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INTERPOLATION BY SPLINES ON FINITE AND INFINITE PLANAR SETS

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Abstract

Let q be a complex number, $0<|q|<\infty$. Γ denotes the planar curve $Z=q^x$, $-\infty<x<\infty$.

The author wants to find splines on Γ which interpolate on the set $\{q^v\}_{k_1}^{k_2}$, where k_1, k_2 may be finite integers, or $k_1=-\infty$, $k_2=+\infty$.

For the cases q real, $k_1=-\infty$, $k_2=+\infty$, or $|q|=1$, many authors have dealt with this problem (see [1—7]). But if q is an arbitray complex number and the number of interpolating points is finite or infinite, which classes of splines could be possible?

In the first part of this paper, the author introduces several classes of splines which interpolate on a finite point set. The second part deals with interpolation by splines on infinite sets of data points.

§ 1. Interpolation By Splines On A Finite Planar Set $\{q^v\}_0^N$

1.1. Pseudo-periodic Splines

Let $\Gamma_{v,\mu}$ denote the arc $z=q^x$, $v\leqslant x\leqslant\mu$ on Γ.

The symbol $\mathscr{S}_{A,n}=\{S_A(z)\}$ denotes the class of functions $\{S_A(z)\}$ satisfying the following three conditions

$$\left.\begin{array}{ll} S_A(z)\in\pi_n \text{ on each arc } \Gamma_j, & j=\overline{0,\ N-1} \\ S_A(z)\in C^{n-1}(\Gamma_{0,N}] & \\ S_A^{(i)}(q^N)=q^{N(n-i-1)}S_A^{(i)}(1) & i=\overline{0,\ n-1} \end{array}\right\} \quad (1)$$

where $\Gamma_j=\Gamma_{j,j+1}=\widehat{q^jq^{j+1}}\in\Gamma$, and Π_n is the class of polynomials of degree n over the field of complex numbers, $S_A^{(i)}=\dfrac{d^iS}{dz^i}$.

We call $S_A(z)$ the pseudo-periodic spline.

In case $|q|=1$ and $q^N=1$, the pseudo-periodic splines on Γ are periodic splines on the unit circle [3-7].

Problem A. The interpolation problem P_A may be described as follows:

A sequence of numbers; real or complex

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$$y=(y_\nu) \quad \nu=0,\ 1,\ \cdots,\ N-1 \quad (N\geqslant n+1), \qquad y_N=q^{N(n-1)}y_0 \tag{2}$$

are prescribed. We are to find a function $S_A(z)\in\mathscr{S}_{A,n}$ such that

$$S_A(q^\nu)=y_\nu \text{ for } \nu=0,\ 1,\ \cdots,\ N. \tag{3}$$

If $S_A(z)\in\mathscr{S}_{A,n}$, we may extend $S_A(z)$ to $\Gamma_{N,2N}$ as follows

$$S_A(z)=q^{N(n-1)}S_A(q^{-N}z) \quad z\in\Gamma_{N,2N}. \tag{4}$$

Then

$$S_A(q^{N+\nu})=y_{N+\nu} \quad \nu=0,\ 1,\ \cdots,\ N \tag{5}$$

where $y_{N+\nu}$ denotes the number $q^{N(n-1)}y_\nu$. Evidently $S_A(z)\in C^{n-1}(\Gamma_{0,2N})$.

For any complex number q, $q\neq0$, $|q|\neq1$, we may define the order of two points z_1 and z_2 on Γ as fallows

$$z_1\ll z_2 \text{ if } -\infty<x_1<x_2<+\infty,$$

where $z_i=q^{x_i}$, $i=1,\ 2$.

$$(z_1-z_2)_+^k=\begin{cases}(z_1-z_2)^k, & \text{if } z_1\gg z_2,\\ 0, & \text{otherwise},\end{cases}$$

$$(z_1-z_2)_-^k=\begin{cases}(z_1-z_2)^k, & \text{if } z_2\gg z_1,\\ 0, & \text{otherwise},\end{cases}$$

where k is any positive integer.

By Peano's formula we have

$$S_A(z)=\sum_{k=9}^{n-2}\frac{1}{k!}(z-1)_+^kS_A^{(k)}(1)+\frac{1}{(n-2)!}\int_1^{q^{2N}}S_A^{(n-1)}(t)(z-t)_+^{n-2}\,dt. \tag{6}$$

The integral is along the arc $\Gamma_{0,2N}=\widehat{1,\ q^{2N}}\in\Gamma$.

Applying the method of integration by parts to (6) we obtain, for z on $\Gamma_{0,2N}$

$$\left.\begin{aligned}&S_A(z)=P_{n-2}(z)+\frac{1}{n!}\sum_{j=1}^{2N}M_jH_j^{(n)}(z),\\&P_{n-2}(z)\in\pi_{n-2},\\&H_j^{(n)}(z)=\frac{(z-q^{j+1})_+^n-(z-q^j)_+^n}{b_{j+1}}-\frac{(z-q^j)_+^n-(z-q^{j-1})_+^n}{b_j},\end{aligned}\right\} \tag{7}$$

where $b_j=q^j-q^{j-1}$, $M_j=S_A^{(n-1)}(q^j)$.

$S_A(z)$ may also be written as

$$\left.\begin{aligned}&S_A(z)=P_{n-1}(z)+Y(z), \quad z\in\Gamma_{0,2N},\\&P_{n-1}(z)\in\pi_{n-1},\\&Y(z)=\frac{1}{n!}\left\{\frac{M_1-M_0}{b_1}(z-1)^n+\sum_{i=0}^{2N-1}\left(\frac{M_{i+1}-M_i}{b_{i+1}}-\frac{M_i-M_{i-1}}{b_i}\right)(z-q^i)_+^n\right\}.\end{aligned}\right\} \tag{8}$$

From (7)

$$\left.\begin{aligned}&S_A(q^\nu)=P_{n-2}(q^\nu)+\frac{1}{n!}\sum_{i=1}^{2N}H_j^{(n)}(q^\nu)M_j,\\&H_j^{(n)}(q^\nu)=\omega_{\nu,j}^{(n)}-\omega_{\nu,j+1}^{(n)},\\&\omega_{k,j}^{(n)}=\frac{(q^k-q^{j-1})_+^n-(q^k-q^j)_+^n}{b_j}.\end{aligned}\right\} \tag{9}$$

