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INITIAL VALUE PROBLEMS FOR NONLINEAR DEGENERATE SYSTEMS OF FILTRATION TYPE

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Abstract

In this paper, the periodic boundary problems and the initial value problems for the nonlinear system of parabolic type $u_t=(\mathrm{grad}\,\varphi(u))_{xx}$ are studied, where $u=(u_1, \cdots, u_N)$ is an N-dimensional vector valued function, $\varphi(u)$ is a strict convex function of vector variable u, and its matrix of derivatives of second order is zero-definite at $u=0$. This system is degenerate. The definition of the generalized solution of the problems: $u(x, t)\in L_\infty((0, T); L_2(\mathbf{R}))$, $\mathrm{grad}\,\varphi(u)\in L_\infty((0, T); W_2^{(1)}(\mathbf{R}))$ and it satisfies appropriate integral relation. The existence and uniqueness of the generalized solution of the problem are proved. When $N=1$, the system is the commonly so-called degenerate partial differential equation of filtration type.

§ 1

The purpose of this paper is to study the existence and uniqueness of the solutions of the nonlinear systems of partial differential equations

$$u_t=(\mathrm{grad}\,\varphi(u))_{xx} \tag{1}$$

for the periodic boundary problems

$$\begin{aligned} &u(x+2D, t)=u(x, t),\\ &u(x, 0)=u_0(x) \quad (-D\leqslant x\leqslant D) \end{aligned} \tag{2}$$

and for the initial value problems

$$u(x, 0)=u_0(x) \quad (-\infty<x<\infty), \tag{3}$$

where $u(x, t)=(u_1(x, t), \cdots, u_N(x, t))$ is an N-dimensional vector valued function, $\varphi(u)=\varphi(u_1, \cdots, u_N)$ is a scalar function of the vector variable u, $u_0(x)=(u_{01}(x), \cdots, u_{0N}(x))$ is an N-dimensional vector valued function which is periodic with period $2D$ in the case of periodic boundary problems and is given in $\mathbf{R}=(-\infty, \infty)$ in the case of initial value problems, "grad" denotes the gradient opertor with respect to the vector variable u. Suppose that $\varphi(u)$ is a convex function. This means that the Hessian matrix $H(\varphi)=(\varphi_{ij})$, composed of all the derivatives of $\varphi(u)$ of second order, is an $N\times N$ non-negatively definite matrix where $\varphi_{ij}=\dfrac{\partial^2\varphi(u)}{\partial u_i\partial u_j}$ $(i, j=1, \cdots, N)$. Hence

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(1) is a nonlinear degenerate parabolic system of partial differential equations.

When $N=1$, the above system (1) takes the form

$$u_t=f(u)_{xx}, \tag{4}$$

where $f(u)=\varphi'(u)$, $f'(u)=\varphi''(u)\geqslant 0$. This equation is commonly called the equation of filtration type. From yrars of the fifties, there have been a great amount of works[1-7] contributed to the deep research of the properties of solutions for the so-called equations of filtration type. By this means, we call the systems (1) the systems of filtration type.

The component expressions for system (1) are

$$u_{it}=\sum_{j=1}^{N}\varphi_{ij}u_{jxx}+\sum_{jk=1}^{N}\varphi_{ijk}u_{jx}u_{kx}\quad (i=1, \cdots, N), \tag{5}$$

where $\varphi(u)$ with indeces i, j, k denotes the derivative of $\varphi(u)$ with respect to variables u_i, u_j, u_k, i. e., $\varphi_{ij}=\dfrac{\partial^2\varphi(u)}{\partial u_i\partial u_j}$, $\varphi_{ijk}=\dfrac{\partial^3\varphi(u)}{\partial u_i\partial u_j\partial u_k}$ $(i, j, k=1, \cdots, N)$ and so forth. The coefficient matrix $H(u)=(\varphi_{ij})$ for the terms of second order derivatives on the right hand side of system (5) is non-negative definite, i. e., $(\xi, H(u)\xi)\geqslant 0$, for any N-dimensional vector $\xi\in\mathbf{R}^N$ and for $u\in\mathbf{R}^N$.

When the Hessian matrix $H(u)$ of $\varphi(u)$ is positively definite, i. e., there exists a constant $s>0$, such that for any N-dimensional vector ξ, $(\xi, H(u)\xi)\geqslant s(\xi, \xi)$, and system (1) is a nonlinear parabolic system. In this time, the system is non-degenerate.

We introduce the lemma[8] on the periodic boundary problem for linear parabolic systems for the following discussion.

Lemma 1. *Suppose that the linear parabolic systems*

$$u_t-A(x, t)u_{xx}+B(x, t)u_x+C(x, t)u=f(x, t) \tag{6}$$

and the periodic boundary conditions (2) satisfy the following conditions, where $u=(u_1, \cdots, u_N)$ *is an N-dimensional vector valued function.*

(1) $A(x, t)$ *is an* $N\times N$ *bounded positively definite matrix in* $Q_T=\{-D\leqslant x\leqslant D, 0\leqslant t\leqslant T\}$, *i. e., the elements* $a_{ij}(x, t)$ *of the matrix are bounded in* Q_T $(i, j=1, \cdots, N)$ *and for any N-dimensional vector* $\xi\in\mathbf{R}^N$, $(\xi, A_0\xi)\geqslant a_0(\xi, \xi)$, *where* $a_0>0$.

(2) $B(x, t)$ *and* $C(x, t)$ *are* $N\times N$ *matrices bounded in* Q_T, *i. e., their elements* $b_{ij}(x, t)$ *and* $c_{ij}(x, t)$ $(i, j=1, 2, \cdots, N)$ *are all bounded in* Q_T.

(3) *The free term* $f(x, t)$ *is an N-dimensional vector valued function, quadratic integrable in* Q_T, *i. e., all of its components* $f_i(x, t)$ $(i=1, \cdots, N)$ *are quadratic integrable in* Q_T.

