Chin. Ann. of Math,
5B (4) 1984

SUBMANIFOLDS WITH NONNEGATIVE SECTIONAL CURVATURE*

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Abstract

This paper gives some sufficient conditions for a compact submanifold with nonnegative sectional curvature in a space form to be totally umbilical. In particular, for a compact submanifold M with flat normal bundle, if the scalar curvature is proportional to the mean curvature everywhere, then M is totally umbilical or the Riemannian product of several totally umbilical constantly curved submanifolds.

Introduction

Let M be a compact hypersurface with nonnegative sectional curvature in a space form. A theorem of Nomizu and Smyth[1] says that if M has constant mean curvature, then M is either totally umbilical or the Riemannian product of two totally umbilical constantly curved submanifolds. Generalizations of this theorem have been attempted by many authors (cf. [2, 3, 4], etc.). Recently, Huang [5] proved two propositions of this type as follows:

Theorem A. *Let M be an n-dimensional compact and connected hypersurface with nonnegative sectional curvature in a unit sphere. Suppose that there is a constant k such that $R=kH$, where R and H are the scalar curvature and the mean curvature of M, respectively. If $k>2\sqrt{n^3(n-1)}$, then M is eithr totally umbilical or the Riemannian product of two totally umbilical submanifolds.*

Theorem B. *Let M be an n-dimensional compact and connected submanifold with parallel mean curvature vector $\xi(\neq 0)$ in a unit sphere. If the sectional curvature of M is not less than $\frac{1}{2}(1+\|\xi\|^2)$, then M is totally umbilical.*

In this paper, we propose to extend Theorem A to higher codimension and to improve the pinching constant in Theorem B.

Let $S^{n+p}(\tilde{c})$ denote an $(n+p)$-space form with constant curvature $\tilde{c}$. Suppose that M is an n-dimensional compact and connected submanifold with nonnegative sectional

Manuscript received August 4, 1982.
* Projects Supported by the Science Fund of the Chinese Academg of Sciences.

curvature, immersed in $S^{n+p}(\tilde{c})$ with flat normal bundle. Suppose that there is a constant k such that $R=kH$, where R and H are the scalar curvature and the mean curvature of M, respectively. Then we prove that if $k^2>4n^3(n-1)\tilde{c}$, M is either totally umbilical or the Riemannian product of several totally umbilical constantly curved submanifolds. As a corollary, one sees that if M has positive sectional curvature, M must be totally umbilical (and isometric to a standard sphere).

Let M be an n-dimensional compact and connected submanifold with parallel mean curvature vector $\xi(\neq 0)$ in $S^{n+p}(\tilde{c})$ with $\tilde{c}\geqslant 0$ and $p>1$. Then we prove that M is totally umbilical if the sectional curvature of M is larger than $\frac{1}{2}\mu(\tilde{c}+\|\xi\|^2)$, where $\mu=\min\left\{\frac{2p-4}{2p-3}, \frac{n}{n+1}\right\}$. In this case, M must lie in a totally geodesic $S^{n+1}(\tilde{c})$.

It is, of course, possible to generalize these results to submanifolds in complex space forms.

§ 1. Preliminaries

First of all, we begin with the self-contained discussion about Riemannian submanifolds following closely the exposition in [4]. Let $S^{n+p}(\tilde{c})$ be an $(n+p)$-space form of constant curvature $\tilde{c}$ and M an n-dimensional Riemannian manifold isometrically immersed in $S^{n+p}(\tilde{c})$. We choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}$ in $S^{n+p}(\tilde{c})$ such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M. From now on we agree on the following ranges of indices:

$$1\leqslant i, j, k, \cdots, \leqslant n; \quad n+1\leqslant \alpha, \beta, \gamma, \cdots, \leqslant n+p.$$

Let $\omega_1, \cdots, \omega_{n+p}$ be the field of dual frames relative to the frame field of $S^{n+p}(\tilde{c})$ chosen above. We restrict these forms to M. Then (cf. [4])

$$\omega_\alpha=0, \qquad \omega_{\alpha i}=\sum_j h^\alpha_{ij}\omega_j, \qquad h^\alpha_{ij}=h^\alpha_{ji}. \tag{1.1}$$

The second fundamental form of M is definded by $\sum h^\alpha_{ij}\omega_i\otimes\omega_j\otimes e_\alpha$. For each α, let H_α be the matrix (h^α_{ij}). We call $\xi=\frac{1}{n}\sum_\alpha(\mathrm{tr}\ H_\alpha)e_\alpha$ the mean curvature vector, where "tr" denotes the trace of the matrix. The length of ξ

$$\|\xi\|=\frac{1}{n}\sqrt{\sum_\alpha(\mathrm{tr}\ H_\alpha)^2} \tag{1.2}$$

is called the mean curvature (up to a sign). M is said to be totally umbilical if, for each α, all of the eignvalues of H_α are equal. In particular, M is totally geodesic if the second fundamental form vanishes identically.

