Chin. Ann. of Math.
5B (4) 1984

A NOTE ON GENERALIZED DEGREE FOR GENERALIZED GRADIENT MAPPINGS

FAN XIANLING (范先令)
(*Lanzhou University*)

Abstract

In this paper the author studies the generalized degree for Clarke's generalized gradient mappings which are multivalued A-proper in Banach spaces. The results of H. Amann[1] on degree theory for gradient mappings which are compact vector fields in Hilbert spaces are extended.

In this paper we shall adhere to the following notations:

X is a real Banach space, X^* is the conjugate space of X, $\langle x^*, x\rangle$ is the value of $x^*\in X^*$ at $x\in X$.

For any $x\in X$, $A\subset X$, $B\subset X$ and $r>0$ we denote

$$d(x, A)=\inf\{\|x-y\|:y\in A\},$$
$$d(A, B)=\inf\{\|x-y\|:x\in A, y\in B\},$$
$$d^*(A, B)=\sup\{d(x, B):x\in A\},$$
$$B(x, r)=\{y\in X:\|y-x\|<r\},$$
$$B(A, r)=\{x\in X:d(x, A)<r\},$$
$$\bar{B}(A, r)=\{x\in X:d(x, A)\leqslant r\}.$$

$\bar{A}$ and ∂A denote the closure and the boundary of A respectively.

$T:\Omega\subset X\to 2^{X^*}$ is a multivalued mapping from Ω into X^*. G_T is the graph of T. The norm in $X\times X^*$ is defined by $\|(x, y)\|=\max\{\|x\|_X, \|y\|_{X^*}\}$.

If $f:U\to R$ is a locally Lipschitz functional where U is an open subset of X, then we denote by $\partial f(x)$ the Clarke's generalized gradient of f at $x\in U$ (see [2, 3]).

For such a functional f and real numbers $\alpha<\beta$ we denote

$$\{f<\beta\}=\{x\in U:f(x)<\beta\},$$
$$\{\alpha\leqslant f\leqslant\beta\}=\{x\in U:\alpha\leqslant f(x)\leqslant\beta\}.$$

For convenience, according to F. E. Browder[4], we introduce the following notational convention: "If a set V appears several times in a single equation or inequality, the equation or inequality is assumed to hold for each v in V with the

Manuscript received July 10, 1982. Revised June 11, 1983.
* Research supported under CSASF Grant Math-386.

same chosen at all points of occurrence of V in the given equation or inequality." For example the inequality $\langle T(x), x\rangle \geqslant \|T(x)\|^2$ denotes that $\langle y, x\rangle \geqslant \|y\|^2$ for each $y \in T(x)$.

Lemma 1. *Let X be a reflexive Banach space and $T: D\in X \to 2^{X^*}$ u. s. c. with closed convex values. Then for any given $\varepsilon>0$ there exists a continuous single-valued mapping $T_\varepsilon: D \to X$ such that*

(i) *T_ε is a locally Lipschitz mapping,*

(ii) *$\|T_\varepsilon(x)\| \leqslant \|T(x)\|$ for $x\in D$,*

(iii) *$\langle T(x), T_\varepsilon(x)\rangle \geqslant \inf\{\|y\|^2: y\in B(T(B(x, \varepsilon)), \varepsilon)\}$ for $x\in D$.*

In addition, when X is a Hilbert space, we require that

(iv) *$d^*(G_{T_\varepsilon}, G_T) \leqslant \varepsilon$.*

Proof For simplicity we suppose that D is an open subset of X, otherwise consider the relative topology on D.

Let $\varepsilon>0$ be given.

Since T is u. s. c, for each point $x\in D$ there is a positive number $\delta(x)<\varepsilon$ such that $T(B(x, \delta(x)))\subset B(T(x), \varepsilon)$.

$\mathscr{U}=\{B(x, \delta(x)): x\in D\}$ is an open cover of D.

Let $\mathscr{V}=\{v_\alpha: \alpha\in W\}$ be both a locally finite open cover of D and a star refinement of $\mathscr{U}$ where W is some indexing set. (Recall that a cover $\mathscr{V}$ of D is said to be locally finite if each point $x\in D$ has some neighborhood which intersects only finitely many members of $\mathscr{V}$; $\mathscr{V}$ is said to be a star refinement of $\mathscr{U}$ if for each $x\in D$ there is some $u_x\in \mathscr{U}$ such that $u_x \supset \bigcup\{v_\alpha: v_\alpha\in \mathscr{V}, x\in v_\alpha\}$.)