(4) *The initial value* $u_0(x)$ *is an N-dimensional vactor valued function, belonging to* $W_2^{(1)}(-D, D)$ *and periodic with period 2D, i. e., all of its components* $u_{0i}(x)\in W_2^{(1)}(-D, D)$ *are periodic functions with period 2D.*

Then problem (6), (2) has a unique solution $u(x, t)\in Z=L_\infty((0, T); W_2^{(1)}(-D,$

D)) $\cap W_2^{(2,1)}(Q_T)$ *and the estimation*

$$\|u\|_Z^2=\sup_{0\le t\le T}\|u(\cdot,\ t)\|^2_{W_2^{(1)}(-D,D)}+\|u_t\|^2_{L_2(Q_T)}+\|u_{xx}\|^2_{L_2(Q_T)}$$
$$\le K_1\{\|u_0\|^2_{W_2^{(1)}(-D,D)}+\|f\|^2_{L_2(Q_T)}\} \tag{7}$$

holds, where K_1 *is a constant depending on* $a_0>0$ *and the norms* $\|a_{ij}\|_{L_\infty(Q_T)}$, $\|b_{ij}\|_{L_\infty(Q_T)}$, $\|c_{ij}\|_{L_\infty(Q_T)}$ *of the coefficients of system* (1) *in* $L_\infty(Q_T)$.

The square norm of vector valued function is defined to be the sum of the square norms of all its components. If all of the components of a vector valued function belong to certain functional space, then we say simply that the vector valued function belongs to this functional space. Similarly, if all of the elements of a matrix belong to certain functional space, then we say that the matrix belongs to the space.

Corollary. *Suppose that besides the conditions in Lemma 1, there are also the conditions:*

(1) *Derivatives* $D_t^s D_x^r A(x,\ t)$, $D_t^s D_x^r B(x,\ t)$, $D_t^s D_x^r C(x,\ t)$ *of the coefficient matrices are bounded in* $Q_T(r\ge 0,\ s\ge 0)$.

(2) *Derivatives* $D_t^s D_x^r f(x,\ t)$ *of free term vector valued function is quadratic integrable in* Q_T.

(3) *Initial vector valued function* $u_0(x)\in W_2^{(2s+r+1)}(-D,\ D)$. *Then the solution* $u(x,\ t)$ *of problem* (6), (2) *has derivative* $D_t^s D_x^r u(x,\ t)\in Z$.

§ 2

Now we turn to the periodic boundary problem (2) for non-degenerate system (1).

Take $B=\{u\,|\,u\in L_\infty(Q_T),\ u_x\in L_4(Q_T)\}$. For any N-dimensional vector valued function $v\in B$, let us construct a corresponding vector valued function u satisfying the following linear parabolic system

$$u_{it}=\lambda\sum_{j=1}^N\varphi_{ij}(v)u_{jxx}+(1-\lambda)\varepsilon u_{ixx}+\lambda\sum_{j,k=1}^N\varphi_{ijk}(v)v_{jx}v_{kx},\quad (i=1,\ \cdots,\ N) \tag{8}$$

and the periodic boundary conditions

$$u_i(x+2D,\ t)=u_i(x,\ t),\quad (i=1,\ \cdots,\ N),$$
$$u_i(x,\ 0)=\lambda u_{0i}(x), \tag{9}$$

where $0\le\lambda\le 1$ and $\varepsilon>0$. Suppose $u_{0i}(x)\in W_2^{(1)}(-D,\ D)$. Since the Hessian matrix $H(u)=(\varphi_{ij}(u))$ of the function $\varphi(u)$ is positively definite, the coefficient matrix $\tilde{A}(x,\ t)=\lambda H(v(x,\ t))+(1-\lambda)\varepsilon E$ of terms of second order derivatives for linear parabolic system (8) satisfies obviously $(\xi,\ \tilde{A}\xi)\ge\varepsilon(\xi,\ \xi)$, for any N-dimensional vector $\xi\in\mathbf{R}^N$, where E is the $N\times N$ unit matrix. Hence system (8) and periodic boundary conditions (9) satisfy all the conditions in Lemma 1. Thus for every

$0\leqslant\lambda\leqslant1$ and every N-dimensional vector valued function $v\in B$, problem (8), (9) has a unique solution $u(x, t)\in Z\subset B$. Then a functional mapping $T_\lambda: B\to B$ is defined, where $0\leqslant\lambda\leqslant1$.

By means of the compactness of the imbedding mapping $Z\subsetneq B$, for every value $0\leqslant\lambda\leqslant1$, the operator $T_\lambda: B\to B$ is completely continuous.

Let M be any bounded subset of B. For $v\in M\subset B$ and $0\leqslant\lambda, \bar\lambda\leqslant1$, we have $u=T_\lambda v$ and $\bar u=T_{\bar\lambda}v$. The difference vector valued function $w=u-\bar u$ satisfies the linear system

$$w_{it}=\lambda\sum_{j=1}^{N}\varphi_{ij}(v)w_{ixx}+(1-\lambda)\varepsilon w_{ixx}$$
$$+(\lambda-\bar\lambda)\Big[\sum_{j=1}^{N}\varphi_{ij}(v)\bar u_{jxx}-\varepsilon\bar u_{ixx}+\sum_{j,k=0}^{n}\varphi_{ijk}(v)v_{jx}v_{kx}\Big],\quad (i=1,\cdots,N)\tag{10}$$

and the periodic boundary conditions

$$\begin{aligned}&w_i(x+2D, t)=w_i(x, t),\\&w_i(x, 0)=(\lambda-\bar\lambda)u_{0i}(x),\end{aligned}\quad (i=1,\cdots,N).\tag{11}$$

From Lemma 1, we know that u and $\bar u$ belong to Z, hence u_{xx} is quadratic integrable in Q_T. Then the free term part of system (10) is quadratic integrable in Q_T. Owing to the estimation (7), we obtain

$$\|u-\bar u\|_Z\leqslant K_2|\lambda-\bar\lambda|,$$

where K_2 depends on the bounded subset $M\subset B$. Hence, for any bounded subset $M\subset B$, the operator $T_\lambda: M\to B$ is uniformly bounded for λ.

When $\lambda=0$, problem (8), (9) has a unique trivial solution $u(x, t)\equiv0$.