Let R denote the scalar curvature of M. From the Gauss equation for M we have

$$R=n(n-1)\tilde{c}+n^2\|\xi\|^2-\sigma, \tag{1.3}$$

where

$$\sigma=\sum_\alpha \mathrm{tr}\ (H^2_\alpha) \tag{1.4}$$

is called the square of the length of the second fundamental form of M.

Let h^α_{ijk} and h^α_{ijkl} de the first and the second covariant derivative of h^α_{ij}, respectively. Then (cf. [4])

$$h^\alpha_{ijk}=h^\alpha_{ikj}, \tag{1.5}$$

$$h^\alpha_{ijkl}-h^\alpha_{ijlk}=\sum_m h^\alpha_{im}R_{mjkl}+\sum_m h^\alpha_{mj}R_{mikl}+\sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{1.6}$$

We now choose e_{n+p} such that its direction coincides with that of ξ. Then

$$\operatorname{tr} H_\beta=0, \quad (\beta\neq n+p) \tag{1.7}$$

$$\operatorname{tr} H_{n+p}=nH, \tag{1.8}$$

where H is the mean curvature of M so that $H^2=\|\xi\|^2$.

The vector e_{n+p} is parallel in the normal bundle over M if

$$\omega_{\alpha,n+p}=0. \tag{1.9}$$

In this case, we have (cf. [4], § 2 and § 7)

$$R_{n+p,\alpha jk}=0, \tag{1.10}$$

$$\Delta h^\beta_{ij}=\sum_k\Big(\sum_m h^\beta_{mk}R_{mijk}+\sum_m h^\beta_{im}R_{mkjk}+\sum_{\gamma\neq n+p}h^\gamma_{ki}R_{\gamma\beta jk}\Big), \quad (\beta\neq n+p) \tag{1.11}$$

$$\Delta h^{n+p}_{ij}=nH_{ij}+\sum_{k,m}(h^{n+p}_{mk}R_{mijk}+h^{n+p}_{im}R_{mkjk}), \tag{1.12}$$

where $\Delta h^\alpha_{ij}=\sum_k h^\alpha_{ijkk}$ and H_{ij} is the second covariant derivative of H.

We define τ by

$$\tau=\sigma-\operatorname{tr}(H^2_{n+p})=\sum_{\beta\neq n+p}\operatorname{tr}(H^2_\beta), \tag{1.13}$$

which is independent of the choice of the frame fields.

We introduce the quantities σ_{n+p} and ρ_{n+p} by

$$\sigma_{n+p}=\operatorname{tr}(H^2_{n+p}) \tag{1.14}$$

and

$$\rho_{n+p}=n^2H^2-\sigma_{n+p}. \tag{1.15}$$

It is easy to see that $\sigma_{n+p}\geq 0$ with the equality holding if and only if $h^{n+p}_{ij}=0$, i. e., M is totally geodesic relative to the normalized mean curvature vector e_{n+p}, so that M is minimal.

§ 2. Submanifolds with Flat Normal Bundle

Let M be an n-dimensional submanifold in $S^{n+p}(\tilde{c})$ with flat normal bundle. Then, similar to (1.9) and (1.10), we have

$$\omega_{\alpha\beta}=0, \quad R_{\alpha\beta jk}=0. \tag{2.1}$$

In this case, all of H_α's can be simultaneously diagonalizable[11], i. e.,

$$h^\alpha_{ij}=\lambda^\alpha_i\delta_{ij}, \tag{2.2}$$

where λ^α_i are the eigenvalues of H_α. From (1.8) and (1.15) we get

$$\rho_{n+p}=\sum_{i\neq j}\lambda^{n+p}_i\lambda^{n+p}_j. \tag{2.3}$$

Theorem 1. *Let M be an n-dimensional compact and connected submanifold with nonnegative sectional curvature, immersed in $S^{n+p}(\tilde{c})$ with flat normal bundle. Suppose that ther is a constant k such that $R=kH$, where R and H are the scalar curvature and the mean curvature for M, respectively. If $k^2>4n^3(n-1)\tilde{c}$, then M is either totally umbilical or the Riemannian product of several totally umbilical constantly curved submanifolds, i. e.,*

$$M=S^{q_1}\times\cdots\times S^{q_r},\quad \left(\sum_{i=1}^{r}q_i=n\right),$$

where each $S^{q_i}(i=1,\cdots,r)$ is a q_i-dimensional totally umbilical constantly curved submanifold.