As for the existence of the cover $\mathscr{V}$ see [5].

Let $\{\varphi_\alpha: \alpha\in W\}$ be a locally Lipschitz partition of unity subordinate to $\{v_\alpha: \alpha\in W\}$, that is, for each $\alpha\in W$ $\varphi_\alpha: D\to [0, 1]$ is locally Lipschitzian,

$$\operatorname{supp}\varphi_\alpha \triangleq \{x: x\in D, \varphi_\alpha(x)\neq 0\}\subset v_\alpha \quad \text{and} \quad \sum_{\alpha\in W}\varphi_\alpha(x)=1 \quad \text{for each } x\in D.$$

For each fixed $v_\alpha\in \mathscr{V}$, let

$$S_\alpha = \bigcap\{B(T(x), \varepsilon): x\in D, B(x, \delta(x))\supset v_\alpha\}.$$

Then each S_α is a nonempty convex set with $S_\alpha\supset T(v_\alpha)$.

Let $y_\alpha\in \bar{S}_\alpha$ be such that $\|y_\alpha\|=\inf\{\|y\|: y_\alpha\in \bar{S}_\alpha\}$ and choose $z_\alpha\in X$ such that $\|z_\alpha\|=\|y_\alpha\|$ and $\langle y_\alpha, z_\alpha\rangle=\|y_\alpha\|^2$. Define $T_\varepsilon: D\to X$ by

$$T_\varepsilon(x)=\sum_{\alpha\in W}\varphi_\alpha(x)z_\alpha \quad \text{for} \quad x\in D.$$

Since $\mathscr{V}$ is locally finite, it may be seen that the single-valued mapping T_ε is well defined.

We now verify that T_ε satisfies the required conditions.

(i) Since $\mathscr{V}$ is locally finite and each φ_α is locally Lipschitzian, it follows that T_ε is locally Lipschitzian on D.

(ii) Let $x_0 \in D$ be given arbitrarily and $W_0 = \{\alpha \in W : x_0 \in v_\alpha\}$. Then W_0 is a finite set because $\mathscr{V}$ is locally finite. It is obvious that $T(x_0) \subset T(v_\alpha) \subset S_\alpha \subset \bar{S}_\alpha$ for each $\alpha \in W_0$. Thus $\|z_\alpha\| = \|y_\alpha\| = \inf\{\|y\| : y \in \bar{S}_\alpha\} \leqslant \|T(x_0)\|$ for each $\alpha \in W_0$. Hence

$$\|T_\varepsilon(x_0)\| = \|\sum_{\alpha \in W} \varphi_\alpha(x_0) z_\alpha\| = \|\sum_{\alpha \in W_0} \varphi_\alpha(x_0) z_\alpha\| \leqslant \sum_{\alpha \in W_0} \varphi_\alpha(x_0)\|z_\alpha\| \leqslant \|T(x_0)\|.$$

So T_ε satisfies condition (ii).

(iii) Let $x_0 \in D$ and W_0 be as above. Since $\mathscr{V}$ is a star refinement of $\mathscr{U}$, it follows that there is some $u_* = B(x_*, \delta(x_*)) \in \mathscr{U}$ such that $u_* \supset \bigcup_{\alpha \in W_0} v_\alpha$. By the definition of S_α, it implies that $S_\alpha \subset B(T(x_*), \varepsilon)$ for each $\alpha \in W_0$. Noting that $\|x_* - x_0\| < \delta(x_*) < \varepsilon$, we have $B(T(x_*), \varepsilon) \subset B(T(B(x_0, \varepsilon)), \varepsilon)$. Hence $y_\alpha \in \bar{S}_\alpha \subset \bar{B}(T(B(x_0, \varepsilon)), \varepsilon)$ and $\|y_\alpha\| \geqslant \inf\{\|y\| : y \in B(T(B(x_0, \varepsilon)), \varepsilon)\}$ for each $\alpha \in W_0$.

