Chin. Ann. of Math.
5B (3) 1984

THE PERTURBATION OF ALMOST PERIODIC SOLUTION OF ALMOST PERIODIC SYSTEM

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Abstract

By using the exponential dichotomy and the averaging method, a perturbation theory is established for the almost periodic solutions of an almost differential system.

Suppose that the almost periodic differential system

$$\frac{dx}{dt}=f(x,\ t)+\varepsilon^2 g(x,\ t,\ \varepsilon) \tag{1}$$

has an almost periodic solution $x=x_0(t,\ M)$ for $\varepsilon=0$, where $M=(m_1,\ \cdots,\ m_k)$ is the parameter vector. The author discusses the conditions under which (1) has an almost periodic solution $x=x(t,\ \varepsilon)$ such that

$$\lim_{\varepsilon\to 0} x(t,\ \varepsilon)=x_0(t,\ M)$$

holds uniformly. The results obtained are quite complete.

§1. Introduction

Until now there is no perturbation theory for the almost periodic solutions of an almost periodic differential system. In this note we shall give the perturbation theory by the exponential dichotomy and the averaging method. The results obtained are quite complete.

Let us consider the uniformly almost periodic differential system

$$\frac{dx}{dt}=f(x,\ t)+\varepsilon^2 g(x,\ t,\ \varepsilon), \tag{1}$$

where $x\in G$, $-\infty<t<\infty$, G is a domain in n-dimensional Euclidian space, the vector functions $f(x,\ t)$ and $g(x,\ t,\ \varepsilon)$ are continuous in all variables x, t and ε, almost periodic in t uniformly for x and ε, and in class C^3 for x on G. At the same time we assume that (1) has the almost periodic solution $x_0(t,\ M)$ for $\varepsilon=0$, where $M(m_1, m_2, \cdots, m_k)$ is the parameter vector. Furthermore, we assume that the variational system of $x_0\ (t,\ M)$ is

$$\frac{dx}{dt}=A(t)x, \quad A(t)=\left(\frac{\partial f_i(x_0(t,\ M),\ t)}{\partial x_j}\right) \tag{2}$$

which has only k zero characteristic exponents in the extensive sense [1], and

$$\frac{\partial x_0(t,\ M)}{\partial m_1},\ \frac{\partial x_0(t,\ M)}{\partial m_2},\ \cdots,\ \frac{\partial x_0(t,\ M)}{\partial m_k}$$

Manuscript received October 4, 1982. Revised April 27, 1983.

are the independent almost periodic solutions of (2); i, e. there is a fundamental matrix

$$X(t)=\left(\frac{\partial x_0(t, M)}{\partial m_1}, \cdots, \frac{\partial x_0(t, M)}{\partial m_k}, x_1^{(1)}(t), \cdots, x_{n_1}^{(1)}(t), x_1^{(2)}(t), \cdots, x_{n_2}^{(2)}(t)\right)$$

of (2) with upper characteristic exponents[1] $\bar{\lambda}(x_j^{(1)}(t))<-\alpha<0$, and lower characteristic exponents $\underline{\lambda}(x_s^{(2)}(t))>\alpha>0$, $j=1, 2, \cdots, n_1$; $s=1, 2, \cdots, n_2$.

Now take the transformation $y=x-x_0(t, M)$ for(1). Then we can transform (1) into the following form

$$\frac{dy}{dt}=A(t)y+F_0(y, t)+\varepsilon^2G_0(y, t, \varepsilon), \tag{3}$$

where $F_0(y, t)=O(\|y\|^2)$, as $\|y\|$ tends to zero. Put $y=\varepsilon z$, we can reduce (3) into

$$\frac{dz}{dt}=A(t)z+\varepsilon G(z, t, \varepsilon), \tag{4}$$

where $G(z, t, 0)\not\equiv 0$. In this note we shall detail the existence of the almost periodic solution of(4).