Now we are going to establish the a priori estimations in space B, uniformly for $0\leqslant\lambda\leqslant1$ for all possible solutions for the periodic boundary problem

$$\begin{aligned}&u(x+2D, t)=u(x, t),\\&u(x, 0)=\lambda u_0(x)\end{aligned}\tag{9}$$

of the nonlinear parabolic system

$$u_t=(\operatorname{grad}\tilde\varphi(u))_{xx},\tag{12}$$

where

$$\tilde\varphi(u)=\lambda\varphi(u)+(1-\lambda)\varepsilon|u|^2\tag{13}$$

and $|u|^2=\sum_{j=1}^{N}u_j^2$. The Hessian matrix of the function is $\tilde H(u)=\lambda H(u)+(1-\lambda)\varepsilon E$. Since for any N-dimensional vector $\xi\in\mathbf{R}^N$, $(\xi, H(u)\xi)\geqslant\varepsilon(\xi,\xi)$, we see that

$$(\xi, \tilde H(u)\xi)\geqslant\varepsilon(\xi,\xi).$$

Making the scalar product of a vector u and system (12), we have

$$(u, u_t)=(u, (\operatorname{grad}\tilde\varphi(u))_{xx}).$$

Integrating in Q_τ and simplifying, we obtain the relation

$$\|u(\cdot,\tau)\|^2_{L_2(-D,D)}-\lambda^2\|u_0\|^2_{L_2(-D,D)}+\iint_{Q_\tau}(u_x, \tilde H(u)u_x)dx\,dt=0.$$

Since for $0\leqslant\lambda\leqslant 1$, matrix $\tilde{H}(u)$ is positively definite, there is

$$\|u(\cdot,\ \tau)\|^2_{L_2(-D,D)}+\varepsilon\|u_x\|^2_{L_2(Q_\tau)}\leqslant\lambda^2\|u_0\|^2_{L_2(-D,D)}, \tag{14}$$

where $\tau>0$.

Multiplying system (12) by the matrix $\tilde{H}(u)$, we have

$$(\operatorname{grad}\tilde{\varphi}(u))_t=\tilde{H}(u)(\operatorname{grad}\tilde{\varphi}(u))_{xx}.$$

Then taking the scalar product with the vector $(\operatorname{grad}\tilde{\varphi}(u))_{xx}$, we get

$$((\operatorname{grad}\tilde{\varphi}(u))_{xx},\ (\operatorname{grad}\tilde{\varphi}(u))_t)=((\operatorname{grad}\tilde{\varphi}(u))_{xx},\ \tilde{H}(u)(\operatorname{grad}\tilde{\varphi}(u))_{xx}).$$

Again integrating in $Q_\tau(\tau>0)$ and simplifying, the inequality

$$\begin{aligned}&\|(\operatorname{grad}\tilde{\varphi}(u(\cdot,\ t))_x\|^2_{L_2(-D,D)}+\varepsilon\|(\operatorname{grad}\tilde{\varphi}(u))_{xx}\|^2_{L_2(Q_\tau)}\\&\leqslant\|(\operatorname{grad}\tilde{\varphi}(\lambda u_0))_x\|^2_{L_2(-D,D)}\end{aligned} \tag{15}$$

holds.

Because

$$|(\operatorname{grad}\tilde{\varphi}(u))_x|^2=|(\lambda H(u)+(1-\lambda)\varepsilon E)u_x|^2\geqslant\varepsilon^2\|u_x|^2,$$

from (14) and (15) it follows that

$$\sup_{0\leqslant t\leqslant T}\|u(\cdot,\ t)\|_{W_2^{(1)}(-D,D)}\leqslant K_3, \tag{16}$$

where K_3 is independent of $T>0$, $\lambda>0$ and $D>0$, but depends on $\varepsilon>0$.

From (15), we have

$$\|(\operatorname{grad}\tilde{\varphi}(u))_{xx}\|^2_{L_2(Q_T)}\leqslant K_4$$

or

$$\iint_{Q_T}\sum_{i=1}^N\Big(\sum_{j=1}^N\tilde{\varphi}_{ij}(u)u_{jxx}+\sum_{j,k=1}^N\tilde{\varphi}_{ijk}u_{ix}u_{kx}\Big)^2dx\,dt\leqslant K_4.$$

It is easily seen that the inequality can be rewritten as

$$\varepsilon^2\|u_{xx}\|^2_{L_2(Q_T)}-K_5\|u_x\|^4_{L_4(Q_T)}\leqslant K_4. \tag{17}$$

By means of interpolation formula, there is

$$\begin{aligned}\|u_x\|^4_{L_4(Q_T)}&=\int_0^T\|u_x(\cdot,\ t)\|^4_{L_4(-D,D)}dt\leqslant C\int_0^T\|u_x(\cdot,\ t)\|^3_{L_2(-D,D)}\|u_{xx}(\cdot,\ t)\|_{L_2(-D,D)}dt\\&\leqslant CK_3^3\sqrt{T}\|u_{xx}\|_{L_2(Q_T)}.\end{aligned}$$

Thus (17) becomes

$$\varepsilon^2\|u_{xx}\|^2_{L_2(Q_T)}-K_5CK_3^3\sqrt{T}\|u_{xx}\|_{L_2(Q_T)}\leqslant K_4.$$

From this it can be derived that

$$\|u_{xx}\|_{L_2(Q_T)}\leqslant K_6,$$

where constant K_6 is independent of $\lambda>0$ and $D>0$, but depends on $T>0$ and $\varepsilon>0$.

Following the previous discussion, we see that all possible solutions of problem (12), (9) are uniformly bounded for $0\leqslant\lambda\leqslant 1$ in the norm of the space B. Therefore, periodic boundary problem (2) of nondegenerate system (1) has at least one solution $u(x,\ t)\in B$. From Lemma 1, we know that this solution belongs to Z.

Theorem 1. *Suppose that for periodic boundary problem (2) of non-degenerate system (1), the following assumptions hold.*

(1) *The function $\varphi(u)$ is three-times continuously differentiable with respect to the vector variable u, and its Hessian matrix $H(u)$ is positively definite, i. e., for any N-dimensional vector $\xi\in\mathbf{R}^N$, $(\xi, H(u)\xi)\geqslant s(\xi,\xi)$, where $s>0$.*

(2) *The initial vector valued function $u_0(x)\in W_2^{(1)}(-D, D)$ is periodic with period $2D$.*

Then periodic boundary problem (2) of non-degenerate system (1) has at least one solution $u(x, t)\in Z$.