By the choices of y_α and z_α, it may be seen that $\langle y, z_\alpha \rangle \geqslant \langle y_\alpha, z_\alpha \rangle = \|y_\alpha\|^2$ for each $\alpha \in W$ and each $y \in \bar{S}_\alpha$. Especially, for each $\alpha \in W_0$ we have

$$\langle T(x_0), z_\alpha \rangle \geqslant \|y_\alpha\|^2 = \inf\{\|y\|^2 : y \in \bar{S}_\alpha\} \geqslant \inf\{\|y\|^2 : y \in B(T(B(x_0, \varepsilon)), \varepsilon)\}$$

since $T(x_0) \subset \bar{S}_\alpha$. Hence

$$\langle T(x_0), T_\varepsilon(x_0) \rangle = \langle T(x_0), \sum_{\alpha \in W_0} \varphi_\alpha(x_0) z_\alpha \rangle \geqslant \inf\{\|y\|^2 : y \in B(T(B(x_0, \varepsilon)), \varepsilon)\}.$$

Thus T_ε satisfies condition (iii).

If X is a Hilbert space, then for any $x_0 \in D$, under the notations mentioned above, we have $z_\alpha = y_\alpha \in \bar{S}_\alpha \subset \bar{B}(T(x_*), \varepsilon)$. By convexity of $B(T(x_*), \varepsilon)$, $T_\varepsilon(x_0) = \sum_{\alpha \in W_0} \varphi_\alpha(x_0) z_\alpha \in \bar{B}(T(x_*), \varepsilon)$. Noting that $\|x_* - x_0\| < \delta(x_*) < \varepsilon$, we have $d((x_0, T_\varepsilon(x_0)), G_T) \leqslant \varepsilon$ and therefore $d^*(G_{T_\varepsilon}, G_T) \leqslant \varepsilon$ since $x_0 \in D$ is arbitrary.

Lemma 2. *Let $X = R^n$ is an n-dimensional Euclidean space, U an open subset of X, $f: U \to R$ a locally Lipschitz functional and $T = \partial f: U \to 2^X$ the generalized gradient of f. Suppose that there are real numbers $\alpha < \beta$ and $r > 0$ and a point $x_0 \in U$ such that $\Omega \triangleq \{f < \beta\}$ is bounded with $\bar{\Omega} \subset U$ and $\{f \leqslant \alpha\} \subset \bar{B}(x_0, r) \subset \Omega$. Moreover suppose that $0 \bar{\in} T(\{\alpha \leqslant f \leqslant \beta\})$.*

Then $\deg(T, \Omega, 0) = 1$, *where* $\deg(T, \Omega, 0)$ *is the Cellina-Lasota degree* (see [6]).

Proof Note that $T: U \to 2^X$ is u. s. c. with compact convex values (see [2]), so $\deg(T, \Omega, 0)$ is well defined and $\deg(T, \Omega, 0) = \deg(T, B(x_0, r), 0)$ since $0 \bar{\in} T(\{\alpha \leqslant f \leqslant \beta\})$.

Without loss of generality we may assume that $\{f \leqslant \alpha\} \subset B(x_0, r)$, otherwise replace r by r' which is slightly large than r.

Let $\varepsilon_1 = \inf\{\|y\| : y \in T(\{\alpha \leqslant f \leqslant \beta\})\}$. Then $\varepsilon_1 > 0$ since $T(\{\alpha \leqslant f \leqslant \beta\})$ is compact (see [7]). By the continuity of f, there exist real numbers α' and β' such that $\alpha < \alpha' < \beta' < \beta$ and $\{f \leqslant \alpha'\} \subset B(x_0, r) \subset \{f < \beta'\}$.

Let $\delta_1 = d(\partial\{f \leqslant \beta'\}, \partial\{f < \beta\})$ and $\delta_2 = d(\partial\{f \leqslant \alpha\}, \partial\{f < \alpha'\})$. It is obvious that $\delta_1 > 0$ and $\delta_2 > 0$.

We take a positive number $\varepsilon<\min\left\{\delta_1, \delta_2, \frac{\varepsilon_1}{2}\right\}$. Let $T_\varepsilon:\overline{\Omega}\to X$ be a mapping satisfying conditions (i)—(iv) of Lemma 1 (for $D=\overline{\Omega}$). Without loss of generality we suppose that the ε is already so small that

$$\deg(T, B(x_0, r), 0)=\deg(T_\varepsilon, B(x_0, r), 0)$$

(see [6]).