§ 2. The structure of the variational equation of $x_0(t, M)$

In this paragraph, we give some properties of almost periodic solutions of the almost periodic linear system as follows:

Property 1. *If $x(t)$ is the nontrivial almost periodic solution of (2), then there is a positive constant α_0 such that $\|x(t)\|\geqslant\alpha_0>0$.*

Proof If $x(t_n)\to 0$, then we may suppose that $A(t+t_n)$ and $x(t+t_n)$ converge uniformly to $B(t)$ and $y(t)$ respectively, and $y(t)$ is the almost periodic solution of the almost periodic linear system

$$\frac{dy}{dt}=B(t)y.$$

$y(0)=\lim\limits_{n\to\infty} x(t_n)=0$, so $y(t)\equiv 0$. On the other hand $x(t)=\lim\limits_{n\to\infty} y(t-t_n)\equiv 0$, which contradicts the assumption that $x(t)$ is not the trivial solution of (2). This proves Property 1 completely.

Property 2. *Suppose that $x(t)$ satisfies the condition in Property 1. Then there is an elementary matrix $p(t)$ which transforms $x^*(t)$ into*

$$y^*(t)=(y_1(t), \cdots, y_n(t))=(x_1(t), \cdots, x_n(t))p(t)=x^*(t)p(t) \tag{\#}$$

with $y_1(t)\geqslant\varepsilon>0$, where $p(t)$ is the almost periodic matrix with bounded inverse.

Proof Suppose that $\|x(t)\|\geqslant\alpha_0>0$, $\alpha_0=4n\varepsilon$. By the almost periodicity of $x(t)$ there is a positive number $L(\varepsilon)$, any interval of length $L(\varepsilon)$ on the real axis containing a point t_0 such that

$$\|x(t+t_0)-x(t)\| \leqslant \varepsilon, \text{ for all } t.$$

Here we only prove that there is an almost periodic matrix $p(t)$ in (#) such that $y_1(t) \geqslant 2\varepsilon$ for $0 \leqslant t \leqslant L(\varepsilon)$. If this conclusion does not hold, then there is a sequence of matrices $p_m(t)$, $(m=1, 2, \cdots)$ which are almost periodic with bounded inverses and transform $x^*(t)$ into

$$(y_1^{(m)}(t), \cdots, y_n^{(m)}(t)) = (x_1(t), \cdots, x_n^{(m)}(t))p_m(t), \qquad (\#)_m$$

$y_1^{(m)}(t) \geqslant 2\varepsilon$ in the maximum interval $[0, L_m]$, $L_m \to L_0$, $L(\varepsilon) \geqslant L_0$, $L_m < L_0$.

Let us take integer m_0 sufficiently large, and transform $y^{(m_0)*}(t)$ by adding the function $\sum_{j=2}^{n} |y_j^{(m_0)}(t)|^2$ to $y_1^{(m_0)}(t)$, i. e., there is an almost periodic matrix $p_m(t)$ with bounded inverse which transforms $x^*(t)$ into $y^{(m')}(t)$ with

$$y_1^{(m')}(t) = y_1^{(m_0)}(t) + \sum_{j=2}^{n} |y_j^{(m_0)}(t)| \geqslant 2\varepsilon, \ 0 \leqslant t \leqslant L_{m'}, \ L_{m'} > L_0.$$

This gives a contradiction to $(\#)_m$, so we can assume that $y_1(t) \geqslant 2\varepsilon$ in (#) when $0 \leqslant t \leqslant L(\varepsilon)$. For any t there is a t_0 such that

$$\|y_1(t+t_0) - y_1(t)\| \leqslant \varepsilon \text{ and } 0 \leqslant t+t_0 \leqslant L(\varepsilon),$$

i. e., $y_1(t) \geqslant \varepsilon$ for all t.

Crollary. *If $x_1(t), \cdots, x_k(t)$ are the independent almost periodic solutions of (2), there are almost periodic and elementary matrices $Q(t)$ and $p(t)$ which transform $x_1(t), \cdots, x_k(t)$ into*

$$Y_k^*(t) = Q(t) X_k^*(t) p(t),$$

where $Q(t)$ is $n \times n$ matrix and $p(t)$ is $k \times k$ matrix, $x_k(t) = (x_1(t), \cdots, x_k(t))$, the upper $k \times k$ matrix of $Y_k^*(t)$ has bounded inverse.

