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ON THE PROBLEM OF NECESSARY CONDITIONS ENSURING UNIFORM CONVERGENCE OF KERNEL DENSITY ESTIMATES

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Abstract

Let $X_1, \cdots, X_n$ be a sequence of p-dimensional iid. random vectors with a common distribution $F(x)$. Denote the kernel estimate of the probability density of F(if it exists) by

$$\hat{f}_n(x)=n^{-1}h_n^{-p}\sum_{i=1}^{u}K\left(\frac{x-X_i}{h_n}\right).$$

Suppose that there exists a measurable function $g(x)$ and $h_n>0$, $h_n\to 0$ such that

$$\lim_{n\to\infty}\sup|\hat{f}_n(x)-g(x)|=0 \quad \text{a. s.}$$

Does $F(x)$ have a uniformly continuous density function $f(x)$ and $f(x)=g(x)$? This paper deals with the problem and gives a sufficient and necessary condition for general p-dimensional case.

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$$\hat{f}_n(x)=n^{-1}h_n^{-p}\sum_{i=1}^{n}K\left(\frac{x-X_i}{h_n}\right). \tag{1}$$

Suppose that there exists a measurable function $g(x)$ and $h_n>0$, $h_n\to 0$ shch that

$$\lim_{n\to\infty}\sup_{x}|\hat{f}_n(x)-g(x)|=0 \quad \text{a. s.} \tag{2}$$

Does $F(x)$ have a uniformly continuous density function $f(x)$ and $f(x)=g(x)$? Schuster[1] is the first to consider this problem in case $p=1$, he proved that under a set of complicated conditions, the answer is positive. Chen[2] obtained the same conclusion under simplified conditions that $K(x)$ is a probability density with bounded variation on $(-\infty, \infty)$ and $nh_n^2\to\infty$. He pointed out that these conditions may not be necessary and gave some examples. In particular, he conjectured that the assumption of bounded variation can be ejected. In the present paper Chen's conjecture is proved and a sufficient and necessary condition for general p-dimensional case is given. Specifically, we have the following two theorems:

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Theorem 1. *Suppose that* (i) $K(x)$ *is a bounded and integrable function on* R^p, $\int_{R^p} K(x)dx=1$; (ii) $h_n>0$, $h_n\to 0$, $nh_n^p\to\infty$. *Then whenever* (2) *holds*, $F(x)$ *hasanuniformly continuous density function* $f(x)$ *and* $f(x)=g(x)$.

Theorem 2. *Let* $K(x)$ *be a non-negative and integrable function on* R^p. *Then the necessary and sufficient condition for the following assertion being true:*

"Whenever there is a function $g(x)$, and $h_n>0$, $h_n\to 0$, such that (2) holds, then $F(x)$ must possess a density which is exactly $g(x)$ and $g(x)$ is uniformly continuous on R^p *is that* $K(x)$ *is bounden*, $\int_{R^p} K(x)dx=1$ *and* $nh_n^p\to\infty$.

In order to prove these theorems, we give some lemmas. Our proof of Theorem 1 is different from that of Schuster and Chen, mainly in that we avoid to prove $\sup_x|\hat{f}_n(x)-E\hat{f}_n(x)|\to 0$, a. s. directly. For simplicity, we sometimes write h for h_n, $\hat{f}(x)$ for $\hat{f}_n(x)$ and put $L=\sup_x|K(x)|$, $G=\sup_x|g(x)|$ and $M=\int_{R^p}|K(x)|dx$.

Lemma 1. *Let* $K(x)$ *be an integrable fvnction. Then for any bounded open interval* (a, b) *on* R^p, *we have*

$$D_n(a, b)\triangleq\int_{(a,b)}\hat{f}_n(x)dx-\int_{(a,b)}E\hat{f}_n(x)dx\to 0, \qquad \text{a. s.} \tag{3}$$

Proof From (1) it follows that

$$\int_{(a,b)}\hat{f}_n(x)dx=n^{-1}\sum_{i=1}^{n}\int_{\left(\frac{a-X_i}{h},\ \frac{b-X_i}{h}\right)}K(x)dx\triangleq n^{-1}\sum_{i=1}^{n}Y_{i,h} \tag{4}$$

Since $K(x)$ is integrable, we have

$$|Y_{i,h}|\leqslant\int_{R^p}|K(x)|dx=M<\infty$$

Notice that $Y_{i,h}(i=1, 2, \cdots, n)$ are iid. by direct calculation, we obtain

$$ED_n^4(a, b)\leqslant 3M^4/n^2 \tag{6}$$

Hence $\sum_{n=1}^{\infty} D_n^4(a, b)<\infty$. Finally, by Markov's inequality and Borel-cantelli lemma, we conclude that the lemma is true.