Now we consider the following initial value problem of ordinary differential equation of abstract functions:

$$\begin{cases}\dfrac{du(t, x)}{dt}=-T_\varepsilon(u(t, x)),\\ u(0, x)=x.\end{cases} \tag{I}$$

It is well known that for each $x\in\overline{B}(x_0, r)$ Problem (I) has a unique solution $u(t, x)$ on $[0, t^+(x))$ where $t^+(x)=\sup\{t>0$: Problem (I) has the solution on $[0, t)\}$.

If for a fixed point $x\in\overline{B}(x_0, r)$ we denote $F(t)=f(u(t, x))$, then $F:[0, t^+(x))\to R$ is locally Lipschitzian. By Chain Rule (see [3]) we have

$$\partial F(t)\subset\langle T(u(t, x)), -T_\varepsilon(u(t, x))\rangle. \tag{1}$$

By condition (iii) and (1), F is monotone decreasing on $[0, t^+(x))$. It follows that $u(t, x)\in\{f\leqslant\beta'\}$ for $t\in[0, t^+(x))$ and therefore $t^+(x)=+\infty$.

Let $A=\{\alpha'<f\leqslant\beta\}$. Then

$$\inf\{\|y\|:y\in B(T(B(x, \varepsilon)), \varepsilon)\}\geqslant\varepsilon_1-\varepsilon>\varepsilon_1-\frac{\varepsilon_1}{2}=\frac{\varepsilon_1}{2}$$

for $x\in A$. By condition (iii) we have

$$\langle T(x), T_\varepsilon(x)\rangle\geqslant\frac{\varepsilon_1^2}{4} \quad \text{for} \quad x\in A. \tag{2}$$

From (1) and (2), it follows that if $x\in\overline{B}(x, r)$ is given and $u(t, x)\in A$ for $t\in[t_1, t_2]$ where $0\leqslant t_1<t_2$, then

$$f(u(t_1, x))-f(u(t_2, x))\geqslant\frac{\varepsilon_1^2}{4}(t_2-t_1). \tag{3}$$

Now we define $\Phi:[0, \infty)\times\overline{B}(x_0, r)\to X$ by $\Phi(t, x)=x-u(t, x)$ for $t\in[0, \infty)$ and $x\in\overline{B}(x_0, r)$, where $u(t, x)$ is the solution of (I). Then Φ is continuous.

Denote $\Phi_t(\cdot)=\Phi(t, \cdot)$. Then $0\overline{\in}\Phi_t(\partial B(x_0, r))$ for all $t>0$ by (3). According to the homotopy invariance property of Brouwer's degree, we obtain

$$\deg(\Phi_{t_1}, B(x_0, r), 0)=\deg(\Phi_{t_2}, B(x_0, r), 0), \quad \forall t_1, t_2>0.$$

From (3) it follows that there exists a sufficiently large positive number t_2 such that

$$u(t_2, x)\in\{f<\alpha'\}\subset B(x_0, r) \quad \text{for all} \quad x\in\partial B(x_0, r).$$

For $x\in\overline{B}(x_0, r)$ let $I_{x_0}(x)=x-x_0$. Then it is easy to verify that $0\overline{\in}((1-\lambda)I_{x_0}+\lambda\Phi_{t_2})(\partial B(x_0, r))$ for any $\lambda\in[0, 1]$. It follows that

$$\deg(\Phi_{t_2}, B(x_0, r), 0)=\deg(I_{x_0}, B(x_0, r), 0)=1$$

(see [8], Chapter 3).

On the other hand, by [9] (p. 101) there is a sufficiently small positive number t_1 such that

$$\Phi_{t_1}(x) \neq -\mu T_s(x) \text{ for each } x \in \partial B(x_0, r) \text{ and all } \mu > 0.$$

Because of the well-known property of Brouwer degree, it follows that

$$\deg(\Phi_{t_1}, B(x_0, r), 0) = \deg(T_s, B(x_0, r), 0).$$

Thus $\deg(T, B(x_0, r), 0) = \deg(T_s, B(x_0, r), 0) = \deg(\Phi_{t_1}, B(x_0, r) = \deg(\Phi_{t_1}, B(x_0, r), 0) = 1.$

Remark 3. In Lemma 2 (and in the following Theorem 4), if we replace the ball $B(x_0, r)$ by an open convex set, the assertion remains true.

In the following we always assume that X is a Banach space and U is an open subset of X.