Proof First of all, for any $\beta(\neq n+p)$, from (2.1), (2.2) and (1.11) we have

$$\sum_{i,j}h^\beta_{ij}\Delta h^\beta_{ij}=\sum_{i,j,k,l}h^\beta_{ij}(h^\beta_{kl}R_{lijk}+h^\beta_{li}R_{lkjk})=\frac{1}{2}\sum_{i,j}(\lambda^\beta_i-\lambda^\beta_j)^2R_{ijij}, \tag{2.4}$$

which gives rise to

$$\frac{1}{2}\Delta\tau=\sum_{\substack{i,j,k\\ \beta\neq n+p}}(h^\beta_{ijk})^2+\frac{1}{2}\sum_{\substack{i,j\\ \beta\neq n+p}}(\lambda^\beta_i-\lambda^\beta_j)^2R_{ijij}, \tag{2.5}$$

where τ is defined by (1.13). Since M is compact and $R_{ijij}\geqslant0$, it follows from Hopf's maximum principle that

$$\tau=\text{constant}. \tag{2.6}$$

Since $R=kH$, it is easy to see from (1.3), (1.13) and (1.14) that

$$kH=n(n-1)\tilde{c}+n^2H^2-\sigma_{n+p}-\tau, \tag{2.7}$$

which together with (1.15) implies that

$$kH=\rho_{n+p}+n(n-1)\tilde{c}-\tau, \tag{2.8}$$

from which and (2.6) we get

$$k\,dH=d\rho_{n+p}. \tag{2.9}$$

On the other hand, from (1.12), (2.1) and (1.15) we have

$$\frac{1}{2}n^2\Delta H^2=\frac{1}{2}\Delta\rho_{n+p}+\sum_{i,j,k}(h^{n+p}_{ijk})^2+n\sum_i\lambda^{n+p}_iH_{ii}+\frac{1}{2}\sum_{i,j}(\lambda^{n+p}_i-\lambda^{n+p}_j)^2R_{ijij}. \tag{2.10}$$

Since the normal bundle over M is flat, it can be seen from (1.5) and (2.1) that the operator $\square$ defined by

$$\square f=\sum_{i,j}(nH\delta_{ij}-h^{n+p}_{ij})f_{ij}$$

is self-adjoint (cf. [2], § 1). Then, from (2.10) we obtain (cf. [2], § 2)

$$n\square H=\frac{1}{2}\Delta\rho_{n+p}+\sum_{i,j,k}(h^{n+p}_{ijk})^2-n^2\|dH\|^2+\frac{1}{2}\sum_{i,j}(\lambda^{n+p}_i-\lambda^{n+p}_j)^2R_{ijij}, \tag{2.11}$$

from which we conclude that

$$\int_M\left\{\sum_{i,j,k}(h^{n+p}_{ijk})^2-n^2\|dH\|^2+\frac{1}{2}\sum_{i,j}(\lambda^{n+p}_i-\lambda^{n+p}_j)^2R_{ijij}\right\}*1=0. \tag{2.12}$$

Exterior differentiating (1.15) and using (2.1), we get

$$\rho_{n+p,k}=2n^2HH_{,k}-2\sum_{i,j}h^{n+p}_{ij}h^{n+p}_{ijk},$$

from which together with (2.9) it follows that

$$\sum_k\Big(\sum_{i,j} h_{ij}^{n+p}h_{ijk}^{n+p}\Big)^2=\Big(n^4H^2-n^2kH+\frac{1}{4}k^2\Big)\|dH\|^2. \tag{2.13}$$

Making use of the Schwarz inequality

$$\sum_k\Big(\sum_{i,j} h_{ij}^{n+p}h_{ijk}^{n+p}\Big)^2\leqslant\sigma_{n+p}\sum_{i,j,k}(h_{ijk}^{n+p})^2, \tag{2.14}$$

we get from (2.13)

$$\sigma_{n+p}\Big\{\sum_{i,j,k}(h_{ijk}^{n+p})^2-n^2\|dH\|^2\Big\}\geqslant\Big\{n^2(n^2H^2-kH-\sigma_{n+p})+\frac{1}{4}k^2\Big\}\|dH\|^2, \tag{2.15}$$

which together with (2.7) gives

$$\sigma_{n+p}\{\sum_{i,j,k}(h_{ijk}^{n+p})^2-n^2\|dH\|^2\}\geqslant\frac{1}{4}\{k^2-4n^2[n(n-1)\tilde{c}-\tau]\}\|dH\|^2. \tag{2.16}$$

If M is minimal (and, hence $\tilde{c}>0$), then the theorem is trivial according to Theorem 5.2 in [3].