We form the divided difference $[y_\nu, \cdots, y_{\nu+n-1}]$ of order $n-1$. From (3), (5), (9) we obtain

$$[y_\nu, \cdots, y_{\nu,n-1}] = \frac{1}{n!} \sum_{y=\nu}^{\nu+n-1} M_j [H_j^{(n)}(q^\nu), \cdots, H_j^{(n)}(q^{\nu+n-1})]. \tag{10}$$

From (1) and (4)

$$M_{N+k} = M_k, \quad 0 \leqslant k \leqslant N. \tag{11}$$

From (9)

$$H_{j+l}^{(n)}(q^{k+l}) = q^{(n-1)l} H_j^{(n)}(q^k). \tag{12}$$

The determinant definition of divided difference gives

$$\begin{aligned} H_j^{(n)}[q^\nu, \cdots, q^{\nu+n-1}] &= [H_j^{(n)}(q^\nu), \cdots, H_j^{(n)}(q^{\nu+n-1})] \\ &= H_{j-\nu}^{(n)}[1, q, \cdots, q^{n-1}]. \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned} &[y_\nu, \cdots, y_{\nu+n-1}] = \frac{1}{n!} \sum_{l=0}^{n-1} W_l^{(n)} M_{\nu+l}, \quad \nu = \overline{0, N-1}, \\ &W_l^{(n)} = H_l^{(n)}[1, q, \cdots q^{n-1}]. \end{aligned} \tag{14}$$

The system of equations given above may be written as

$$(W)_n \begin{pmatrix} M_0 \\ \vdots \\ M_{N-1} \end{pmatrix} = n! \begin{pmatrix} [y_0, \cdots, y_{n-1}] \\ \vdots \\ [y_{N-1}, \cdots, y_{N+n-2}] \end{pmatrix}, \tag{15}$$

where $(W)_n$ is an $N \times N$ circulant matrix

$$(W)_n = \begin{pmatrix} W_0^{(n)} & W_1^{(n)} & \cdots & W_{n-1}^{(n)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & W_0^{(n)} & \cdots & & W_{n-1}^{(n)} & 0 & \cdot & \cdot & 0 \\ W_1^{(n)} & W_2^{(n)} & \cdots & W_{n-1}^{(n)} & 0 & \cdot & \cdot & 0 & W_0^{(n)} \end{pmatrix} \tag{16}$$

It is easy to prove that

$$W_{k+1}^{(n+1)} = \frac{q^n - q^k}{q^n - 1} W_k^{(n)} + \frac{q^{k+2} - 1}{q^n - 1} W_{k+1}^n \quad \text{for } |q| \neq 1. \tag{17}$$

Let $W_{-1}^{(n)} = W_n^{(n)} = 0$. From (7), we then have

$$W_0^{(1)} = W_0^{(2)} = W_1^{(2)} = 1, \; W_0^{(3)} = \frac{1}{q+1}, \quad W_1^{(3)} = 2, \; W_2^{(3)} = \frac{q}{q+1}.$$

The eigenvalues of the matrix $(W)_n$ are

$$\xi_j = \sum_{l=0}^{n-1} W_l^{(n)} \lambda_j^l, \; \lambda_j = e^{\frac{2\pi}{N} ij}, \; j = \overline{0, N-1}.$$

If no λ_i is a zero of the polynomial

$$W_{n-1}^{(n)} z^{n-1} + W_{n-2}^{(n)} z^{n-2} + \cdots + W_0^{(n)},$$

then $(W)_n$ is non-singular.

If $(W)_n$ is non-singular, $(M_j)_0^{N-1}$ and then, from (11) and (8), $Y(z)$ are determined, since $S_A(q^\nu) = y_\nu$. Using the induction method, we obtain $P_{n-1}(z)$ uniquely, thus the interpolation spline function $S_A(z)$ is determined.

We now introduce the following polynomial of degree n in z.

$$F_n(z) = z \sum_{l=0}^{n-1} W_l^{(n)} z^l. \tag{18}$$

From (17) we have

$$F_{n+1}(z)=\frac{1}{q^n-1}\left\{(q^nz-1)F_n(z)+\left(1-\frac{z}{q}\right)F_n(qz)\right\}.$$

By induction we obtain

$$F_n(z)=\frac{z}{\prod\limits_{j=0}^{n-2}(q^{n-1-j}-1)}\sum_{m=0}^{n-1}\binom{n-1}{m}\prod_{k=m+1}^{n-1}(q^kz-1)\prod_{l=0}^{m-1}(q-q^lz)$$

$$=\frac{z}{K_n}\prod_{l=-1}^{n-1}(q^lz-1)G_n(z), \tag{19}$$

where

$$K_n=\frac{1-q}{q}\prod_{j=0}^{n-2}(q^{n-j-1}-1), \tag{20}$$

$$G_n(z)=\sum_{j=0}^{n}\binom{n}{j}\frac{(-q)^j}{q-zq^j}. \tag{21}$$

We see that $G_n(z_0)=0$ if and only if $F_n(z_0)/z_0=0$.

We set

$$A_{n,l}(x,\lambda)=\frac{(-1)^n}{n!\zeta^n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{q^{(l+j)x}}{q^{(l+j)}-\lambda},\quad -\infty<x<+\infty. \tag{22}$$

$q=e$, l is any real number.

The function $A_{n,l}(z,\lambda)$ satisfies the following system of equations

$$\left.\begin{aligned}&y^{(j)}(1)=\lambda y^{(j)}(0), \qquad j=\overline{0,\ n-1},\\ &y^{(n)}(1)=\lambda y^{(n)}(0)+1,\end{aligned}\right\} \tag{23}$$

where $y^{(j)}(x)=\frac{d^jy}{dx^j}$, $j=\overline{0,\ n}$.

Evidently, we have

$$G_n(z)=\frac{(-1)^{n+1}n!\zeta^n}{z}A_{n,0}(1,qz^{-1})=\frac{(-1)^{n+1}n!\zeta^nq}{z^2}A_{n,0}\left(0,\frac{q}{z}\right). \tag{24}$$

Thus, we obtain

Theorem 1. *ProblemP$_A$ can be solved uniquely in $\mathscr{S}_{A,n}$ iff*

$$A_{n,0}(0,q\lambda_j^{-1})\neq 0,$$

$$\lambda_j=e^{\frac{2\pi i}{N}j},\ \textit{for all } j=0,\ \cdots,\ N-1,$$

where $A_{n,0}(x,\lambda)$ is given by (22) for $l=0$.