Proof It may be proved by induction on the rank k of $x_1(t), \cdots, x_k(t)$. We omit the details here.

By these properties we obtain immediately

Lemma 1. *Suppose that $x_1(t), \cdots, x_k(t)$ are the independent almost periodic solutions of (2). Then there are almost periodic vector functions $q_1(t), \cdots, q_{n-k}(t)$ such that*

(1) $q_i(t) \perp q_j(t)$, $\|q_i(t)\| = 1$, $i, j = 1, 2, \cdots, n-k$, $j \neq i$;

(2) $q_j(t) \perp x_s(t)$, $j = 1, \cdots, n-k$, $s = 1, 2, \cdots, k$;

(3) *The vector functions $q_1(t), \cdots, q_{n-k}(t)$ are in class C^1.*

Proof (omitted.)

Lemma 2. *There is an almost periodic linear transformation*

$$\begin{pmatrix} u \\ v \end{pmatrix} = p(t)x \qquad (5)$$

which reduces the system (2) into the diagonal blocks:

$$\begin{aligned} \frac{du}{dt} &= c_1(t)u, \quad (6)_1 \\ \frac{dv}{dt} &= c_2(t)v, \quad (6)_2 \end{aligned} \tag{6}$$

where the first linear system has the almost periodic fundamental matrix, the second one admits the exponential dichotomy.

Proof Take the almost periodic matrix

$$p_0(t) = \left(\frac{\partial x_0(t, M)}{\partial m_1}, \cdots, \frac{\partial x_0(t, M)}{\partial m_k}, q_1(t), \cdots, q_{n-k}(t) \right),$$

where the meaning of $q_j(t)$ and $x_s(t) = \frac{\partial x_0(t, M)}{\partial m_s}$ have been stated in Lemma 1. By Schmidt's method there is a unitary matrix

$$Q_0(t) = P_0(t) R_0(t),$$

where $R_0(t)$ is the upper triangular and almost periodic matrix.

Take the regular transformation $y = Q_0^{-1}(t)x$ for (2). Then we have

$$\begin{aligned} \frac{dy}{dt} &= \left((Q_0^{-1}(t)A(t)Q_0(t) - Q_0^{-1}(t)\, \frac{dQ_0(t)}{dt} \right) y = \left(\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} - R_0^{-1}(t) \frac{dR_0(t)}{dt} \right) y \\ &= \begin{pmatrix} c_1(t) & c_{12}(t) \\ 0 & c_2(t) \end{pmatrix} y. \end{aligned} \tag{7}$$

Put

$$\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = Q_0^{-1}(t) \left(\frac{\partial x_0(t, M)}{\partial m_1}, \cdots, \frac{\partial x_0(t, M)}{\partial m_k} \right).$$

Then we have

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = R_0^{-1}(t) = Q_0^{-1}(t) P_0(t) = \begin{pmatrix} U(t) & * \\ V(t) & * \end{pmatrix},$$

so that $V(t) \equiv 0$, and

$$\frac{dU(t)}{dt} = c_1(t) U(t),$$

which proves that the first linear system in (6) has the almost periodic fundamental matrix.

If we transform (7) by the linear transformation

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} \eta^2 E_k & 0 \\ 0 & \eta E_{n-k} \end{pmatrix} y,$$

where η is a sufficiently small positive number, then we have

$$\begin{aligned} \frac{du^*}{dt} &= c_1(t) u^* + o(\eta) v^*, \\ \frac{dv^*}{dt} &= c_2(t) v^*. \end{aligned} \tag{8}$$

Therefore we may consider system (7) or (8) as the small perturbation system of (6).