Lemma 2. *If* $K(x)$ *satisfies the conditions of Theorem* 1, *and* (2) *holds, then* $F(x)$ *has a density function* $f(x)$ *and* $f(x)=g(x)$.

Proof Because $K(x)$ is bounded and for any fixed $n, X_1, \cdots, X_n$, $\hat{f}_n(x)$ is bounded function of x, we see tnat $g(x)$ is bounded. By (2) we have

$$\int_{(a,b)}\hat{f}_n(x)dx\longrightarrow\int_{(a,b)}g(x)dx \qquad \text{a. s.} \tag{7}$$

for any finite interval (a, b).

Now from Lemma 1 and formula (7), it follows that

$$\int_{(a,b)}E\hat{f}_n(x)dx\longrightarrow\int_{(a,b)}g(x)dx \tag{8}$$

On the other hand, according to Fubini's theorem, we obtain

$$\int_{(a,b)} E\hat{f}_n(x)dx = h^{-p}\int_{(a,b)} dx \int_{R^p} K\left(\frac{x-y}{h}\right)dF(y)$$
$$= \int_{R^p} dF(y) \int_{\left(\frac{a-y}{h},\, \frac{b-y}{h}\right)} K(z)dz \tag{9}$$

From $\int_{R^p} \int K(z)dz = 1$, it is evident that

$$\int_{\left(\frac{a-y}{h},\, \frac{b-y}{h}\right)} K(z)dz \longrightarrow \begin{cases} 1 & \text{for } y\in(a,\, b) \\ 0 & \text{for } y\notin[a,\, b] \\ Q(y) & \text{for } y\in[a,\, b]-(a,\, b) \end{cases} \tag{10}$$

where $|Q(y)| \leqslant \int_{R^p} |K(z)|dz < \infty$.

Now choose $a<b$ such that a, b are continuity points of, F. From (9), (10). and $F(\{a\}) = F(\{b\}) = 0$. We have

$$\lim_{n\to\infty} \int_{(a,b)} E\hat{f}_n(x)dx \longrightarrow \int_{(a,b)} dF(x). \tag{11}$$

From (8) and (11)

$$\int_{(a,b)} dF(x) = \int_{(a,b)} g(x)dx. \tag{12}$$

Since (12) holds for any continuity points a, b of F, a routine argnment shows that it holds for any $a<b$. Hence g is the density of F

Lemma 3. *Suppose that $W(x)$ and $Q(x)$ are bounded and integrable on R^p, then*

$$R(x) \triangleq \int_{R^p} W(x-z)Q(z)dz$$

is uniformly continuous.

Proof Denote $|x| = \max_{1\leqslant i\leqslant p} |x_i|$, where $x=(x_1, \cdots, x_p)^\tau$. Since $Q(x)$ is integrable, we can find a sufficiently large A such that

$$\int_{|z|>A} |Q(z)|dz \leqslant \varepsilon/4 \sup_x |W(x)|.$$

According to Luzin's theorem, for any fixed x, there exists a continuous function $\widetilde{W}(y)$ on the interval $\{y\colon |y|\leqslant |x|+1+A\}$, snch that $|\widetilde{W}(y)| \leqslant \sup_y |W(y)|$ and $L(y\colon \widetilde{W}(y) \neq W(y),\ |y| \leqslant |x|+1+A) \leqslant (\sup_x |W(x)| \sup_x |Q(x)|)^{-1}\varepsilon/4$. So we see that for $|\Delta x|<1$

$$|R(x) - R(x+\Delta x)| \leqslant 2\sup_x |W(x)| \int_{|z|>A} |Q(z)|dz$$
$$+ \left|\int_{|z|\leqslant A} (W(x-z) - W(x+\Delta x-z)Q(z)dz\right|$$
$$\leqslant \varepsilon/2 + \left|\int_{|z|\leqslant A} (\widetilde{W}(x-z) - \widetilde{W}(x+\Delta x-z))Q(z)dz\right|$$
$$+ \left|\int_{|z|\leqslant A} (\widetilde{W}(x-z) - W(x-z))Q(z)dz\right|$$

$$+\left|\int_{|z|\leqslant A}(\overline{W}(x+\Delta x-z)-W(x+\Delta x-z))Q(z)dz\right|$$

$$\leqslant \varepsilon+\left|\int_{|z|\leqslant A}(\overline{W}(x-z)-\overline{W}(x+\Delta x-z))Q(z)dz\right|. \tag{14}$$

Letting $|\Delta x|\to 0$ and then $\varepsilon\to 0$, we obtain

$$\lim_{\Delta x\to 0} R(x+\Delta x)=R(x) \tag{15}$$

for all $x\in R^p$.