Since

$$A_{n,l-\nu}(0,\lambda)=q^\nu A_{n,l}(0,q^\nu\lambda),$$

letting $l=\nu=-\frac{n}{2}$, then

$$A_{n,0}(0,\lambda)=q^{-\frac{n}{2}}A_{n,-\frac{n}{2}}(0,q^{-\frac{n}{2}}\lambda). \tag{25}$$

If $A_{n,-\frac{n}{2}}(0,\xi_1)=0$, then $A_{n,-\frac{n}{2}}(0,\xi_1^{-1})=0$.

Let

$$\hat{\pi}_n(\xi)=A_{n,-\frac{n}{2}}(0,\xi)\prod_{j=0}^{n}\left(e^{\left(j-\frac{n}{2}\right)\zeta}-\xi\right), \tag{26}$$

$q=e^{\zeta}$. $\hat{\pi}_n(\xi)$ is a polynomial of degree $n-1$ in ξ.

If $\hat{\pi}_n(\xi)=0$, then

$$\hat{\pi}_n(\xi^{-1})=F_n(q^{1-\frac{n}{2}}\xi)=F_n(q^{1-\frac{n}{2}}\xi^{-1})=0. \tag{27}$$

We now deal with the problem P_A for the cases $1\leqslant n\leqslant 4$.

(i) $n=1$, $0<|q|<\infty$.

There is always a solution in $\mathscr{S}_{A,1}$.

(ii) $n=2$, $0<|q|<\infty$.

$F_2(z)=Cz(z+1)$, C is a constant.

We then have

$$\left.\begin{array}{l} F_2(-1)=0, \\ \dfrac{1}{z}F_2(z)\neq 0, \quad z\neq -1. \end{array}\right\} \tag{28}$$

If N is odd, then $e^{\frac{2k\pi}{N}i}\neq -1$, for $0\leqslant k\leqslant N-1$. From (28) we have

$$F_2(e^{\frac{2k\pi}{N}i})\neq 0,\ 0\leqslant k\leqslant N-1.$$

$(W)_2$ is non-singular.

If N is even, then $(W)_2$ is singular.

(iii) $n=3$, $0<|q|<\infty$.

(a) If $q=e^{\frac{2\pi}{N}i}$, $N(\geqslant 2)$ being a positive integer.

We then reduce the problem P_A to the corresponding interpolation problem on the unit circle [3-7] and conclude that P_A can be solved uniquely in $\mathscr{S}_{A,3}$.

(b) If $|q|\neq 1$.

Let $\xi_i^{(3)}$ $(i=1, 2)$ denote the roots of the equation $\hat{\pi}_3(\xi)=0$,

$$\xi_1^{(3)}=-V_1+\sqrt{V_1^2-1},\ \xi_2^{(3)}=-V_1-\sqrt{V_1^2-1}, \tag{29}$$

where $V_1=t+t^{-1}$, $t=q^{\frac{1}{2}}$.

If $V_1^2=1$, then $q=e^{\frac{2\pi}{3}i}$(or $q=e^{\frac{4\pi}{3}i}$), but this is contrary to the hypothesis $|q|\neq 1$.

$F_3(z)$ has two distinct roots $q^{-\frac{1}{2}}\xi_1^{(3)}$ and $q^{-\frac{1}{2}}\xi_2^{(3)}$.

If $q^{-\frac{1}{2}}\xi_1^{(3)}=e^{i\theta}$, then from (29) we obtain

$$q^{\frac{1}{2}}e^{i\theta}+q^{-\frac{1}{2}}e^{-i\theta}=-2(q^{\frac{1}{2}}+q^{-\frac{1}{2}}),$$

$$q=-\frac{2+e^{-i\theta}}{2+e^{i\theta}},$$

but this leads to $|q|=1$.

Therefore, $F_3(z)$ has no roots on the unit circle, thus $(W)_3$ is non-singular.

(iv) $n=4$.

(a) $q=e^{\frac{2\pi i}{N}}$.

$(W)_3$ is non-singular if N is odd, and singular if N is even (see [5, 6]).

(b) $|q|\neq 1$.

The zeros of $F_4(z)$ are $q^{-1}\xi_i^{(4)}$ $(i=1, 2, 3)$, namely

$$-q^{-1},\ q^{-1}\left(-2-\frac{3V_2}{2}\pm\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right),\ \text{where } V_2=q+q^{-1}.$$

If $q^{-1}\left(-2-\frac{3V_2}{2}+\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right)=e^{i\theta}$, then

$$q=[-2\pm i\sqrt{6(1+\cos\theta)}]/[3+\cos\theta+i\sin\theta],$$

but this leads to $|q|=1$.

We conclude that

$$|q^{-1}\xi_i^{(4)}|\neq 1,\quad i=1, 2, 3.$$

We have the following result:

Theorem 2. *If q is a complex number, $0<|q|<\infty$, then problem P_A has a unique solution in $\mathscr{S}_{A,n}$ when $n=1$ and $n=3$ $(q=e^{\frac{2\pi}{N}i})$.*

For $n=2$, P_A has a unique solution in $\mathscr{S}_{A,2}$ iff N is odd, for all q, $0<|q|<\infty$.

For the case $n=4$, P_A has a unique solution in $\mathscr{S}_{A,4}$ iff one of the following conditions is satisfied

(1) $q=e^{\frac{2\pi i}{N}}$, *N odd,*

(2) $|q|\neq 1$, *N odd or even.*

1.2. Cubic splines of the classes $\mathscr{S}_{B,3}$, $\mathscr{S}_{C,3}$

The symbol $\mathscr{S}_{B,3}=\{S_B(z)\}$ $(\mathscr{S}_{C,3}=\{S_C(z)\})$ denotes the class of functions satisfying the following three conditions

$$S_B(z)\,(S_C(z))\in\pi_n \text{ on each arc } \Gamma_j,\quad j=\overline{0, N-1}.$$
$$S_B(z)\,(S_C(z))\in C^{n-1}(\Gamma_{0,N}).$$
$$S'_B(1)=y'_0,\ S'_B(q^N)=y'_N,$$
$$(S_C^{(2)}(1)=y''_0,\ S_C^{(2)}(q^N)=y''_N),$$

where y'_0, y'_N, y''_0 and y''_N are prescribed numbers, and

$$\frac{d^iS_B(z)}{dz^i}=S_B^{(i)}(z)\left(\frac{d^iS_C(z)}{dz^i}=S_C^{(i)}(z)\right).$$

We propose the interpolation problem $P_B(P_C)$ as follows.