From now on we assume that M is not minimal. By using the continuity of H and the connectedness of M, it follows that the connected component of $\sigma_{n+p}>0$ becomes M itself. Hence, $\sigma_{n+p}>0$ on M everywhere.

Substituting (2.16) into (2.12), we get

$$\frac{1}{4}\int_M\{k^2-4n^2[n(n-1)\tilde{c}-\tau]\}\frac{\|dH\|^2}{\sigma_{n+p}}*1+\frac{1}{2}\int_M\{\sum_{i,j}(\lambda_i^{n+p}-\lambda_j^{n+p})^2R_{ijij}\}*1\leqslant 0. \tag{2.17}$$

On the other hand, since $k^2>4n^3(n-1)\tilde{c}\geqslant 4n^2[n(n-1)\tilde{c}-\tau]$ and $R_{ijij}\geqslant 0$, then the left hand side of (2.17) is nonnegative so that (2.17) yields that

$$\|dH\|^2=0,\quad H=\text{constant}. \tag{2.18}$$

Because M has the flat normal bundle, The mean curvature vector ξ of M is parallel. Thus, our theorem is followed immediately from Lemma 2.8 of [3]. This completes the proof of Theorem 1.

Remark. In 1971, Yano and Ishihara[3] proved that a compact submanifold in $S^{n+p}(\tilde{c}\geqslant 0)$ with flat normal bundle, constant mean curvature and nonnegative sectional curvature is either totally umbilical or the Riemannian product of several totally umbilical submanifolds. In 1973—1974, Smyth[12] and Yau[4] generalized the above proposition, independently. In 1981, we shown[10] that the hypothesis of constant mean curvature in [3] can be replaced by that of constant scalar curvature. Now, we obtain the same conclusion when the scalar curvature is proportional to the mean curvature everywhere. The similar propositions for the compact hypersurfaces have been obtained by Cheng and Yau[2] and Huang[5].

From (2.17) it is sufficient that $k^2>4n^2[n(n-1)\tilde{c}-\tau]$. Thus, we have

Corollary 1.1. *Under the same conditions as in Theorem 1, if*

$$k^2>4n^2[n(n-1)\tilde{c}-\tau],$$

then the same conclusion holds.

Moreover, we have

Corollary 1.2. *Under the same conditions as in Theorem 1, if the sectional curvature of M is positive everywhere, then M must be totally umbilical (and isometric to a standard sphere).*

Proof Since $R_{ijij}>0$, it is easy to see from (2.5) that for any $\beta(\neq n+p)$, all of λ_i^{β}'s are equal, i. e., M is totally umbilical with respect to each $e_\beta(\beta\neq n+p)$.

On the other hand, it follows from the second integral in the left hand side of (2.17) that all of λ_i^{n+p}'s are equal, i. e., M is totolly umbilical with respect to e_{n+p}, too. Hence, the corollary holds.

§ 3. Submanifolds with Parallel Mean Curvature Vector

Let M be a submanifold in $S^{n+p}(\tilde{c})$. It is said that M has parallel mean curvature vector ξ if the normalized vector of ξ is parallel in the normal bundle over M and $\|\xi\|=\text{constant}$.

Theorem 2. *Let M be an n-dimensional compact and connected submanifold with parallel mean curvature vector $\xi\,(\neq 0)$ in $S^{n+p}(\tilde{c})$ with $\tilde{c}\geqslant 0$ and $p>1$. Then, M is totally umbilical if the sectional curvatures of M are larger than $\frac{1}{2}\mu(\tilde{c}+\|\xi\|^2)$, where $\mu=\min\left\{\frac{2p-4}{2p-3},\ \frac{n}{n+1}\right\}$. In this case, M lies in a totally geodesic $S^{n+1}(\tilde{c})$.*

Proof We take $e_{n+p}=\xi/\|\xi\|$ as in § 1. Since, by the hypothesis, the sectional curvatures of M are positive, it follows from Theorem 9 in [4] that M is pseudo-umbilical, i. e.,

$$h_{ij}^{n+p}=H\delta_{ij}. \tag{3.1}$$

Combining this with the fact that e_{n+p} is parallel, we see from Theorem 1 and Theorem 9 in [4] that M is a minimal submanifold in an $(n+p-1)$-dimensional totally umbilical hypersurface $S^{n+p+1}(c')$ of $S^{n+p}(\tilde{c})$ with constant sectional curvature $c'=\tilde{c}+\|\xi\|^2$ in such a way that e_{n+p} is everywhere perpendicular to $S^{n+p-1}(c')$.