On the Other hand

$$|R(x)|\leqslant\sup_x|W(x)|\int_{|z|>A}|Q(z)|dz+\sup_z|Q(z)|\int_{|z|\leqslant A}|W(x-z)|dz$$

$$\leqslant\varepsilon/4+\sup_x|Q(z)|\int_{(x-\mathbf{1}A,x+\mathbf{1}A)}|W(z)|dz \tag{16}$$

Letting $|x|\to\infty$ and then $\varepsilon\to 0$, also using (16) we obtain $\lim_{|x|\to\infty}R(x)=0$. where $\mathbf{1}=(1,\cdots,1)^\tau$. This completes the proof of Lemma 3.

Lemma 4. *If $g(x)$ is a probability density function and $g(x)$ is uniformly continuous in R^p, then $g(x)$ is bounded and* $\lim_{|x|\to\infty}g(x)=0$

The proof is obvions and therefore omitted

Proof of theorem 1.

From Lemma 2 we know that $g(x)$ is a density function of $F(x)$, we have only to prove that $g(x)$ is uniformly continuous. For any sequence $\{x_n\}$, we have

$$\operatorname{Var}\hat{f}_n(x_n)=n^{-1}\operatorname{Var}\left\{h^{-p}K\left(\frac{x_n-X_1}{h}\right)\right.$$

$$\leqslant n^{-1}E\left\{h^{-p}K\left(\frac{x_n-X_1}{h}\right)\right\}^2$$

$$\leqslant n^{-1}h^{-p}\sup_z|K(z)|\int_{R^p}|K(z)|g(x_n-hz)dz\leqslant GMLn^{-1}h_n^{-p}. \tag{17}$$

Because the final term in (17) is indepenednt of $\{x_n\}$, and $n^{-1}h_n^{-p}\to 0$, we can find a subsequence $\{n_k\}$ (independent of $\{x_n\}$), snch that

$$\lim_{k\to\infty}|\hat{f}_{n_k}(x_{n_k})-E\hat{f}_{n_k}(x_{n_k})|=0 \qquad \text{a. s.}$$

Then from (2), it is seen that

$$|\hat{f}_{n_k}(x_{n_k})-g(x_{n_k})|\to 0 \qquad \text{a. s. (as } k\to\infty).$$

Therefore

$$E\hat{f}_{n_k}(x_{n_k})-g(x_{n_k})\to 0$$

or

$$\sup_x|E\hat{f}_{n_k}(x)-g(x)|\to 0.$$

Because

$$E\hat{f}_n(x)=h^{-p}\int_{R^p}K\left(\frac{x-z}{h}\right)g(z)dz$$

from Lemma 3 we know that $E\hat{f}_n(x)$ is uniformly continuous in x for any n. Thus $g(x)$ is uniformly continuous. This end the proof of of Theorem 1.

Proof of Theorem 2 Since the sufficiency is an immediate consequence of Theorem 1, we have only to deal with the necessity part.

(i) $K(x)$ is bounded. For otherwise, from (2), it would follow that $g(x)$ is unbounded, which is in contradiction with Lemma 4.

(ii) $\int_{R^p}$ For if $K(x)dx=1$. $\int_{R^p} K(x)dx=c\neq 1$, then from the proof of Lemma 2. it wonld follow that

$$\int_{(a,b)} E\hat{f}_n(x)dx \to \int_{(a,b)} g(x)dx$$

for any finite interval. On the other hand, since $g(x)$ is a uniformly continuous density function of $F(x)$, we have $E\hat{f}_n(x)\to cg(x)$. Hence

$$\int_{(a,b)} E\hat{f}_n(x)dx \to c\int_{(a,b)} g(x)dx.$$

This would be a contradiction unless we have $c=1$.