In order to prove that the second linear system in (6) admits the exponential dichotomy, we shall introduce some knowledge[4] about the spectral points of the linear differential system as follow

Suppose that the linear system

$$\frac{dv}{dt}=(c_2(t)-\lambda E)v,\ \lambda \text{ is the real number,}$$

does admit the exponential dichotomy. Then we call λ the spectral point of $(6)_2$. The set of spectral points of $(6)_2$ forms a finite numbers of closed intervals. By Theorem 3 in [2], it is easy to prove that the exponential dichotomy is preserved under the small perturbation, so that the set of spectral points is stable under the small perturbation.

If the second linear system of (6) does not admit the exponential dichotomy, then the original point of real axis is the spectral point of $(6)_2$. Here we may suppose that $(6)_2$ has the spectral interval $[\alpha_0, \beta_0]$ containing the original point O. The system (7) or (8) may be considered as the small perturbation system of (6), so that one of the following three cases must take place:

(I) $\alpha_0<0<\beta_0$. The number of zero characteristic exponent in the extensive sense of $(8)>k$;

(II) $\alpha_0<0\leqslant\beta_0$, $(\alpha_0=0<\beta_0)$. There is a solution $(u^*(t), v^*(t))$ of (8) with upper (lower) characteristic exponent

$\bar{\lambda}=\bar{\lambda}(u^*(t), v^*(t))$ $(\underline{\lambda}=\underline{\lambda}(u^*(t), v^*(t)))$ in the interval $-\alpha<\bar{\lambda}(\underline{\lambda})<\alpha$.

(III) $\alpha_0=\beta_0=0$. I) or II) takes place.

Any one of these facts contradicts the assumption of (2). Therefore we conclude that the second linear system in (6) admits the exponential dichotomy.

At last we transform (7) with the linear transformation

$$\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} E_k & S(t) \\ 0 & E_{n-k} \end{pmatrix}y,$$

so that

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} c_1(t) & s'(t)-c_1(t)S(t)+S(t)c_2(t)+c_{12}(t) \\ 0 & c_2(t) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

In the following paragraph we shall prove that there is an almost periodic matrix $S(t)$ fatisfying the equation

$$S'(t)=c_1(t)S(t)-S(t)c_2(t)-c_{12}(t). \tag{9}$$

Put $w=U^{-1}(t)S$, where $U^{-1}(t)$ is the almost periodic fundamental matrix of the first linear system in (6). Then there is a system

$$\frac{dw}{dt}=-wc_2(t)+c_0(t),\ c_0(t)=-U^{-1}(t)c_{12}(t). \tag{10}$$

We have proved that the second linear system in (6), i. e. the linear system

$$\frac{dw}{dt}=-wc_2(t) \tag{11}$$

admits the exponential dichotomy, so that it satisfies the Favard's property[3].

Therefore there is an almost periodic $k\times(n-k)$ matrix $w_0(t)$ satisfying equation (11), and the almost periodic matrix $S(t)=U(t)w_0(t)$ is the solution of equation (9). The proof of Lemma 2 is complete.

Corollary. *There is an almost periodic regular linear transformation which reduces* (4) *into the following form:*

$$\begin{aligned}\frac{du}{dt}&=c_1(t)u+\varepsilon F_1^*(u,\ v,\ t,\ \varepsilon),\\ \frac{dv}{dt}&=c_2(t)v+\varepsilon F_2^*(u,\ v,\ t,\ \varepsilon).\end{aligned}\tag{12}$$

In the following paragraph we shall only discuss the existence of the almost periodic solution of (12).

§ 3. perturbation problem

By the above statement there is a fundamental matrix $V(t)$ of the second linear system in $(6)_2$ such that

$$\begin{aligned}&V(t)=V_1(t)+V_2(t),\quad V^{-1}(s)=Z_1(s)+Z_2(s),\\ &V(t)V^{-1}(s)=V_1(t)Z_1(s)+V_2(t)Z_2(s),\\ &\|V_1(t)Z_1(s)\|\leqslant\beta\exp(-\alpha(t-s)),\quad t\geqslant s,\\ &\|V_2(t)Z_2(s)\|\leqslant\beta\exp(\alpha(t-s)),\quad s\geqslant t.\end{aligned}$$

Where α, β are the positive constants.