A sequence of numbers (real or complex)

$$y=(y_\nu)\quad \nu=\overline{0, N}\,(N\geqslant n+1),$$
$$y'_0,\ y'_N\quad (y''_0,\ y''_N)$$

is prescribed. We are to find a function $S_B(Z)\in\mathscr{S}_{B,3}$ $(S_C(Z)\in\mathscr{S}_{C,3})$ such that

$$\left.\begin{aligned}&S_B(q^\nu)=y_\nu,\quad \nu=\overline{0, N},\\ &S'_B(1)=y'_0,\quad S'_B(q^N)=y'_N,\\ &(S_C(q^\nu)=y_\nu,\quad \nu=\overline{0, N},\ S''_C(1)=y''_0,\ S''_C(q^N)=y''_N).\end{aligned}\right\}\tag{30}$$

Let $A_j=S'(q^j)=\dfrac{dS(q^j)}{dZ}$, $b_j=q^j-q^{j-1}$, $S(z)$ denotes the spline belonging to $\mathscr{S}_{B,3}$ or $\mathscr{S}_{C,3}$.

We have

$$S(z)=A_{l-1}\frac{(q^l-z)^2(z-q^{l-1})}{b_l^2}-A_l\frac{(z-q^{l-1})^2(q^l-z)}{b_l^2}$$
$$+y_{l-1}\frac{(q^l-z)^2[2(z-q^{l-1})+b_l]}{b_l^3}+y_l\frac{(z-q^{l-1})^2[2(q^l-z)+b^l]}{b_l^3}. \quad (31)$$

The continuity condition is here imposed on $S^{(2)}(z)(=S''(z))$ at $z=q^l$ $(l=\overline{1, N-1})$. This leads to the requirement

$$\omega_l A_{l-1}+2A_l+\xi_l A_{l+1}=C_l, \quad l=\overline{1, N-1}, \quad (32)$$

where $\omega_l=\dfrac{b_{l+1}}{b_l+b_{l+1}}$, $\xi_l=1-\omega_l$.

For P_B, we add the following conditions

$$A_0=y_0', \quad A_N=y_N'. \quad (33)$$

For P_C the additional conditions are

$$\left.\begin{aligned} y_0''&=\frac{-4A_0}{b_1}-\frac{2A_1}{b_1}+\frac{6(y_1-y_0)}{b_1^2},\\ y_N''&=\frac{2A_{N-1}}{b_N}+\frac{4A_N}{b_N}-\frac{6(y_N-y_{N-1})}{b_N^2}. \end{aligned}\right\} \quad (34)$$

(33) and (34) may be written into

$$\left.\begin{aligned} 2A_0+\xi_0A_1&=C_0,\\ \omega_NA_{N-1}+2A_N&=C_N. \end{aligned}\right\} \quad (30)$$

For P_B, P_C we assign the values of ξ_0 and ω_N as follows

$$P_B:\quad \xi_0=\omega_N=0, \quad (36)$$
$$P_C:\quad \xi_0=\omega_N=1. \quad (37)$$

Then the defining equations are

$$\begin{pmatrix} 2 & \xi_0 & 0 & \cdots & \cdots & 0\\ \omega_1 & 2 & \xi_1 & 0 & \cdots & 0\\ 0 & \omega_2 & & \cdots & \cdots & 0\\ & & \vdots & & & \\ 0 & \cdots & & 2 & \xi_{N-2} & 0\\ 0 & \cdots & & \omega_{N-1} & 2 & \xi_{N-1}\\ 0 & \cdots & & 0 & \omega_N & 2 \end{pmatrix}\begin{pmatrix} A_0\\ A_1\\ \vdots\\ A_N\end{pmatrix}=\begin{pmatrix} C_0\\ C_1\\ \vdots\\ C_N\end{pmatrix}. \quad (38)$$

We may also write (38) as

$$D_{N+1}A=C. \quad (38)'$$

Let $D_{N+1,1}$ be the determinant of the matrix D_{N+1} with $\xi_0=\xi_N=0$.

Then we have

$$D_{N+1,1}=4\Delta_{N-1},$$

where

$$\Delta_{N-1}=\begin{vmatrix} 2 & \xi_1 & 0 & \cdots & 0 & 0 & 0 \\ \omega_2 & 2 & \xi_2 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \omega_{N-2} & 2 & \xi_{N-2} \\ 0 & \cdots & & & 0 & \omega_{N-1} & 2 \end{vmatrix}. \tag{39}$$

Since

$$\left.\begin{aligned} \omega_l&=\frac{q^l(q-1)}{q^{l-1}(q^2-1)}=\frac{q}{q+1}, \qquad 1\leqslant l\leqslant N-1, \\ \xi_l&=1-\omega_l=\frac{1}{q+1}, \qquad 1\leqslant l\leqslant N-1, \end{aligned}\right\} \tag{40}$$

we have

$$\Delta_{N-1}=2\Delta_{N-2}+\tau\Delta_{N-3}, \tag{41}$$

where $\tau=-q/(1+q)^2$.

Let $\Delta_{-1}=0$, $\Delta_0=1$, $\Delta_1=2$, then

$$\Delta_2=4+\tau,\ \Delta_3=8+4\tau,\ \Delta_4=16+12\tau+\tau^2,$$
$$\Delta_5=32+32\tau+16\tau^2.$$

From (41), by induction we obtain

$$\Delta_{N-1}=\frac{(1+\xi)^N-(1-\xi)^N}{2\xi},\quad \xi=\frac{\sqrt{q^2+q+1}}{1+q}. \tag{42}$$

If $\Delta_{N-1}=0$, then $|1+\xi|^2=|1-\xi|^2$, with $\xi=x+iy$. We obtain $\xi=iy$, and $q=\frac{1}{2(1+y^2)}[-(1+2y^2)\pm i\sqrt{4y^2+3}]$, but this leads to $|q|=1$.

