}$$

If $f(t)\in C^3$, we may analogously get

$$\|T_L(f^{(p)}-Q^{(p)})\|_\infty \leqslant B_\varepsilon K_\Delta\omega_3(\delta)\delta^{2-p-\varepsilon}, \quad p=0,\ 1. \tag{3.4}$$

We may not estimate $\|T_L(f''-Q'')\|_\infty$ even if f'' or $f'''\in H^\mu$, since $R''(a)$ and $R''(b)$ in general do not vanish simultaneously.

Now let $Q(t)$ be the spline introduced in § 2. Here L may be closed or open. It is easy to prove that

$$\|R\|_\infty\leqslant B\omega_1(\delta)\delta,$$
$$M_\varepsilon(R)\leqslant B\omega_1(\varrho)\delta^{1-\varepsilon}.$$

Using the results in [2], we get

$$\|T_L(f-Q)\|_\infty\leqslant B_\varepsilon\omega_1(\delta)\delta^{1-\varepsilon}. \tag{3.5}$$

If $f'(t)\in H^\mu(0<\mu<1)$, we may easily prove

$$\|R'\|_\infty\leqslant B\delta^\mu,$$
$$M_\varepsilon(R')\leqslant B\delta^{\mu-\varepsilon},$$

and, by [2]

$$\|T_L(f'-S')\|_\infty\leqslant B_\varepsilon K_\Delta^\varepsilon\delta^{\mu-\varepsilon}.$$

Hence

$$\|T_L(f'-Q')\|_\infty\leqslant B_\varepsilon K_\Delta^\varepsilon\delta^{\mu-\varepsilon}. \tag{3.6}$$

It is obvious that (3.6) is also valid for $\mu=1$.

If $f(t)\in C^r$ ($r=2$ or 3), in a similar way we may get

$$\|T_L(f^{(p)}-Q^{(p)})\|_\infty\leqslant B_\varepsilon\omega_r(\delta)\delta^{r-p-\varepsilon},\quad p=0,\ 1. \tag{3.7}$$

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