Let $\Gamma = (\{X_n\}, \{P_n\}; \{X_n^*\}, \{P_n^*\})$ be an admissible injective scheme of (X, X^*) (see [10]).

$T: U \to 2^{X^*}$ is said to be A-proper (w. r. t. Γ) if for any closed set $D \subset U$ $T|_D: D \to 2^{X^*}$ is A-proper.

About the multivalued A-proper mappings and their generalized degree see [7].

Theorem 4. *Suppose that all the hypotheses of Lemma 2 except $X = R^n$ are satisfied. Moreover suppose that $\partial f: U \to 2^{X^*}$ is A-proper. Then* $\deg(\partial f, \Omega, 0) = \{1\}$.

Proof Denote $T = \partial f$, $U_n = U \cap X_n$, $\Omega_n = \Omega \cap X_n$, $f_n = f P_n: U_n \to R$, $T_n = P_n^* T P_n: U_n \to 2^{X^*}$.

Note that each bounded linear mapping $P_n: X_n \to X$ is continuously Fréchet differentiable and $dP_n(x) = P_n$ where $dP_n(x)$ is the Fréchet derivative of P_n at x (see [8], Chapter 1). By using Chain Rule (see [3]) to f_n, we obtain $\partial f_n(x) \subset T_n(x)$.

In addition, ∂f_n is u. s. c. since X is finite dimensional.

Since T is A-proper and $0 \overline{\in} T(\{\alpha \leqslant f \leqslant \beta\})$, it follows that there exists a positive integer n_0 such that $0 \overline{\in} T_n(\{\alpha \leqslant f_n \leqslant \beta\})$ for all $n \geqslant n_0$ and so $0 \overline{\in} \partial f_n(\{\alpha \leqslant f_n \leqslant \beta\})$ for $n \geqslant n_0$.

It is easy to verify that for each $n \geqslant n_0$ $f_n: U_n \to R$ satisfies all the conditions of Lemma 2. Hence $\deg(\partial f_n, \Omega_n, 0) = 1$ for $n \geqslant n_0$ by Lemma 2 and the invariance property of Brouwer degree under nonsingular C^1-coordinate transformation (see [11]).

Note that $\deg(T_n, \Omega_n, 0) = \deg(\partial f_n, \Omega_n, 0)$ since $\partial f_n(x) \subset T_n(x)$. It follows that $\mathrm{Deg}(T, \Omega, 0) = \{1\}$ by the definition of $\mathrm{Deg}(T, \Omega, 0)$ (see [7]).

Theorem 5. *Let $f: X \to R$ be locally Lipschitzian and $\partial f: X \to 2^{X^*}$ A-proper. Suppose that $f(x) \to \infty$ as $\|x\| \to \infty$. If there is some $r_0 > 0$ such that $0 \overline{\in} \partial f(x)$ for $\|x\| \geqslant r_0$, then* $\mathrm{Deg}(\partial f, B(0, r), 0) = \{1\}$ *for $r \geqslant r_0$.*

Proof Note that $\mathrm{Deg}(\partial f, B(0, r), 0) = \mathrm{Deg}(\partial f, B(0, r_0), 0)$ for $r \geqslant r_0$ since

$0\overline{\in}\partial f(x)$ for $\|x\|\geqslant r_0$. It suffices to show that Deg $(\partial f, B(0, r_0), 0)=\{1\}$.

Let T, T_n, f_n be as in the proof of Theorem 4. It is obvious that for each n, $f_n: X_n \to R$ is locally Lipschitzian, ∂f_n is u. s. c, $\partial f_n(x)\subset T_n(x)$ and $f_n(x)\to\infty$ as $\|x\|\to\infty$.

Denote $B_n=B(0, r_0)\cap X_n$. Since $T=\partial f$ is A-proper, it implies that there is $n_0\geqslant 1$ such that $0\overline{\in}T_n(X_n\backslash B_n)$ for $n\geqslant n_0$, so $0\overline{\in}\partial f_n(X_n\backslash B_n)$ for $n\geqslant n_0$.