Therefore we have

$$D_{N+1,1}\neq 0 \quad \text{if } |q|\neq 1. \tag{43}$$

For the problem P_C, the relevant determinant is $D_{N+1,2}$.

$$D_{N+1,2}=\begin{vmatrix} 2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \omega_1 & 2 & \xi_1 & 0 & \cdots & \cdots & 0 \\ 0 & \omega_2 & 2 & \xi_2 & \cdots & \cdots & 0 \\ & & \vdots & & & & \\ 0 & \cdots & \cdots & \cdots & \omega_{N-1} & 2 & \xi_{N-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 2 \end{vmatrix}.$$

It is easy to prove that

$$D_{N+1,2}=4\Delta_{N-1}-2\Delta_{N-2}+\frac{q}{q^2-1}\Delta_{N-3}=4\Delta_{N-1}-\Delta_{N-1}=3\Delta_{N-1}\neq 0 \text{ if } |q|\neq 1.$$

We then have

Theorem 3. *If $0<|q|<\infty$, $|q|\neq 1$, then the interpolation problems P_B and P_C can be solved uniquely in $\mathscr{S}_{B,3}$ and $\mathscr{S}_{C,3}$, respectively.*

§ 2. Interpolation by Splines on Infinite Planar Point Sets

The symbol $\mathscr{S}_n(\Gamma)=\{S(z)\}$ denotes the class of functions satisfying the following conditions

$$\left.\begin{array}{l} S(z)\in\pi_n,\ z\in\Gamma_j,\quad j=0,\ \pm1,\ \pm2,\ \cdots, \\ S(z)\in C^{n-1}(\Gamma), \end{array}\right\} \tag{44}$$

where $\Gamma: z=q^x,\ -\infty<x+\infty,\quad 0<|q|<\infty.$

The interpolation problem for the infinite planar set may be described as follows.

If a sequence of numbers (complex or real)

$$y=(y_\nu),\quad \nu=0,\ \pm1,\ \cdots$$

are prescribed, then we attempt to find $S(z)\in\mathscr{S}_n(\Gamma)$ such that

$$S(q^\nu)=y_\nu,\ \nu=0,\ \pm1,\ \cdots. \tag{45}$$

Since for the case $|q|=1$, the interpolation problem has beed discussed in many papers, we consider only the case: $|q|\neq1$. Using the notation described in part 1, every $S(z)\in\mathscr{S}_n(\Gamma)$ may be represented in the form

$$\begin{aligned} S(z)=P_n(z)+a_1(z-q)_+^n+a_2(z-q^2)_+^n+\cdots \\ +a_0(z-1)_-^n+a_{-1}(z-q^{-1})_-^n+\cdots, \end{aligned} \tag{46}$$

where $P_n(z)\in\pi_n$, $\{a_\nu\}$ are constants.

Let us determine (44) so as to satisfy the relation (45).

Select $P_n(z)$ arbitrary such that the relations

$$P_n(1)=y_0,\ P_n(q)=y_1 \tag{47}$$

are satisfied. $P_n(z)$ being selected, we see that the coefficients $a_1,\ a_2,\ \cdots$ (and $a_0,\ a_{-1},\ \cdots$) are successively and uniquely determined by conditions (45). We have

Lemma 1. *The problem (45) with the requirement that $S(z)\in\mathscr{S}_n(\Gamma)$ where $n\geqslant2$ has infinitely many solutions forming a linear manifold in $\mathscr{S}_n(\Gamma)$ of dimension $n-1$.*

We define three sub-classes of $S_n(\Gamma)$ as follows.

(i) $\mathscr{S}_{A,n}^\infty=\{S(z)\}$.

If $S(z)\in\mathscr{S}_{A,n}^\infty$, then

$$\begin{array}{l} S(z)=S_A(z)\in\mathscr{S}_{A,n}\text{ for } z\in\Gamma_{0,N}, \\ S(z)\in\mathscr{S}_n(\Gamma). \end{array} \tag{48$_1$}$$

(ii) $\mathscr{S}_\gamma^\infty=\{S_\gamma(z)\},\ \gamma\geqslant0.$

If $S(z)\in\mathscr{S}_\gamma^\infty$, then

$$\begin{array}{l} S(z)\in\mathscr{S}_n(\Gamma), \\ S(z)=O(|\ln z|^\gamma),\quad |q|\neq1. \end{array} \tag{48$_2$}$$

(iii) $\mathscr{S}_n^*(\Gamma)$

If $S(z)\in\mathscr{S}_n^*(\Gamma)$, then

$$S(z)\in\mathscr{S}_n(\Gamma),\qquad (48)_3$$
$$S(z)=O(|z|^{\xi}),$$

for some $\xi>0$, $|q|\neq 1$.

Using Theorem 2 and Lemma 1, we have

Theorem 4. *Let values* $y=(y_\nu)$, $\nu=0,\ \pm1,\ \cdots(\nu\neq N)$, $y_N=q^{N(n-1)}y_0$ be given. *If the conditions of Theorem 2 are valid, then there is one and only one function* $S(Z)$ *in* $\mathscr{S}^{\infty}_{A,n}$, $1\leqslant n\leqslant 4$, such that

$$S(q^\nu)=y_\nu,\quad \nu=0,\ \pm1,\ \cdots.$$

The problem P^{∞}_{γ} may be described as follows.

Given data $y=(y_\nu)$, $y_\nu=O(|\nu|^\nu)$, $\nu=0,\ \pm1,\ \cdots$, we wish to find the function $S(z)\in\mathscr{S}^{\infty}_{\gamma}$ such that

$$S(q^\gamma)=y_\gamma,\ \nu=0,\ \pm1,\ \cdots.\qquad (49)$$

This interpolation problem is called problem P^{∞}_{γ}.

We now discuss the zeros of $\tilde{\pi}_n(\xi)$ (see(26)). We shall follow the theory of cardinal $\mathscr{L}$-splines [1] to solve the problem P^{∞}_{γ} for $n\leqslant 4$.

From section 1 we have

$$A_{n,0}(0,\ \lambda)=K_n\tilde{\pi}_n(\lambda)/\prod_{j=0}^{n}(q^{j-\frac{n}{2}}-\lambda q^{-\frac{n}{2}}),\qquad (50)$$

where

$$\left.\begin{aligned} K_n&=(-1)^n(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^n q^{-\frac{n}{2}}/n!\zeta^n,\qquad q=e^{\zeta},\\ \tilde{\pi}_n(\lambda)&=\frac{(-1)^n n!\zeta^n}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^n}\,\hat{\Pi}_n(q^{-\frac{n}{2}}\lambda).\end{aligned}\right\}\qquad (51)$$

$$\tilde{\pi}_n(\lambda)=\begin{cases}1, & n=1,\\ q^{-1}\lambda+1, & n=2,\\ (q^{-\frac{3}{2}}\lambda)^2+2(q^{\frac{1}{2}}+q^{-\frac{1}{2}})(q^{-\frac{3}{2}}\lambda)+1, & n=3,\\ (q^{-2}\lambda+1)[(q^{-2}\lambda)^2+(4+3q+3q^{-1})(q^{-2}\lambda)+1], & n=4.\end{cases}\qquad (52)$$

Therefore

$\tilde{\pi}_1(\lambda)$ no zero, (53)

$\tilde{\pi}_2(\lambda)$ one zero $\lambda=-q$, (54)

$\tilde{\pi}_3(\lambda)$ two zeros:

$$\left.\begin{aligned}\lambda_1&=-q^{\frac{3}{2}}[V_1+\sqrt{V_1^2-1}],\\ \lambda_2&=-q^{\frac{3}{2}}[V_1-\sqrt{V_1^2-1}],\end{aligned}\right\}\qquad (55)$$

$\tilde{\pi}_4(\lambda)$ three zeros:

$$\left.\begin{aligned}\lambda_1&=-q^2,\\ \lambda_2&=-q^2\left[2+\frac{3}{2}V_2+\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right],\\ \lambda_3&=-q^2\left[2+\frac{3}{2}V_2-\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right],\end{aligned}\right\}\qquad (56)$$

where $V_1=q^{\frac{1}{2}}+q^{-\frac{1}{2}}$, $V_2=q+q^{-1}$.