(iii) $\lim\limits_{n\to\infty} nh_n^p=\infty$. Choose x_0 such that $g(x_0)>0$. Since $K(x)$ is integrable and $g(x)$ is uniformly continuous, we have for each $y>0$

$$P\left(n^{-1}h^{-p}K\left(\frac{x_0-X_1}{h}\right)\geqslant y\right)\leqslant (ny)^{-1}Eh^{-p}K\left(\frac{x_0-X_1}{h}\right)\to 0 \quad \text{as } n\to\infty. \tag{18}$$

This shows that the median μ_n of the random variable $n^{-1}h^{-p}K\left(\frac{x_0-X_1}{h}\right)$ is of the order $O(n^{-1})$.

If $nh_n^p \not\to \infty$, then there is a subsequence $\{n_k\}$ such that $\lim\limits_{k\to\infty} n_k h_{n_k}^p=c<\infty$.

According to the sufficient and necessary condition of the weak Law of large numbers (see Chapter 5 of [3]) noticing that (2) is satisfied. We have

$$\begin{aligned} W_n&=n\int_{R^p}[Q_n^2(y)/(n^2+Q_n^2(y))]dy \\ &=nh^p[h^{2p}Q_n^2(x_0-hz)/(n^2h^{2p}+h^{2p}Q_n(x_0-hz))]g(x_0-hz)dz\to 0 \end{aligned} \tag{19}$$

$$\hat{f}_n(x_0)-\left(n\mu_n+\int_{Q_n(y)<n\tau} Q_n(y)g(y)dy\right)\Rightarrow 0, \tag{20}$$

where

$$Q_n(y)=h^{-p}K\left(\frac{x_0-y}{h}\right)-n\mu_n, \quad \tau>0.$$

Notice that

$$h_{n_k}^p Q_{n_k}(x_0-h_{n_k}z)=K(z)-n_k\mu_{n_k}h_{n_k}^p\to K(z),$$

$$g(x_0-h_{n_k}z)\to g(x_0),$$

$$n_k h_{n_k}^p\to c,\ \mu_{n_k}=O(n_k^{-1}).$$

$K^2(z)$ is integrable and $g(z)$ is bounded. Applying the Lebesgue's bounded convergence theorem, we obtain that for $c>0$

$$W_{n_k}=n_k h_{n_k}^p\int_{R^p}[(K^2(z)-2n_k\mu_{n_k}h_{n_k}^p K(z)+n_k^2\mu_{n_k}^2 h_{n_k}^{2p})$$
$$\div(n_k^2 h_{n_k}^{2p}+Q_{n_k}^2(x_0-h_{n_k}z))]g(x_0-h_{n_k}z)dz$$
$$\to cg(x_0)\int_{R^p}[K^2(z)/(c^2+K^2(z))]dz \quad (\neq 0).$$

This is in contradiction with (19).

If $c=0$, we have

$$A_{n_k}\triangleq n_k\mu_{n_k}+\int_{|Q_{n_k}(y)|<\tau n_k}Q_{n_k}(y)g(y)dy$$
$$=\int_{|K(z)-n_k\mu_{n_k}h_{n_k}^p|<n_k h_{n_k}^p\tau}K(z)g(x_0-h_{n_k}z)dz$$
$$+n_k\mu_{n_k}h_{n_k}^p\int_{|K(z)-n_k\mu_{n_k}h_{n_k}^p|>n_k h_{n_k}^p\tau}g(x_0-h_{n_k}z)dz$$
$$\leqslant G\int_{|K(z)-n_k\mu_{n_k}h_{n_k}^p|<n_k h_{n_k}^p\tau}K(z)dz+n_k\mu_{n_k}\int_{K(z)>n_k h_{n_k}^p\tau}(n_k\tau)^{-1}K(z)g(x_0-h_{n_k}z)dz$$
$$\leqslant G\left[\int_{K(z)\leqslant n_k h_{n_k}^p(\tau+\mu_{n_k})}K(z)dz+\mu_{n_k}\tau^{-1}\right]\to 0 \text{ as } K\to\infty. \quad (21)$$

From (20), (21) we obtain $\hat{f}_{n_k}(x_0)\Rightarrow 0$. This contradicts (2) from which we have

$$\hat{f}_n(x_0)\to g(x_0)(\neq 0) \text{ a. s.}$$

This completes the proof of Theorem 2.

Note we can prove that the conclusion of Theorem 2 still holds under the conditions that $\exists\ c>0$, such that $K(x)\geqslant -c$ for all $x\in R^p$ and $K(x)$ is integrable on R^p.

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