If $(u(t,\ \varepsilon),\ v(t,\ \varepsilon))$ is the almost periodic solution of (12) with

$$\lim_{\varepsilon\to0}u(t,\ \varepsilon)=u(t,\ 0),\ \lim_{\varepsilon\to0}v(t,\ \varepsilon)=0,\ u(0,\ 0)=x_0\tag{13}$$

uniformly, then we can express $(u(t,\ \varepsilon),\ v(t,\ \varepsilon))$ in the following form:

$$u(t,\ \varepsilon)=U(t)U^{-1}(o)(x_0+\beta(\varepsilon))+\varepsilon\int_0^t U(t)U^{-1}(s)F_1^*(u(s,\ \varepsilon),\ v(s,\ \varepsilon),\ s,\ \varepsilon)ds,$$

$$v(t,\ \varepsilon)=\left(\varepsilon\int_{-\infty}^t V_1(t)Z_1(s)-\varepsilon\int_t^\infty V_2(t)Z_2(s)\right)F_2^*(u(s,\ \varepsilon),\ v(s,\ \varepsilon),\ s,\ \varepsilon)ds.$$

Theorem 1. *If* (12) *has an almost periodic solution* $(u(t,\ \varepsilon),\ v(t,\ \varepsilon))$ *with property* (13), *then*

$$P(x_0)=\lim_{t\to\infty}\frac{1}{t}\int_0^t U(t)U^{-1}(s)F_1^*(u(s,\ 0),\ 0,\ s,\ 0)ds=0.$$

Proof $(u(t,\ \varepsilon),\ v(t,\ \varepsilon))$ converges uniformly to $(u(t,\ 0),\ v(t,\ 0))$, as $\varepsilon\to0$. For any $\eta>0$ there is an $\varepsilon(\eta)>0$ such that

$$\|F_1^*(u(t,\ \varepsilon),\ v(t,\ \varepsilon),\ t,\ \varepsilon)-F_1^*(u(t,\ 0),\ 0,\ t,\ 0\|<\frac{\eta}{2},\tag{14}$$

when $0\leqslant\varepsilon\leqslant\varepsilon(\eta)$. Since the matrix $U(t)$ is regular and almost periodic, there is a constant k_0 such that $\|U(t)U^{-1}(s)\|<k_0$.

By (14) we have

$$\frac{1}{t}\int_0^t \|U(t)U^{-1}(s)\| \|F_1^*(u(s, \varepsilon), v(s, \varepsilon), s, \varepsilon) - F_1^*(u(s, 0), 0, s, 0)\| ds < k_0\eta$$

and

$$P(x_0, \varepsilon) = \lim \frac{1}{t}\int_0^t U^{-1}(s) F_1^*(u(s, \varepsilon), v(s, \varepsilon), s, \varepsilon) ds = 0,$$

then we conclude that $P(x_0) = 0$.

Theorem 2. *If the vector x_0 satisfies the equation $P(x_0) = 0$, and the characteristic root of $\left(\frac{\partial P_i(x_0)}{\partial x_j}\right)$ has no real part equal to zero, then (12) has a unique almost periodic solution $(u(t, \varepsilon), v(t, \varepsilon))$ such that $\lim_{\varepsilon\to 0} u(t, \varepsilon) = u(t, x_0, 0)$, $\lim_{\varepsilon\to 0} v(t, \varepsilon) = 0$ tuniformly.*

The proof of this theorem will be given in § 5.

§ 4. Averaging method

In order to prove Theorem 2, we give three Lemmas as follows:

Lemma 3. *Suppose that $f(x, t)$ is an almost periodic function of t uniformly for x with the following properties:*

(i) $\frac{\partial f(x, t)}{\partial x_j}$, $j = 1, 2, \cdots, n$ *are the uniformly continuous vector functions of* (x, t);

(ii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t f(x, s) ds = 0$.

Then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\partial f(x, s)}{\partial x_j} ds = 0, \ j = 1, 2, \cdots, n$$

uniformly for x on the given domain.