Corollary. *If $\varphi(u)$ is four-times continuously differentiable, and*

$$u_0(x)\in W_2^{(2)}(-D, D),$$

then the solution $u(x, t)$ of periodic boundary problem (2) for non-degenerate system (1) is classical, and u, $u_x\in Z$.

Theorem 2. *Under the conditions of Theorem 1 and its corollary, the classical solution of periodic boundary problem (2) for non-degenerate system (1) is unique.*

Proof Suppose that the non-degenerate problem has two classical solutions $u(x, t)$ and $\bar{u}(x, t)$, where u, u_x, $\bar{u}$, $\bar{u}_x\in Z$ and u_{xx}, $\bar{u}_{xx}\in L_\infty(Q_T)$. The difference vector valued function $w(x, t)=u(x, t)-\bar{u}(x, t)$ satisfies the system

$$\begin{aligned} w_{it}=&\sum_{j=1}^N\varphi_{ij}(u)w_{jxx}+\sum_{j,k=1}^N\varphi_{ijk}(u)(u_{jx}+\bar{u}_{jx})w_{kx}\\ &+\sum_{l=1}^N\Big[\sum_{j=1}^N\Big(\int_0^1\varphi_{ijl}(\tau u+(1-\tau)\bar{u})\,d\tau\Big)\bar{u}_{jxx}\\ &+\sum_{j,k=1}^N\Big(\int_0^1\varphi_{ijkl}(\tau u+(1-\tau)\bar{u})\,d\tau\Big)\bar{u}_{jx}\bar{u}_{kx}\Big]w_l \end{aligned}$$

and the homogeneous boundary conditions

$$w_i(x+2D, t)=w_i(x, t),$$
$$w_i(x, t)=0\quad(i=1, \cdots, N).$$

From the estimation (7) of Lemma 1, we see that $w(x, t)\equiv 0$, i. e., $u(x, t)\equiv\bar{u}(x, t)$. So that the classical solution of non-degenerate problem is unique.

The obtained generalized solution $u(x, t)\in Z$ in Theorem 1 is also unique. The uniqueness of weak solution will be proved later. Thus naturally the uniqueness of the generalized solution is also available.

§ 3

Now we are going to consider initial value problem (3) of non-degenerate system (1) in domain $Q_T^*=\{x\in\mathbf{R}, 0\leqslant t\leqslant T\}$, where $\mathbf{R}=(-\infty, \infty)$.

Let us take a sequence $\{D_s\}$ such that $D_s\to\infty$, as $s\to\infty$[9]. For every s, let us construct an N-dimensional vector valued function $u_0^{[s]}(x)$ of x, periodic with period $2D_s$, such that $u_0^{[s]}(x)\equiv u_0(x)$, as $x\in[-(D_s-1), (D_s-1)]$, $u_0^{[s]}(x)$ possesses the same smoothness as $u_0(x)$ and has uniformly bounded appropriate norm for s and D_s, i.

e., if $u_0(x)\in W_2^{(k)}(\mathbf{R})$, then $u_0^{[s]}(x)\in W_2^{(k)}(-D_s, D_s)$ and $\|u_0^{[s]}\|_{W_2^{(k)}(-D_s,D_s)}$ are uniformly bounded for s, where $k\geqslant 1$.

For every s, let us consider the periodic boundary problem

$$\begin{aligned} &u(x+2D_s,\ t)=u(x,\ t), \qquad (-D_s\leqslant x\leqslant D_s)\\ &u(x,\ 0)=u_0^{[s]}(x), \end{aligned} \tag{18}$$

for non-degenerate system (1) in domain $Q_T^{[s]}=\{-D_s\leqslant x\leqslant D_s,\ 0\leqslant t\leqslant T\}$, where $s=1, 2, \cdots$. On account of Theorem 1, problem (1), (18) has at least one solution $u^{[s]}(x, t)\in Z(Q_T^{[s]})$. From the discussion of previous section, it can be seen that the following estimation holds

$$\|u^{[s]}\|_{Z(Q_T^{[s]})}\leqslant K_7\|u_0\|_{W_2^{(1)}(-D_s,D_s)}, \tag{19}$$

where K_7 depends on $s>0$ and the behavior of function $\varphi(u)$, but is independent of $s>0$ and $D_s>0$.

In the sequence $\{u^{[s]}(x, t)\}$, a subsequence can be selected still denoted by $\{u^{[s]}(x, t)\}$; when $s\to\infty$ and $D_s\to\infty$, the subsequence $\{u^{[s]}(x, t)\}$ converges to the limiting vector valued function $u(x, t)$ in every point of Q_T^*. Also the subsequences $\{u^{[s]}(x, t)\}$ and $\{u_x^{[s]}(x, t)\}$ uniformly converge to $u(x, t)$ and $u_x(x, t)$ in any domain $\{-L\leqslant x\leqslant L,\ 0\leqslant t\leqslant T\}$ respectively; subsequences $\{u_{xx}^{[s]}(x, t)\}$ and $\{u_t^{[s]}(x, t)\}$ are weakly convergent to generalized derivatives $u_{xx}(x, t)$ and $u_t(x, t)$ in Q_T^* respectively. The obtained limiting vector valued function $u(x, t)\in Z$ is just the solution of initial value problem (3) of non-degenerate system (1).

Theorem 3. *Suppose that the function $\varphi(u)$ is threetimes continuously differentiable, its Hessian matrix $H(u)$ is positively definite, and $u_0(x)\in W_2^{(1)}(\mathbf{R})$. Then initial value problem (3) of non-degenerate system (1) has at least one generalized solution $u(x, t)\in Z(Q_T^*)$. When the function $\varphi(u)$ is four-times continuously differentiable and $u_0(x)\in W_2^{(2)}(\mathbf{R})$, initial value problem (3) of non-degenerate system (1) has a unique classical solution $u(x, t)$, where $u, u_x\in Z(Q_T^*)$.*

Similarly, the generalized solution $u(x, t)\in Z(Q_T^*)$ is also unique. As the uniqueness of weak solution is proved, the generalized solution is unique.