Lemma 2. *Let λ_1, λ_2 be the zeros of $A_{3,0}(0, \lambda)$. Let $D_{3,1}$, $D_{3,2}$ denote the regions $0<|q|<r_1$ and $r_2<|q|<\infty$ respectively, where*

$$\left.\begin{aligned} r_1&=\frac{1}{2}(3-\sqrt{5})<0.382, \\ r_2&=\frac{1}{2}(3+\sqrt{5})>2.618. \end{aligned}\right\} \tag{57}$$

Then

$$|\lambda_i|<1,\ i=1,\ 2, \quad q\in D_{3,1}, \tag{58}$$

$$|\lambda_i|>1,\ i=1,\ 2, \quad q\in D_{3,2}, \tag{59}$$

$$|\lambda_2|<|\lambda_1|, \quad q\in D_{3,1}\cup D_{3,2}. \tag{60}$$

Proof Suppose $\lambda_i=e^{i\theta}$, then

$$2(q^{\frac{1}{2}}+q^{-\frac{1}{2}})=-(q^{\frac{3}{2}}e^{-i\theta}+q^{-\frac{3}{2}}e^{i\theta}). \tag{61}$$

Let $q=re^{i\varphi}$. From (61) we obtain

$$\left.\begin{aligned} 2\cos\frac{\varphi}{2}&=-(r+r^{-1}-1)\cos\left(\frac{3\varphi}{2}-\theta\right), \\ 2\sin\frac{\varphi}{2}&=-(r+r^{-1}-1)\sin\left(\frac{3\varphi}{2}-\theta\right). \end{aligned}\right\} \tag{62}$$

Then we obtain

$$\cos(3\varphi-2\theta)=\frac{l^2-3}{2l}. \tag{63}$$

Let $l=r+r^{-1}$, $f(l)=\frac{l^2-3}{2l}$, $f(l)$ is a monotone increasing function of l. Since $\frac{dl}{dr}>0$ $(r>1)$ and $\frac{dl}{dr}<0(r<1)$, we have

$$f(l)>1 \text{ for } 0<r<r_1 \text{ and } r_2<r<\infty. \tag{64}$$

We see that (64) contradicts (63). Therefore, $|\lambda_1|\neq 1$ for $q\in D_{3,1}\cup D_{3,2}$.

Wemay also prove $|\lambda_2|\neq 1$ for $q\in D_{3,1}\cup D_{3,2}$. Since

$$\lim_{q\to 0}\lambda_i=0, \quad i=1,\ 2$$

and

$$\lim_{q\to\infty}\lambda_i=\infty,\ i=1,\ 2,$$

therefore, (58) and (59) are true.

It is easy to prove that

$$|\lambda_1|\neq|\lambda_2| \text{ for } q\in D_{3,1}\cup D_{3,2}. \tag{65}$$

Now we choose $q=0.1\in D_{3,1}$. Then $\lambda_1\doteq -0.215$, $\lambda_2\doteq -0.005$. We conclude

$$|\lambda_2|<|\lambda_1| \text{ for } q\in D_{3,1}. \tag{66$_1$}$$

If we choose $q=10\in D_{3,2}$. Then $\lambda_1\doteq -215.35$, $\lambda_2\doteq -4.64$. We thus have

$$|\lambda_2|<|\lambda_1| \text{ for} q\in D_{3,2}, \tag{66$_2$}$$

and (60) is proved. Q. E. D.

Lemma 3. *Let λ_1, λ_2, λ_3 be the roots of $A_{4,0}(0, \lambda)$. Let $D_{4,1}$, $D_{4,2}$ denote the* *regions* $0<|q|<r_1$, $r_2<|q|<\infty$ *respectively, where*

$$r_1=\frac{1}{2}(5-\sqrt{21})\,(\doteq 0.21),\qquad r_2=\frac{1}{2}(5+\sqrt{21})\,(\doteq 4.80).$$

Then

$$|\lambda_i|<1\quad i=1,\ 2,\ 3 \text{ for } q\in D_{4,1}, \tag{67}$$

$$|\lambda_i|>1\quad i=1,\ 2,\ 3 \text{ for } q\in D_{4,2}, \tag{68}$$

$$|\lambda_1|<|\lambda_3|<|\lambda_2| \text{ for } q\in D_{4,1}\cup D_{4,2}. \tag{69}$$

Proof

$$\lambda_1=-q^2,\ \lambda_2=-q^2\left[2+\frac{3}{2}V_2+\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right],$$

$$\lambda_3=-q^2\left[2+\frac{3}{2}V_2-\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right],\ V_2=q+q^{-1}.$$

If we set $\lambda_2=e^{i\theta}$ (or $\lambda_1=e^{i\theta}$), then

$$4+3(q+q^{-1})=-(q^{-2}e^{i\theta}+q^2w^{-i\theta}).$$

Let $q=re^{i\varphi}$. We have

$$\left.\begin{aligned}\sqrt{3}\,(r^{\frac{1}{2}}+r^{-\frac{1}{2}})\cos\frac{\varphi}{2}&=\mp(r+r^{-1})\sin\left(\varphi-\frac{\theta}{2}\right),\\ \sqrt{3}\,(r^{\frac{1}{2}}-r^{-\frac{1}{2}})\sin\frac{\varphi}{2}&=\pm(r-r^{-1})\cos\left(\varphi-\frac{\theta}{2}\right).\end{aligned}\right\} \tag{70}$$

From (70) we obtain

$$l^2-3l-\left[4\cos^2\left(\varphi-\frac{\theta}{2}\right)+6\cos\varphi\right]=0,$$

where $l=r+r^{-1}$. The roots of this equation are

$$l_1,\ l_2=\frac{3}{2}\pm\sqrt{\frac{9}{4}+4\cos^2\left(\varphi-\frac{\theta}{2}\right)+6\cos\varphi}. \tag{71}$$