For each n let $\alpha_n=\sup\{f_n(x): x\in \overline{B}_n\}$. Then $\alpha_n<+\infty$ since f_n is continuous. Let $r_n=\sup\{\|x\|: x\in\{f_n\leqslant\alpha_n\}\}$. Then $r_n<+\infty$ since $f_n(x)\to\infty$ as $\|x\|\to\infty$. Let $\beta_n=1+\sup\{f_n(x): x\in X_n, \|x\|\leqslant r_n\}$ and $\Omega_n=\{f_n\leqslant\beta_n\}$. Then similarly $\beta_n<+\infty$ and Ω_n is bounded.

Thus for each $n\geqslant n_0$ f_n satisfies all the conditions of Lemma 2. By Lemma 2 $\deg(\partial f_n, \Omega_n, 0)=1$, so $\deg(\partial f_n, B_n, 0)=1$ and $\deg(T_n, B_n, 0)=1$. Hence $\deg(T, B(0, r_0), 0)=\{1\}$.

Lemma 6. *Let $f: U\to R$ be locally Lipschitzian and $D\subset U$. Suppose that $\|\partial f(x)\|\geqslant b$ for $x\in D$ where b is a positive constant. Then there exists a locally Lipschitz mapping $P: D\to X$ such that*

(i) $\|P(x)\|\leqslant 1$ for $x\in D$,

(ii) $\langle\partial f(x), P(x)\rangle>\frac{1}{2}b$ for $x\in D$.

The proof of Lemma 6 is similar to the corresponding proof of Lemma 3.3 in [12].

Let $f: U\to R$ be locally Lipschitzian.

$x_0\in U$ is said to be a critical point of f if $0\in\partial f(x_0)$.

We say that f satisfies Condition $(P.S)$ if each sequence $\{x_n\}$ along which $\{f(x_n)\}$ is bounded and

$$\|\partial f(x_n)\|_*\triangleq\inf\{\|y\|: y\in\partial f(x_n)\}\to 0 \quad (n\to\infty)$$

has a convergent subsequence (see [12]).

If x_0 is an isolated critical point of f and ∂f is A-proper, then we denote by $I(\partial f, x_0)$ the $\mathrm{Deg}(\partial f, \Omega, 0)$, where Ω is a sufficiently small open neighborhood of x_0.

Lemma 7. *Let $f: U\to R$ be locblly Lipschitzian. Suppose that $x_0\in U$ is an isolated critical point of f at which f has a local minimum. Moreover suppose that f satisfies Condition $(P.S)$. Then there exists some $R_0>0$ such that*

$$\inf\{f(x): r_1\leqslant\|x-x_0\|\leqslant r_2\}>f(x_0) \quad \textit{whenever} \quad 0<r_1<r_2<R_0.$$

Proof We can assume that $x_0=0$ and $f(0)=0$.

By our hypotheses there exests $R_0>0$ such that

(1) $x_0=0$ is the only critical point of f in $B(0, R_0)\subset U$ at which f has the global minimum in $B(0, R_0)$,

(2) $f(x)\leqslant M$ for $x\in B(0, R_0)$ where M is a positive constant.

Let $0<r_1<r_2<R_0$. Choose $\delta>0$ such that $r_1-2\delta>0$ and $r_2+2\delta<R_0$. Denote

$$D=\{x\in U: r_1-2\delta\leqslant\|x\|\leqslant r_2+2\delta\}.$$

Since f satisfies Condition (*P. S.*), it implies that there exists a constant $b>0$ such that $\|\partial f(x)\|\geqslant b$ for $x\in D$.

Let $P: D\to X$ be as mentioned in Lemma 6.

Now we consider the following Problem (II):

$$\begin{cases}\dfrac{du(t,x)}{dt}=-P(u(t,x)),\\ u(0,x)=x.\end{cases}\tag{II}$$

Let x be given with $r_1\leqslant\|x\|\leqslant r_2$. For this initial value x, Problem (II) has a unique solution $u(t,x)$ on $[0, t^+(x))$ where $t^+(x)=\sup\{t>0$: (II) has the solution on $[0, t)\}$.

We claim that there exists $t_*\in(0, t^+(x))$ such that either i) $\|u(t_*, x)\|<r_1-\delta$ or ii) $\|u(t_*, x)\|>r_2+\delta$. Otherwise $r_1-\delta\leqslant\|u(t,x)\|\leqslant r_2+\delta$ for $t\subset[0, t^+(x))$. It follows that $t^+(x)=+\infty$ since $\|p(x)\|\leqslant 1$ for $x\in D$. But $f(x)-f(u(t,x))\geqslant\frac{b}{2}t$ (see [12]). This contradicts the fact that $f(x)\leqslant M$ for $x\in B(0, R_0)$, so the claim is true.