§ 4

Before the consideration of periodic boundary problem (2) and initial value problems (3) for nonlinear degenerate parabolic systems (1), let us make some simple discussions on the properties of function $\varphi(u)$ in system (1).

Suppose that the function $\varphi(u)$ satisfies the following conditions:

(I) The function $\varphi(u)$ it three-times continuously differentiable with respect to vector variable u in $\mathbf{R}_u^N$. As $|u|\neq 0$, $\varphi(u)>0$, and $\varphi(0)=0$.

(II) For any N-dimensional vector ξ, holds the inequality

$$(\xi,\ H(u)\xi)\geqslant\sigma(u)(\xi,\ \xi), \tag{21}$$

where $H(u)$ is the Hessian matrix of $\varphi(u)$, $\sigma(u)$ is a function possessing the following properties:

(i) In a certain neighborhood of $|u|=0$, $\sigma(u)=\sigma_0|u|^\mu$, where $\sigma_0>0$, $\mu\geqslant 0$;

(ii) When $|u|$ is sufficiently large, $\sigma(u)=\sigma_\infty|u|^{-\nu}$, where $\sigma_\infty>0$, $\nu<1$;

(iii) As $u\in\mathbf{R}_u^N\backslash 0$ or $0<|u|<\infty$, $\sigma(u)>0$.

(III) For $u\in\mathbf{R}_u^N$, the relations

$$\left|\frac{\partial\varphi(u)}{\partial u_j}\right|^{\lambda_j}=O(\varphi(u)) \tag{22}$$

hold, where $1\leqslant\lambda_j\leqslant 2$, $j=1,\ \cdots,\ N$.

Lemma 2. *Assume that the function $\varphi(u)$ satisfies conditions (I), (II) and (III). Then the continuously differentiable mapping $\Phi:\mathbf{R}_u^N\to\mathbf{R}_v^N$, defined by the correspondence of any $u\in\mathbf{R}_u^N$ to $\Phi(u)\equiv\operatorname{grad}\varphi(u)=v\in\mathbf{R}_v^N$, has an inverse mapping $\Phi^{-1}:\mathbf{R}_v^N\to\mathbf{R}_u^N$, which is locally Hölder continuous with index $\frac{1}{1+\mu}$.*

Proof Firstly, we are going to prove that for any $v\in\mathbf{R}_v^N$, the solution of system

$$v=\operatorname{grad}\varphi(u) \tag{23}$$

is unique. Let $u_1,\ u_2\in\mathbf{R}_u^N$ be two different solutions of (23), $u_1\neq u_2$. On account of $\operatorname{grad}\varphi(u_1)=\operatorname{grad}\varphi(u_2)$, we then have

$$\begin{aligned}0&=(\operatorname{grad}\varphi(u_2)-\operatorname{grad}\varphi(u_1),\ u_2-u_1)\\&=\int_0^1(H(\tau u_2+(1-\tau)u_1)(u_2-u_1),\ (u_2-u_1))d\tau\\&\geqslant\int_0^1\sigma(\tau u_2+(1-\tau)u_1)d\tau\cdot|u_2-u_1|^2.\end{aligned}$$

From the properties of $\sigma(u)$ given in condition (II), the integral

$$\int_0^1\sigma(\tau u_2+(1-\tau)u_1)d\tau$$

has positive value, as $u_1\neq u_2$. Then the right hand side of the inequality is positive. This contradiction shows that $u_1=u_2$.

In order to prove the existence of solution of systems (23) for any given $v\in\mathbf{R}_v^N$, it is sufficient to verify that all solutions of the system

$$v=(1-\eta)u+\eta\operatorname{grad}\varphi(u) \tag{24}$$

are uniformly bounded with respect to $0\leqslant\eta\leqslant 1$.

From (24) there is

$$|u|\cdot|v|\geqslant(v,\ u)\geqslant(1-\eta)|u|^2+\eta\left(\int_0^1\sigma(\tau u)d\tau\right)|u|^2. \tag{25}$$

We know from (IIii) that $\sigma(u)=\sigma_\infty|u|^{-\nu}$ as $|u|\geqslant M_0$. Then For any chosen $0<\tau_0<1$, when $|u|\geqslant\frac{M_0}{\tau_0}$,

$$\int_0^1 \sigma(\tau u)d\tau=\int_0^{\tau_0} \sigma(\tau u)d\tau+\int_{\tau_0}^1 \sigma(\tau u)d\tau$$
$$\geqslant \sigma_\infty \int_{\tau_0}^1 |\tau u|^{-\nu} d\tau=\sigma_\infty \frac{1-\tau_0^{1-\nu}}{1-\nu}|u|^{-\nu}.$$

Thus, when $|u|\geqslant \frac{M_0}{\tau_0}$,

$$|v|\cdot|u|\geqslant(1-\eta+\eta\varepsilon)|u|^2+\sigma_\infty \frac{1-\tau_0^{1-\nu}}{1-\nu}\eta|u|^{2-\nu},$$

where $\varepsilon>0$, $0\leqslant\eta\leqslant1$ and $0<\tau_0<1$, $\nu<1$. When $|u|\geqslant\left(\sigma_\infty\frac{1-\tau_0^{1-\nu}}{1-\nu}\right)^{\frac{1}{\nu}}$,

$$|v|\cdot|u|\geqslant(1-\eta)|u|^2+\sigma_\infty\frac{1-\tau_0^{1-\nu}}{1-\nu}\eta|u|^{2-\nu}\geqslant\sigma_\infty\frac{1-\tau_0^{1-\nu}}{1-\nu}|u|^{2-\nu}.$$

So, when $|u|\geqslant\left\{\left(\frac{|v|}{\sigma_\infty}\cdot\frac{1-\nu}{1-\tau_0^{1-\nu}}\right)^{\frac{1}{1-\nu}}\right\}$, the above inequality cannot be available. Therefore, when $|u|\geqslant M^*=\max\left\{\frac{M_0}{\tau_0},\ \sigma_\infty\left(\frac{1-\tau_0^{1-\nu}}{1-\nu}\right)^{\frac{1}{\nu}},\ \left(\frac{|v|}{\sigma_\infty}\cdot\frac{1-\nu}{1-\tau_0^{1-\nu}}\right)^{\frac{1}{1-\nu}}\right\}$, the contradiction takes place. This shows that all possible solutions of system (24) satisfy the estimation $|u|\leqslant M^*$, where M^* is independent of η. Hence, system (23) always has solution u.