Let q belong to $D_{4,1}\cup D_{4,2}$. Then

$$l=|q|+|q|^{-1}>5. \tag{72}$$

But (71) tells us that $\max(|l_1|,\ |l_2|)\leqslant 5$, this contradicts (72). Therefore we conclude

$$|\lambda_i|\neq 1\ i=1,\ 2,\ 3 \text{ for } q\in D_{4,1}\cup D_{4,2}.$$

Since

$$\lim_{q\to\infty}\lambda_i=0,\qquad \lim_{q\to\infty}\lambda_i=\infty$$

and since λ_i is a continuous function of q, we have

$$|\lambda_i|<1 \text{ for } q\in D_{4,1},$$

$$|\lambda_i|>1 \text{ for } q\in D_{4,2},$$

$i=1,\ 2,\ 3$. Thus, (67), (68) are true.

Suppose that $|\lambda_2|=|\lambda_3|$. Then

$$\left|-2-\frac{3}{2}V_2-\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right|=\left|-2-\frac{3}{2}V_2+\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}\right|.$$

This leads to

$$\left.\begin{aligned}-2-\frac{3}{2}V_2-\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}=e^{i\xi},\\ -2-\frac{3}{2}V_2+\sqrt{\left(2+\frac{3}{2}V_2\right)^2-1}=e^{-i\xi}.\end{aligned}\right\} \tag{73}$$

Thus we have

$$-4-3(q+q^{-1})=2\cos\xi. \tag{74}$$

Let $q=re^{i\theta}$. From (74) we have

$$(r-r^{-1})\sin\theta=0.$$

If $r-r^{-1}=0$, then $r=1$. This contradicts $q\in D_{4,1}\cup D_{4,2}$. If $\sin\theta=0$, then from (74)

$$\cos\xi=-2-\frac{3}{2}(r+r^{-1})<-2-\frac{15}{3}. \tag{75}$$

(75) cannot be valid, since $|\cos\xi|\leqslant 1$.

Thus

$$|\lambda_2|\neq|\lambda_3|. \tag{76$_1$}$$

If $|\lambda_1|=|\lambda_2|$, we still obtain (73), so we conclude that

$$|\lambda_1|\neq|\lambda_2| \text{ and } |\lambda_1|\neq|\lambda_3|. \tag{76$_2$}$$

It is easy to prove that

$$|\lambda_1|<|\lambda_3|<|\lambda_2| \text{ for } q\in D_{4,1}\cup D_{4,2}.$$

(69) is proved. Q. E. D.

Lemma 4. *The system of equations*

$$\left.\begin{aligned}&q^iW^{(i)}(q)=\lambda W^{(i)}(1),\ 0\leqslant i\leqslant n-1,\ q\neq 0,\ |q|\neq 1,\\ &W(1)=0,\end{aligned}\right\} \tag{77}$$

has a non-trivial solution in π_n *iff* $\tilde{\pi}_n(\lambda)=0$, *where* $W^{(i)}=\frac{d^iW}{dz^i}$. $\tilde{\pi}_n(\lambda)$ *is defined by* (51).

Proof If is easy to find that the determinant of (77) is

$$\Delta_n=(-1)^n n\cdot\zeta^n\prod_{k=0}^{n}(q^k-\lambda)A_{n,0}(0,\lambda)=(q^{\frac{n+1}{2}}-q^{\frac{n-1}{2}})^n\tilde{\pi}_n(\lambda). \tag{78}$$

Q. E. D.

Let $P_j(z)$ be a polynomial of degree n in z, such that

$$P_j(0)=P_j(1)=0,\ P_j^{(i)}(0)=\delta_{ij},\ i=\overline{1,\ n-1},$$
$$j=1,\ 2,\ \cdots,\ n-1.$$

Let Ω denote the $(n-1)\times(n-1)$ matrix

$$\Omega=(\lambda_{ij}),\ \lambda_{ij}=P_j^{(i)}(1)q^i. \tag{79}$$

We have

Lemma 5.

$$\operatorname{Det}(\Omega-\lambda I)=C\tilde{\pi}_n(\lambda),\quad C\neq 0. \tag{80}$$

$\tilde{\pi}_n(\lambda)$ *is defined by* (51). $q\neq 0$, $|q|\neq 1$.

Proof Let $P(z)$ be the polynomial of degree n which satisfies (77), then

$$P(z)=\sum_{j=0}^{n} a_j z^j=\sum_{k=1}^{n-1} P^{(k)}(1)\,(q-1)^k P_k\left(\frac{z-1}{q-1}\right)^k. \tag{81}$$

From (79), (81), we have

$$(\Omega-\lambda I)W=0, \tag{82}$$

where

$$W=\begin{pmatrix} W_1 \\ \vdots \\ W_{n-1}\end{pmatrix},\ W_k=P^{(k)}(1)\,(q-1)^k.$$

Since

$$W_k=(q-1)^k\sum_{j=k}^{n}\frac{j!}{(j-k)!}a_j, \quad 0\leqslant k\leqslant n,$$

and

$$a_n+a_0=-\sum_{j=1}^{n-1}a_j,$$

$$q^n a_n+a_0=-\sum_{j=1}^{n-1}q^j a_j,$$

$(|q|\neq 1)$, hence $W_1, \cdots W_{n-1}$ may be represented by $\{a_j\}_1^{n-1}$ uniquely. Thus

$$W=YA,\ A=\begin{pmatrix} a_n \\ \vdots \\ a_{n-i}\end{pmatrix},\ \det(Y)\neq 0. \tag{83}$$

From (82), (83) we have

$$(\Omega-\lambda I)YA=0. \tag{84}$$

But (84) may also be obtained directly from the following system of equations

$$\left.\begin{aligned}&\sum_{l=0}^{n-j}\binom{n-l}{j}(q^{n-l}-\lambda)\,a_{n-l}=0, \quad j=\overline{0,\ n-1},\\ &\sum_{l=0}^{n}a_l=0.\end{aligned}\right\} \tag{85}$$

By eliminating a_0 and a_n from (85) we have

$$zA=0, \tag{86}$$

where $z=(z_{ij})$ is an $(n-1)\times(n-1)$ matrix,

$$Z_{ij}=\sum_{j=1}^{n-1}\left[\binom{j}{i}(q^j-\lambda)\,(i-j)_+^0-\binom{n}{i}\frac{q^j-1}{q^n-1}(q^n-\lambda)\right],$$

$$(i-j)_+^0=\begin{cases}1, & j\leqslant i-1,\\ 0, & j\geqslant i.\end{cases}$$

It is easy to prove that

$$\det(z)=(q^n-1)\det(D_n)=(q^n-1)\Delta_n, \tag{87}$$

where D_n is the matrix of the system of equations (85). From (84), (86), (87) and (78), Lemma 5 is proved.