In case i) there are two numbers t_1 and t_2 with $0\leqslant t_1<t_2<t^+(x)$ such that $\|u(t_1,x)\|=r_1$ and $\|u(t_2,x)\|=r_1-\delta$. Then

$$f(u(t_1,x))-f(u(t_2,x))\geqslant\frac{b}{2}(t_2-t_1)\geqslant\frac{b}{2}\|u(t_1,x)-u(t_2,x)\|\geqslant\frac{b}{2}\delta.$$

It implies that $f(x)=f(u(0,x))\geqslant\frac{b}{2}\delta$.

Similarly, in case ii) we have also $f(x)\geqslant\frac{b}{2}\delta$. Hence

$$\inf\{f(x): r_1\leqslant\|x\|\leqslant r_2\}\geqslant\frac{b}{2}\delta>0.$$

Theorem 8. *Suppose that all the hypotheses of Lemma 7 are satisfied. If $\partial f: U\to 2^{X^*}$ is A-proper, then $I(\partial f, x_0)=\{1\}$.*

Proof We assume still that $x_0=0$ and $f(x_0)=0$.

Let R_0 be as mentioned in the proof of Lemma 7. Let r_1 and r_2 be fixed with $0<r_1<r_2<R_0$. Then $\inf\{f(x): r_1\leqslant\|x\|\leqslant r_2\}=\beta>0$ by Lemma 7.

Noting that $\{f<\beta\}$ is an open set containing $0\in X$, we may choose an $r>0$ such that $\overline{B}(0, r)\subset\{f<\beta\}$.

Let $\alpha=\frac{1}{2}\inf\{f(x): r\leqslant\|x\|\leqslant r_2\}$. By using Theorem 4 to $f|B(0, r_2)$ we obtain

$$I(\partial f, 0)=\mathrm{Deg}(\partial f, B(0, r), 0)=\{1\}.$$

Lemma 9. *Let Ω be a bounded open subset of X, $f:\overline{\Omega}\to R$ locally Lipschitzian and $\partial f:\overline{\Omega}\to 2^{X^*}$ A-proper. Suppose that f is bounded from below on $\partial\Omega$. Then f is also bounded from below on $\overline{\Omega}$.*

Proof Suppose that the assertion of Lemma 9 is false. Then there exists a sequence $\{x_k\}\subset\Omega$ such that $f(x_k)\to-\infty$ as $k\to\infty$. By the definition of injective scheme (see [10]) and the continuity of f, we may find a sequence $\{x_{n(k)}\}$ such that

i) $n(k)\to\infty$ as $k\to\infty$,

ii) $x_{n(k)}\in\Omega_{n(k)}=\Omega\cap X_{n(k)}$,

iii) $|f_{n(k)}(x_{n(k)})-f(x_k)|<\frac{1}{k}$, where $f_{n(k)}=f|_{\bar{\Omega}_{n(k)}}$.

It is clear that $f_{n(k)}(x_{n(k)})\to-\infty$ as $k\to\infty$.

For each $n(k)$, $f_{n(k)}:\bar{\Omega}_{n(k)}\to R$ attains its global minimum in $\bar{\Omega}_{n(k)}$ at some point $x'_{n(k)}\in\bar{\Omega}_{n(k)}$. Of course, $f_{n(k)}(x'_{n(k)})\leqslant f_{n(k)}(x_{n(k)})$ and consequently $f_{n(k)}(x'_{n(k)})\to-\infty$ as $k\to\infty$. Without loss of generality we can assume that $x'_{n(k)}\in\Omega_{n(k)}$ for erery $n(k)$ since f is bounded from below on $\partial\Omega$. Then $0\in\partial f_{n(k)}(x'_{n(k)})$ (see [3]), therefore $0\in(\partial f)_{n(k)}(x'_{n(k)})$ for each $n(k)$.

Since ∂f is A-proper, it follows that there exists a subsequence $\{x'_{n(k)(j)}\}\subset\{x'_{n(k)}\}$ such that $n(k)(j)\to\infty$ and $P_{n(k)(j)}x'_{n(k)(j)}\to x\in\bar{\Omega}$. By the continuity of f we obtain

$$f(x)=\lim f_{n(k)(j)}(x'_{n(k)(j)})=-\infty.$$

This is absurd. Hence the assertion of Lemma 9 is true.