Now we prove that the inverse mapping $\Phi^{-1}:\mathbf{R}_v^N\to\mathbf{R}_u^N$ is Hölder continuous with index $\frac{1}{1+\mu}$.

Let $\sigma(u)=\sigma_0|u|^\mu$ in a neighborhood $S(0,\delta)$ of zero point $0\in\mathbf{R}_u^N$, where $\sigma_0>0$, $\mu\geqslant0$. When $0<|u|<\delta$ and $v=\operatorname{grad}\varphi(u)$, we have

$$|v|\cdot|u|\geqslant(v,\ u)=(\operatorname{grad}\varphi(u),\ u)\geqslant\frac{\sigma_0}{1+\mu}|u|^{2+\mu}. \tag{26}$$

Hence

$$|u|\leqslant\left(\frac{1+\mu}{\sigma_0}\right)^{\frac{1}{1+\mu}}|v|^{\frac{1}{1+\mu}}.$$

As $|v|\leqslant\bar\delta=\frac{\sigma_0}{1+\mu}\delta^{1+\mu}$, $|u|\leqslant\delta$. When $\bar\delta\leqslant|v|\leqslant\bar K$, there is $\delta\leqslant|u|\leqslant K$. When $\delta\leqslant|u|\leqslant K$, $\sigma(u)\geqslant\eta_0>0$, $H(u)$ is positively definite. Then the Jacobi matrix $\frac{\partial u}{\partial v}=H^{-1}(u)$ for the inverse mapping $\Phi^{-1}:\mathbf{R}_v^N\to\mathbf{R}_u^N$ is bounded. So in any bounded domain of $\mathbf{R}_v^N$, the inverse mapping $\Phi^{-1}:\mathbf{R}_v^N\to\mathbf{R}_u^N$ is Hölder continuous with index $\frac{1}{1+\mu}$.

Corollary. *Suppose that the continuously differentiable mapping $\Phi_\varepsilon:\mathbf{R}_u^N\to\mathbf{R}_v^N$ is defined by the relation $v=\varepsilon u+\operatorname{grad}\varphi(u)$, where $\varepsilon>0$. For any bounded set $V\subset\mathbf{R}_v^N$, $\Phi_\varepsilon^{-1}(V)\subset\mathbf{R}_u^N$ is uniformly bounded for $\varepsilon>0$. Inverse mapping $\Phi_\varepsilon^{-1}:\mathbf{R}_v^N\to\mathbf{R}_u^N$ possesses the local Hölder continuity with index $(1+\mu)^{-1}$ uniformly for $\varepsilon>0$.*

§ 5

Let us consider now periodic boundary problem (2) for the nonlinear degenerate parabolic system

$$u_t=(\operatorname{grad}\varphi(u))_{xx}, \tag{1}$$

where the function $\varphi(u)$ satisfies conditions (I), (II), (III), its Hessian matrix $H(u)$ is non-negatively definite. In this section, let us discuss the existence of the weak solution of problem (1), (2).

Definition 1. *The vector valued function $u(x, t)$ is called the weak solution of periodic boundary problem (2) for nonlinear degenerate parabolic system (1), if $u(x, t)$ satisfies the following conditions:*

(1) *$u(x, t)$ is a bounded function in Q_T and periodic for x with period $2D$.*

(2) *$(\operatorname{grad}\varphi(u))_x$ belongs to $L_\infty((0, T); L_2(-D, D))$.*

(3) *For any test function $\psi(x, t)\in H^1(Q_T)$ periodic for x with period $2D$, the integral relation*

$$\iint_{Q_T}[\psi_t u-\psi_x(\operatorname{grad}\varphi(u))_x]\,dx\,dt+\int_{-D}^{D}\psi(x, 0)u_0(x)\,dx=0, \tag{27}$$

holds where $\psi(x, T)=0$ $(-D\leqslant x\leqslant D)$.

In order to study the existence of the weak solution of periodic boundary problem (2) for nonlinear degenerate system (1), let us construct the solutions of periodic boundary problem for the approximate non-degenerate systems and use the obtained solutions to approach the weak solution. Replacing the $\varphi(u)$ in degenerate system (1) by the function $\varphi_\varepsilon(u)=\varphi(u)+\varepsilon|u|^2$, we obtain the approximate non-degenerate system

$$u_t=(\operatorname{grad}\varphi_\varepsilon(u))_{xx} \tag{28}$$

or

$$u_t=(\operatorname{grad}\varphi(u))_{xx}+\varepsilon u_{xx}, \tag{29}$$

where $\operatorname{grad}\varphi_\varepsilon(u)=\operatorname{grad}\varphi(u)+\varepsilon u$, $H_\varepsilon(u)=H(u)+\varepsilon E$. Now let us consider periodic boundary problem (2) for systems (28) or (29). Since for every $\varepsilon>0$ the Hessian matrix $H_\varepsilon(u)$ is positively definite, i. e., for any N-dimensional vector ξ,

$$(\xi, H_\varepsilon(u)\xi)\geqslant\varepsilon(\xi, \xi),$$

the conditions in Theorem 1 are satisfied. Hence for every $\varepsilon>0$, problem (28), (2) or (29), (2) has at least one vector valued function solution $u_\varepsilon(x, t)\in Z(Q_T)$, where the initial vector valued function $u_0(x)\in W_2^{(1)}(-D, D)$ is assumed. Then the so obtained set $\{u_\varepsilon(x, t)\}$ of vector valued functions can be regarded as the set of approximate vector valued functions for the weak solution of problem (1), (2).