If $\frac{2p-4}{2p-3}\leqslant\frac{n}{n+1}$, then the condition in the hypothesis of Theorem 2 implies that the sectional curvatures of M are larger than $\frac{p-2}{2p-3}c'$. By Theorem 15 in [4] one sees that M is a totally geodesic submanifold in $S^{n+p-1}(c')$, which together with Theorem 1 in [4] yields that M lies in an $(n+1)$-dimensional totally geodesic submanifold $S^{n+1}(\tilde{c})$ such that the fiber of the normal subbundle $e_{n+1}\otimes\cdots\otimes e_{n+p-1}$ to M is perpendicular to $S^{n+1}(\tilde{c})$ everywhere. Hence, combining this with (3.1), we conclude that M is totally umbilical.

If $\frac{2p-4}{2p-3}>\frac{n}{n+1}$, then, in the same way, our theorem follows from the following

Lemma. *Let M be an n-dimensional compact minimal submanifold in $S^N(c')$. If the sectional curvatures of M are larger than $\frac{n}{2(n+1)}c'$ everywhere, then M is totally geodesic.*

Proof Let K denote the function which assigns to each point of M the infimum of the sectional curvatures of M at that point, and let $\sigma'=\sum \operatorname{tr} H_\beta^2$ be the square of the length of the second fundamental form of M. By Yau's formulas (10.5) and (10.9) in [4], we have

$$\begin{aligned}\sum h_{ij}^\beta \Delta h_{ij}^\beta \geqslant (1+a)nK\sigma' - (1-a)\sum\{\operatorname{tr}(H_\beta^2 H_\gamma^2) - \operatorname{tr}(H_\beta H_\gamma)^2\} \\ + a\{\sum[\operatorname{tr}(H_\beta H_\gamma)]^2 - nc'\sigma'\}\end{aligned} \tag{3.2}$$

for any real number $a \geqslant -1$.

On the other hand, according to the proposition 1 in [7] we have

$$\sum[\operatorname{tr}(H_\beta H_\gamma)]^2 \geqslant \frac{2}{n}\sum\{\operatorname{tr}(H_\beta^2 H_\gamma^2) - \operatorname{tr}(H_\beta H_\gamma)^2\}, \tag{3.3}$$

from which together with (3.2), for $0 \leqslant a \leqslant 1$, it follows that

$$\sum h_{ij}^\beta \Delta h_{ij}^\beta \geqslant (1+a)nK\sigma' + 2\left(\frac{a}{n} - \frac{1-a}{2}\right)\sum\{\operatorname{tr}(H_\beta^2 H_\gamma^2) - \operatorname{tr}(H_\beta H_\gamma)^2\} - nac'\sigma'. \tag{3.4}$$

By taking $a=\frac{n}{n+2}$ in (3.4), we get

$$\frac{1}{2}\Delta\sigma' \geqslant \sum (h_{ijk}^\beta)^2 + \frac{2(n+1)}{n+2}n\sigma'\left\{K - \frac{n}{2(n+1)}c'\right\}. \tag{3.5}$$

Thus, under conditions in the hypothesis of Lemma, (3.5) implies that $\sigma'=0$ on M, i. e., M is totally geodesic. Lemma is proved.

Remark. Obviously, $\mu=\min\left\{\frac{2p-4}{2p-3}, \frac{n}{n+1}\right\}<1$. Hence, our theorem improves the pinching constant obtained by Huang in [5].

Corollary 2.1. *Let M be an n-dimensional compact and connected submanifold in $S^{n+2}(\tilde{c}\geqslant 0)$ with nonzero parallel mean curvature vector. Then, M is isometric to an n-sphere if M has positive sectional curvature.*

This corollary may be regarded as a generalization of Hopf's theorem[8] for the convex hypersurfrce to the submanifold of codimension 2.

Corollary 2.2. *Let M be an n-dimensional compact and connected submanifold with parallel mean curvature vector ξ in $S^{n+p}(\tilde{c})$ with $\tilde{c}<0$ and $p>1$. Then, M is totally umbilical if the sectional curvature of M is positive and larger than $\frac{1}{2}\mu(\tilde{c}+\|\xi\|^2)$, where $\mu=\min\left\{\frac{2p-4}{2p-3}, \frac{n}{u+1}\right\}$.*

This is an improvement of the result given by the author in [9].

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