Q. E. D.

Corollary. *The system of equations* (77) *has* $n-1$ *solutions iff* $\tilde{\pi}_n(\lambda)$ *has* $n-1$ *distinct zeros. These solutiont form the eigenvectors of the matrix* Ω (see (79)).

We now suppose that $\tilde{\pi}_n(\lambda)=0$ has $n-1$ distinct zeros $\lambda_1, \cdots, \lambda_{n-1}$.

Consider two cases:

$$\text{(i)} \quad |\lambda_i|<1, \qquad i=\overline{1,\ n-1} \text{ for } 0<|q|<1; \tag{88}$$

$$\text{(ii)} \quad |\lambda_i|>1, \qquad i=\overline{1,\ n-1} \text{ for } 1<|q|<\infty. \tag{89}$$

In either of these two cases, we have the functions $S_i(z)=A_{n,0}(x(z),\ \lambda_i)$ satisfying (77), for $\lambda=\lambda_i$, where $x(z)=\ln z/\ln q$.

Since $\{\lambda_i\}_1^{n-1}$ are distinct, $\{S_i\}_1^{n-1}$ are linearly independent. $S_i(z)$ may be extended to Γ by means of the functional equations

$$\left.\begin{aligned} S(z)&=\lambda_i S(q^{-1}z), && z\in\Gamma, \\ S(z)&=S_i(z), && z\in\Gamma_0, \end{aligned}\right\} \tag{90}$$

$S(Z)\in C^{n-1}(\Gamma)$.

The functions so obtained are called eigensplines and denoted by $S_i(Z)$, $i=\overline{1,\ n-1}$, and

$$S_i(q^\nu z)=\lambda_i^\nu S_i(q^{-\nu}z), \quad z\in\Gamma_0.$$

The symbol $\mathscr{S}_n^0(\Gamma)$ denotes the class of functions satisfying the following conditions:

$$S(z)\in\pi_n,\ z\in\Gamma_j, \quad j=0,\ \pm1,\ \cdots,$$
$$S(z)\in C^{n-1}(\Gamma),$$
$$S(q^\nu)=0, \quad \nu=0,\ \pm1,\ \cdots.$$

From Lemma 1 we obtain

Lemma 6. *If $\{\lambda_i\}_1^{n-1}$ are distinct, then the $n-1$ eigensplines are linearly independent. Every $S(z)\in\mathscr{S}_n^0(\Gamma)$ can be uniquely expressed as a linear combination of eigensplines*

$$S(z)=\sum_1^{n-1} C_j S_j(z), \quad z\in\Gamma,$$

and $S(z)\in\mathscr{S}_n^*(\Gamma)$ (see $(48)_3$).

Following C. A. Micchelli [1], we may prove the following two lemmas.

Lemma 7. *If $\{\lambda_i\}_1^{n-1}$ are distinct and (88) (or (89)) is valid, then there is a function $U_n(Z)$ belonging to $\mathscr{S}_n(\Gamma)$ such that*

$$U_n(q^\nu)=\delta_{\nu 0}, \quad \nu=0,\ \pm1,\ \cdots, \tag{91}$$

$U_n(Z)$ may be constructed as follows.

(i) *If (88) is valid, then*

$$U_n(z)=\begin{cases} \sum\limits_{j=1}^{n-1} a_j S_j(z), & z\in\Gamma_j, \quad j\geqslant 1, \\ Q_1(z), & z\in\Gamma_0, \\ Q_2(z), & z\in\Gamma_{-1}, \\ 0, & z\in\Gamma_j, \quad j\leqslant -2, \end{cases}$$

where $\{a_j\}_1^{n-1}$ *are constants.* $Q_1(z)$ *and* $Q_2(z)$ *are polynomials of dearee n in z.*

(ii) *If* (89) *is valid, then*

$$U_n(z)=\begin{cases}0, & z\in\Gamma_j, \quad j\geqslant 1,\\ P_1(z), & z\in\Gamma_0,\\ P_2(z), & z\in\Gamma_1,\\ \sum\limits_1^{n-1} b_j S_j(z), & z\in\Gamma_j, \quad j\leqslant -2,\end{cases}$$

where $\{b_j\}_1^{n-1}$ *are constants.* $P_1(z)$ *and* $P_2(z)$ *are polynomials of degre en in z.*

Lemma 8. *If* $\{\lambda_i\}_1^{n-1}$ *are distinct and* (88) (*or*(89)) *is valid, then there is one and only one function* $S(z)$ *belonging to* $\mathscr{S}_\gamma^\infty$ *such that*

$$S(z)=O(|\ln z|^\gamma), \quad S(q^\nu)=(Y_\nu), \quad \nu=0, \pm 1, \cdots.$$

$S(z)$ *may be represented as a complex Cardinal series:*

$$S(z)=\sum_{\nu=-\infty}^{\infty} Y_\nu U_n(q^{-\nu}z), \quad z\in\Gamma.$$

The eigenvalues of the matrix Ω for $n=2, 3, 4$ are given by (54), (55) and (56).

Let $D_{n,1}$, $D_{n,2}$ be the regions

$$D_{n,1}: 0<|q|<r_n, \quad D_{n,2}: r_n^{-1}<|q|<\infty \tag{92}$$

respectively, where

$$r_n=\begin{cases}1, & n=2,\\ \frac{1}{2}(3-\sqrt{5}), & n=3,\\ \frac{1}{2}(5-\sqrt{21}), & n=4.\end{cases} \tag{93}$$

From Lemma 2—8, we have

Theorem 5. *Let values* $y=(y_\nu)$, $\nu=0, \pm 1, \cdots$ *be given, where*

$$|y_\nu|=O(|\nu|^\gamma) \quad \gamma\geqslant 0.$$

If $q\in D_{n,1}\cup D_{n,2}$, $2\leqslant n\leqslant 4$, *then there is one and only one function* $S_n(z)\in\mathscr{S}_\gamma^\infty$ *such that*

$$S_n(q^\nu)=y_\nu, \quad \nu=0, \pm 1, \cdots.$$

Corollary. *If* (y_ν) *is bounded, then the interpolation spline is also bounded.*

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