For system (28), making the similar derivatives as (14) and (15), we get the estimation relation for the obtained set $\{u_\varepsilon(x, t)\}$ of vector valued functions

$$\sup_{0\le t\le T}\|u_\varepsilon(\cdot, t)\|^2_{L_2(-D,D)}+2\iint_{Q_T}(u_{\varepsilon x}, (\operatorname{grad}\varphi(u_\varepsilon))_x)dx\,dt+2\varepsilon\|u_{\varepsilon x}\|^2_{L_2(Q_T)}\le 2\|u_0\|^2_{L_2(Q_T)}, \tag{30}$$

$$\begin{aligned}&\sup_{0\le t\le T}\|(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(\cdot, t)))_x\|^2_{L_2(-D,D)}\\ &\quad+2\iint_{Q_T}((\operatorname{grad}\varphi_\varepsilon(u_\varepsilon))_{xx}, H_\varepsilon(u_\varepsilon)(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon))_{xx})dx\,dt\\ &\le 2\|(\operatorname{grad}\varphi(u_0))_x\|^2_{L_2(-D,D)}+2\varepsilon^2\|u_{0x}\|^2_{L_2(-D,D)}.\end{aligned} \tag{31}$$

Without loss of generality, it can be assumed that $0<\varepsilon\le 1$. Then the right hand side of the inequality is uniformly bounded for ε. Taking the scalar product of vector $\operatorname{grad}\varphi_\varepsilon(u_\varepsilon)$ with system (28), we get

$$\begin{aligned}&\sup_{0\le t\le T}\|\varphi_\varepsilon(u_\varepsilon(\cdot, t))\|_{L_1(-D,D)}+2\|(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon))_x\|^2_{L_2(Q_T)}\\ &\le 2\|\varphi(u_0)\|_{L_1(-D,D)}+2\varepsilon\|u_0\|^2_{L_2(-D,D)}.\end{aligned} \tag{32}$$

The right hand side of the inequality is uniformly bounded for $\varepsilon>0$. The following lemma can be obtained from (30), (31) and (32).

Lemma 3. *Suppose that the function satisfies conditions (I), (II) and (III). For the vector valued solution $\{u_\varepsilon(x, t)\}$ of periodic boundary problem (2) for approximate system (28), $\varepsilon>0$, the following estimation relation*

$$\begin{aligned}&\sup_{0\le t\le T}\|u_\varepsilon(\cdot, t)\|^2_{L_2(-D,D)}+\sup_{0\le t\le T}\|(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(\cdot, t)))_x\|^2_{L_2(-D,D)}\\ &\quad+\sup_{0\le t\le T}\|\varphi_\varepsilon(u_\varepsilon(\cdot, t))\|_{L_1(-D,D)}+\sup_{0\le t\le T}\|u_{\varepsilon t}(\cdot, t)\|^2_{H^{-1}(-D,D)}\le K_8\end{aligned} \tag{33}$$

holds, where K_8 depends on the norm of $u_0(x)$ in $W_2^{(1)}(-D, D)$ and the behaviour of $\varphi(u)$, but is independent of $\varepsilon>0$, $T>0$ and $D>0$, moreover $\varphi_\varepsilon(u)=\varphi(u)+\varepsilon|u|^2$, $\operatorname{grad}\varphi_\varepsilon(u)=\operatorname{grad}\varphi(u)+\varepsilon u$.

Proof The boundedness of the first three terms on the left hand side of (33) can be obtained directly from (30), (31) and (32).

For the forth term, let us derive as follows

$$\begin{aligned}\left|\int_{-D}^{D}\psi(x)u_{\varepsilon t}(x, t)dx\right| &= \left|\int_{-D}^{D}\psi(x)(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(x, t)))_{xx}dx\right|\\ &= \left|\int_{-D}^{D}\psi'(x)(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(x, t)))_x dx\right|\\ &\le \|\psi'\|_{L_2(-D,D)}\|(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(\cdot, t)))_x\|_{L_2(-D,D)}\\ &\le C\|\psi\|_{H^1(-D,D)},\end{aligned}$$

where $\psi(x)$ is any test function belonging to $H_0^1(-D, D)$, C is a constant independent of $\varepsilon>0$, $D>0$ and $0\le t\le T$. According to the definition of the norms in Hilbert space with negative order, we know that $\|u_{\varepsilon t}(\cdot, t)\|_{H^{-1}(-D,D)}\le C$. This completes the proof of the lemma.

Corollary. *For the set of vector valued functions $\{u_\varepsilon(x, t)\}$*, there is an estimation

$$\varepsilon\|u_{\varepsilon x}\|^2_{L_2(Q_T)}+\|(\operatorname{grad}\varphi(u_\varepsilon))_x\|^2_{L_2(Q_T)}+\sup_{0\le t\le T}\|\varphi(u_\varepsilon(\cdot,t))\|_{L_1(-D,D)}\le K_9, \tag{34}$$

where K_9 is independent of $\varepsilon>0$, $T>0$ and $D>0$.

Lemma 4. *Let the function $\varphi(u)$ possess properties (I), (II) and (III). Then the vector valued solution $\{u_\varepsilon(x,t)\}$ of periodic boundary problem (2) for approximate system (28) or (29) has the estimation*

$$\|\operatorname{grad}\varphi_\varepsilon(u_\varepsilon)\|_{L_\infty(Q_T)}\le K_{10}, \tag{35}$$

where K_{10} isn't dependent of ε and D individually, but depends on the product εD.

Proof Denote $v_{\varepsilon j}(x,t)=\dfrac{\partial\varphi_\varepsilon(u_\varepsilon(x,t))}{\partial u_j}$ $(j=1,\cdots,N)$ and $v_\varepsilon(x,t)=\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(x,t))$. From property (III), we know

$$\left|\frac{\partial\varphi(u)}{\partial u_j}\right|^{\lambda_j}\le C_1\varphi(u),$$

where $1\le\lambda_j\le 2$, $j=1,2,\cdots,N$. Without loss of generality, it can be assumed that for $0<\varepsilon\le 1$, we have

$$\varepsilon^\lambda|u|^\lambda\le\varepsilon|u|^2+\varepsilon,$$

where $1\le\lambda\le 2$. Then there is

$$\left|\frac{\partial\varphi_\varepsilon(u)}{\partial u_j}\right|^{\lambda_j}\le C_2\left|\frac{\partial\varphi(u)}{\partial u_j}\right|^{\lambda_j}+C_2\varepsilon^{\lambda_j}|u|^{\lambda_j}\le C_3\varphi_\varepsilon(u)+C_2\varepsilon.$$

Integrating the above inequality for x in $[-D,D]$, we get

$$\|v_{\varepsilon j}(\cdot,t)\|_{L_{\lambda_j}(-D,D)}\le C_4\|\varphi_\varepsilon(u_\varepsilon(\cdot,t))\|^{\frac{1}{\lambda_j}}_{L_1(-D,D)}+C_4(\varepsilon D)^{\frac{1}{\lambda_j}}, \tag{36}$$

where $0\le t\le T$. The right hand part depends on εD, but is independent of $\varepsilon>0$ and $D>0$.