Proof For the simplicity we only prove it for the case that x is the real variable. Take the real number sequence $\{h_j | h_j \neq 0, \lim h_j = 0\}$. put

$$\frac{f(x+h_j, t) - f(x, t)}{h_j} = \frac{\partial f(x, t)}{\partial x} + \varepsilon_j(x, t).$$

$\frac{\partial f(x, t)}{\partial x}$ is uniformly continuous for (x, t) on the given domain, so $\varepsilon_j(x, t)$ tends uniformly to zero. For any $\eta > 0$ there is a positive number $\xi(\eta)$ such that $\|\varepsilon_i(x, t)\| \leqslant \frac{\eta}{2}$, when $j \geqslant \xi(\eta)$.

Take T large enough, for j fixed, such that

$$\left| \frac{1}{t}\int_0^t \frac{f(x+h_j, s) - f(x, s)}{h_j} ds \right| < \frac{\eta}{2}, \text{ when } t \geqslant T.$$

Hence we have

$$\left| \frac{1}{t}\int_0^t \frac{f(x, s)}{x} ds \right| < \frac{\eta}{2} + \left| \frac{1}{t}\int_0^t \varepsilon_j(x, s) ds \right| < \varepsilon, \text{ when } t \geqslant T,$$

which proves Lemma 3 completely.

Lemma 4. *Suppose that $f(x, t)$ is the uniformly continuous vector function of t with*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x, s)ds=0$$

uniformly for x. Put

$$J(x, t, \varepsilon)=\int_t^\infty f(x, s)\exp(\varepsilon(t-s))ds, \quad \varepsilon>0.$$

Then we have

(i) $\dfrac{\partial J(x, t, \varepsilon)}{\partial t}=\varepsilon J(x, t, \varepsilon)-f(x, t)$;

(ii) $\lim\limits_{\varepsilon\to 0}\varepsilon J(x, t, \varepsilon)=0$ *uniformly for (x, t) on the given domain.*

Proof Part (i) can be obtained by calculation.

By the almost periodicity of $f(x, t)$, the limit

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t_0+t} f(x, s)ds=0$$

holds uniformly for t_0. Put

$$F(x, t, s)=\int_t^s f(x, r)dr.$$

For any $\eta>0$ there is a $\xi(\eta)>0$ such that

$$|F(x, t, s)|\leqslant\frac{1}{2}|s-t|\eta, \quad \text{when}\ |s-t|>\xi(\eta).$$

Notice that

$$J(x, t, \varepsilon)=\int_t^\infty f(x, s)\exp(\varepsilon t-s))ds$$

$$=\varepsilon\int_t^\infty F(x, t, s)\exp(\varepsilon(t-s))ds$$

$$=\int_t^{\xi(\eta)+t}+\int_{\xi(\eta)+t}^\infty=I_1+I_2.$$

Let us fix the number $\xi(\eta)$, then we have

$$|\varepsilon I_2|<\frac{1}{2}\eta\int_0^\infty \varepsilon^2 s\exp(-\varepsilon s)ds=\frac{1}{2}\eta.$$

Now choose ε_1 so small that $|\varepsilon I_1|<\frac{1}{2}\eta$, when $0\leqslant\varepsilon<\varepsilon_1(\eta)$, which implies that

$|\varepsilon J(x, t, \varepsilon)|<\eta$, when $0\leqslant\varepsilon\leqslant\varepsilon_1(\eta)$.