Lemma 10. *Under the hypotheses of Lemma 9, if* $\inf\{f(x):x\in\bar{\Omega}\}<\inf\{f(x):x\in\partial\Omega\}$, *then f attains its infimum in $\bar{\Omega}$ at some point $x_*\in\Omega$.*

Proof Let $c=\inf\{f(x):x\in\bar{\Omega}\}$. Then $c>-\infty$ by Lemma 9. By the same arguments as in the proof of Lemma 9, we may show that there exists some $x_*\in\Omega$ such that $f(x_*)=c$.

Corollary 11. *Suppose that all the hypotheses of Theorem 4 are satisfied and $x_1\in\Omega$ is a local minimum point of f, which is not a global minimum point of f in Ω. Moreover suppose that f satisfies Condition (P. S) on $\bar{\Omega}$. Then f has at least three critical points in Ω.*

Proof First we note that x_1 is a critical point of f (see [3]).

Since $f(x)=\beta>-\infty$ for $x\in\partial\Omega$, it follows that f is bounded from below on $\bar{\Omega}$ by Lemma 9. Since $\inf\{f(x):x\in\bar{\Omega}\}<\alpha<\beta=\inf\{f(x):x\in\partial\Omega\}$, f attains its infimum in $\bar{\Omega}$ at some $x_*\in\Omega$ by Lemma 10. Thus $x_*\neq x_1$ is another critical point of f. If x_1 and x_* are the only critical points of f in Ω, then

$$\operatorname{Deg}(\partial f,\Omega,0)=I(\partial f,x_1)+I(\partial f,x_*)=\{1\}+\{1\}=\{2\}$$

by Theorem 8 and the additivity of generalized degree (see [7]). This contradicts Theorem 4, so f has at least three critical points in Ω.

Coaollary 12. *Let $f:X\to R$ be locally Lipschitzian and $\partial f:X\to 2^{X^*}$ A-proper. Suppose that $f(x)\to\infty$ as $\|x\|\to\infty$ and x_1 is a local minimum point of f which is not a global minimum point of f. Moreover suppose that f satisfies Condition (P. S). Then f has at least three critical points.*

Proof This follows by using Theorem 5 instead of Theorem 4 in the proof of

Corollary 11.

The results obtained in this paper may be extended to the case that generalized gradient mappings are the uniform limits of multivalued A-proper mappings.

I am very grateful to Professor Chen Wenyuan, Professor Li shujie and Professor Chang Kungching for their useful directions.

Reference

[1] Amann. H., A note on degree theory for gradient mappings, *Proc. Amer. Math. Soc.*, **85** (1982), 591—595.

[2] Clarke. F. H., Generalized gradients and applications, *Trans. Amer. Math. Soc.*, **205** (1975), 247—262.

[3] Clarke. F. H., Generalized gradients of Lipschitz functionals *Adv. Math.*, **40** (1981), 52—67.

[4] Browder. F. E., Multi-valued monotone nonliear mappings and duality mappings in Banach spaces, *Trans. Amer. soc.*, **118** (1965), 338—351.

[5] Willard. S., General topology, Addison-Wesley, Reading, Mass., 1970.

[6] Cellina. A. and Lasota. A., A new approach to the definition of topological degree for multi-valued mappings, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **47**: 8 (1969), 434—440.

[7] Massabò. I. and Nistri. P., A topological degree for multivalued A-proper maps in Banach spaces, *Boll. Un. Mat. Ital.*, **13-B**: 5 (1976), 672—685.

[8] Chen Wenyuan, Nonlinear functional analysis, "Cansu Renmin", (1982) (Ccinese).

[9] Krasnosel'skiĭ. M. A., Translation along trajectories of differential equations, "Nauka". Mosscow. (1966) (Russian).

[10] Browder. F. E. and Petryshyn. W. V., Approximation methods and the generalized topological degree for nonlinear mappings in Banach spaces, *J. Functional Anal.*, **3** (1969), 217—245.

[11] Lioyd. N. G., Degree theory, *Cambridge Tracts in Math.*, **73** (1978).

[12] Chang. K. C., Variational method for non-differentiable functional and their applications to partial differential equations, *J. Math. Anal. Appl.*, **80** (1981), 102—129.