Lemma 5. *Suppose that $f(x, t, \varepsilon)$ satisfies the conditions in Lemma 3. Then there is a vector function $\omega(x, t, \varepsilon)$ having the following properties:*

1) *The continuous vector function $\omega(x, t, \varepsilon)$ is almost periodic and of C^1 for t, of C^∞ for x;*

2) *The following limits hold uniformly:*

$$\lim_{\varepsilon\to 0}\varepsilon\omega(x, t, \varepsilon)=0, \quad \lim_{\varepsilon\to 0}\varepsilon\frac{\partial\omega}{\partial x_i}=0, \quad j=1, 2, \cdots, n;$$

3) *Put* $S(x, t, \varepsilon) \equiv \frac{\partial \omega}{\partial t} + f(x, t, \varepsilon)$, *then we have*

$$\lim_{\varepsilon\to 0} S(x, t, \varepsilon) = 0, \quad \lim_{\varepsilon\to 0} \frac{\partial S(x, t, \varepsilon)}{\partial x_j} = 0, \quad j=1, 2, \cdots, n.$$

Proof Take the real function

$$K(x, \varepsilon) = \begin{cases} k(\varepsilon)\exp\left(\frac{-\varepsilon^2}{\varepsilon^2 - \|x\|^2}\right), & \|x\| < \varepsilon^2, \\ 0 & \|x\| \geq \varepsilon^2. \end{cases}$$

$$\int_{E_n} K(x, \varepsilon)\, dx = 1,$$

where $\|x\|^2 = x_1^2 + x_2^2 +, \cdots, + x_n^2$, E_n is n-dimensional Euclidian space, $dx = dx_1 dx_2 \cdots dx_n$, the volume element of E_n.

Let us define the vector functions

$$J(x, t, \varepsilon) = \int_t^\infty f(x, s, \varepsilon) \exp(\varepsilon(t-s))\, ds,$$

$$\omega(x, t, \varepsilon) = \int_{E_n} K(y, \varepsilon) J(x+Y, t, \varepsilon)\, dy = \int_{E_n} K(y-x, \varepsilon) J(y, t, \varepsilon)\, dy.$$

It is evident that

$$J(x, t, \varepsilon) = \int_0^\infty f(x, t+s, \varepsilon) \exp(-\varepsilon s)\, ds$$

is the almost periodic function of t uniformly for x and ε. The continuity of $\frac{\partial}{\partial t} J(x, t, \varepsilon)$ implies the continuity of $\frac{\partial}{\partial t} \omega(x, t, \varepsilon)$. By the definition of $\omega(x, t, \varepsilon)$, we see that $\omega(x, t, \varepsilon)$ is of C^∞ for x. This proves the part 1).

By Lemmas 3 and 4 the following limits hold uniformly:

$$\lim_{\varepsilon\to\infty} \varepsilon J(x, t, \varepsilon) = 0, \quad \lim_{\varepsilon\to 0} \varepsilon \frac{\partial J(x, t, \varepsilon)}{\partial x_i} = 0, \quad j=1, 2, \cdots, n.$$

From the definition of $\omega(x, t, \varepsilon)$ we get the proof of part 2).

Since

$$\begin{aligned} S(x, t, \varepsilon) &\equiv \frac{\partial \omega}{\partial t} + f(x, t, \varepsilon) \\ &= \int_{E_n} K(y, \varepsilon) \left(\frac{\partial}{\partial t} J(x+Y, t, \varepsilon) + f(x, t, \varepsilon)\right) dy \\ &= \int_E K(Y, \varepsilon)(f(x, t, \varepsilon) - f(x+y, t, \varepsilon))\, dy + \varepsilon\omega(x, t, \varepsilon), \end{aligned}$$

$\lim_{\varepsilon\to 0} S(x, t, \varepsilon) = 0.$

Similarly we have

$$\lim_{\varepsilon\to 0} \frac{\partial}{\partial x_j} S(x, t, \varepsilon) = 0, \quad j=1, 2, \cdots, n.$$

Lemma 5 is proved completely.

§ 5. The proof of Theorem 2

Let us transform (12) by the transformation

$$u=U(t)h, \quad v=v,$$

which reduces (12) into the following form:

$$\begin{aligned}\frac{dh}{dt}&=\varepsilon U^{-1}(t)F_1^*(u, v, t, \varepsilon),\\ \frac{dv}{dt}&=C_2(t)v+\varepsilon F_2^*(u, v, t, \varepsilon).\end{aligned} \tag{15}$$

Proving that (12) has the almost periodic solution $(u(t, \varepsilon), v(t, \varepsilon))\to(u(t, 0), 0)$, as $\varepsilon\to 0$ is equivalent to proving that (15) has the almost periodic solution $(h(t, \varepsilon), v(t, \varepsilon))\to(x_0, 0)$, as $\varepsilon\to 0$. Write

$$U^{-1}(t)F_1^*(u, v, t, \varepsilon)=U^{-1}(t)F_1^*(u(t, 0), 0, t, 0)+\Psi(h, v, t, \varepsilon).$$

The first partial derivative of $\Psi(h, v, t, \varepsilon)$ with respect to h and v are bounded, and $\Psi(h, 0, t, 0)=0$, so the Lipschitz constant of $\Psi(h, v, t, \varepsilon)$ with respect to h is small, when v and ε are small enough. Furthermore we can assume that the Lipschitz constant of $\Psi(h, v, t, \varepsilon)$ with respect to v is sufficiently small. Otherwise, we may apply the transformation $h\to h$, $V\to\frac{1}{\sigma}V$ to system (15), where σ is a small positive number.

Suppose that

$$P(h)=\lim_{t\to\infty}\frac{1}{t}\int_0^t U^{-1}(s)F_1^*(u(s, h, 0), 0, s, 0)ds, \quad u(s, h, 0)=U(s)U^{-1}(0)h.$$

Put

$$f(h, s)=U^{-1}(s)F_1^*(u(s, h, 0), 0, s, 0)-P(h).$$

By Lemma 5, corresponding to the function $f(h, s)$, we can construct the vector function $\omega(h, s, \varepsilon)$. Let us transform (15) by the transformation

$$h=h^*-\varepsilon\omega(h^*, t, \varepsilon), \quad v=v.$$

Then we have

$$\begin{aligned}\frac{dh^*}{dt}&=\varepsilon(p(h^*)+L_1(h^*, v, t, \varepsilon)),\\ \frac{dv}{dt}&=C_2(t)v+\varepsilon L_2(h^*, v, t, \varepsilon).\end{aligned} \tag{16}$$

Noting $P(x_0)=0$ and putting $N=h^*-x_0$, $v=v$, we reduce (16) into

$$\begin{aligned}\frac{dN}{dt}&=\varepsilon(A_0N+L_1^*(N, v, t, \varepsilon)),\\ \frac{dv}{dt}&=C_2(t)v+\varepsilon L_2^*(N, v, t, \varepsilon),\end{aligned} \tag{17}$$

where the constant matrix $A_0=\left(\frac{\partial P_i(x_0)}{\partial x_j}\right)$, the Lipschitz constant of $L_1^*(N, v, t, \varepsilon)$

with respect to N and v are small enough, and $\frac{\partial L^*}{\partial N}$ and $\frac{\partial L^*}{\partial V}$ are bounded. Then there is a unique bounded solution of (17) which can be expressed by the following form:

$$N(t, \varepsilon)=\varepsilon\int_{\infty}^{t} Y_1(\varepsilon(t-s))L_{11}^* ds-\varepsilon\int_{t}^{\infty} Y_2(\varepsilon(t-s))L_{12}^* ds,$$

$$v(t, \varepsilon)=\varepsilon\int_{-\infty}^{t} V_1(t)Z_1(s)L_{21}^* ds-\varepsilon\int_{t}^{\infty} V_2(t)Z_2(s)L_{22}^* ds,$$

where we may assume that

$$A_0=\operatorname{diag}(A_1, A_2),$$

$$Y_1(\varepsilon(t-s))=\exp(\varepsilon A_1(t-s)), \quad Y_2(\varepsilon(t-s))=\exp(\varepsilon A_2(t-s)),$$

the real parts of the characteristic roots of $A_1$are negative, the real parts of the characteristic roots of A_2 are positive.

The almost periodicity of $(N(t, \varepsilon), v(t, \varepsilon))$ has been proved by Theorem 1 in [2]. Therefore, (12) has the almost periodic solution with the property in (13). So the proof of Theorem 2 is complete.

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