On acount of interpolation formula, it can be obtained that

$$\|v_{\varepsilon j}(\cdot,t)\|_{L_2(-D,D)}\le C_5\|v_{\varepsilon j}(\cdot,t)\|^{\frac{2\lambda_j}{2+\lambda_j}}_{L_{\lambda_j}(-D,D)}\cdot\{\|v_{\varepsilon j}(\cdot,t)\|^{\frac{2-\lambda_j}{2+\lambda_j}}_{L_2(-D,D)}+\|v_{\varepsilon jx}(\cdot,t)\|^{\frac{2-\lambda_j}{2+\lambda_j}}_{L_2(-D,D)}\},$$

where $1\le\lambda_j\le 2$, $j=1,2,\cdots,N$. It is not difficult to derive from the inequality the estimation

$$\|v_{\varepsilon j}(\cdot,t)\|_{L_2(-D,D)}\le C_6\{\|v_{\varepsilon j}(\cdot,t)\|_{L_{\lambda_j}(-D,D)}+\|v_{\varepsilon j}(\cdot,t)\|^{\frac{2\lambda_j}{2+\lambda_j}}_{L_{\lambda_j}(-D,D)}\|v_{\varepsilon jx}(\cdot,t)\|^{\frac{2-\lambda_j}{2+\lambda_j}}_{L_2(-D,D)}\}, \tag{37}$$

where

$$\|v_{\varepsilon x}(\cdot,t)\|_{L_2(-D,D)}=\|(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(\cdot,t)))_x\|_{L_2(-D,D)}.$$

Hence from (36) and (37), we can see that the right hand side of the inequality

$$\|v_{\varepsilon j}\|_{L_\infty(Q_T)}=\sup_{0\le t\le T}\|v_{\varepsilon j}(\cdot,t)\|_{L_\infty(-D,D)}$$

$$\le C_7\sup_{0\le t\le T}\|v_{\varepsilon j}(\cdot,t)\|^{\frac12}_{L_2(-D,D)}\|v_{\varepsilon j}(\cdot,t)\|^{\frac12}_{W_2^{(1)}(-D,D)}$$

can be expressed by $\|\varphi_\varepsilon(u_\varepsilon(\cdot,t))\|_{L_1(-D,D)}$ and $\|(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(\cdot,t)))_x\|_{L_2(-D,D)}$ and the upper bound of εD; thus it is bounded. This proves the lemma.

Lemma 5. *Assume that the function $\varphi(u)$ satisfies conditions (I), (II) and (III). Then the set $\{u_\varepsilon(x,t)\}$ of solutions of problem (29), (2) or (28) has the estimation*

$$\sup_{0\leqslant t\leqslant T}\|u_s(\cdot,\ t)\|_{C^{(0,s)}(-D,D)}\leqslant K_{11}, \tag{38}$$

where $s=\frac{1}{2}(1+\mu)^{-1}$ *and* K_{11} *depends on* εD *and* $\|u_0\|_{W_2^{(1)}(-D,D)}$.

Proof Let D be given and εD be bounded. By means of the corollary of Lemma 2, the uniform for ε boundedness of $\{u_\varepsilon\}$ follows from the uniform for ε boundedness of $\{v_\varepsilon\}$. Thus $\|u_\varepsilon\|_{L_\infty(Q_T)}$ is uniformly bounded for ε.

Let $x_1,\ x_2\in[-D,\ D]$. We have

$$|v_\varepsilon(x_2,\ t)-v_\varepsilon(x_1,\ t)|=\left|\int_{x_1}^{x_2}v_{\varepsilon x}(x,\ t)\,dx\right|\leqslant|x_2-x_1|^{\frac{1}{2}}\|v_{\varepsilon x}(\cdot,\ t)\|_{L_2(-D,D)}.$$

Hence there is an estimation

$$\sup_{0\leqslant t\leqslant T}\|v_\varepsilon(\cdot,\ t)\|_{C^{(0,\frac{1}{2})}(-D,D)}\leqslant\sup_{0\leqslant t\leqslant T}\|v_{\varepsilon x}(\cdot,\ t)\|_{L_2(-D,D)}.$$

From Lemma 2, it can be derived that

$$|u_\varepsilon(x_2,\ t)-u_\varepsilon(x_1,\ t)|\leqslant C_1|v_\varepsilon(x_2,\ t)-v_\varepsilon(x_1,\ t)|^{\frac{1}{1+\mu}}$$
$$\leqslant C_2|x_2-x_1|^{\frac{1}{2(1+\mu)}},$$

where the constants C_1, C_2 are independent of $\varepsilon>0$ and $D>0$.

The lemma is proved.

Now we prove the existence of weak solution of periodic boundary problem (2) for nonlinear degenerate system (1), by the method of limiting process $\varepsilon\to0$. Let Ψ be the set of test functions, i. e.,

$$\Psi=\{\psi(x,\ t)\,|\,\psi\in H^1(Q_T),\ \psi(x+2D,\ t)=\psi(x,\ t),\ \psi(x,\ T)\equiv0\}.$$

For any $\psi(x,\ t)\in\Psi$, the vector valued solution $u_\varepsilon(x,\ t)$ of non-degenerate problem (28), (2) or (29), (2) satisfies the integral relation

$$\iint_{Q_T}[\psi_t u_\varepsilon-\psi_x(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon))_x]\,dx\,dt+\int_{-D}^{D}\psi(x,\ 0)u_0(x)\,dx=0. \tag{39}$$

From Lemmas 3, 4 for given $D>0$, the sets $\{v_\varepsilon(x,\ t)\}$ and $\{u_{\varepsilon t}(x,\ t)\}$ of vector valued functions are uniformly bounded for $\varepsilon>0$ in functional space $L_\infty((0,\ T);\ L_2(-D,\ D))$ and $L_\infty((0,\ T);\ H^{-1}(-D,\ D))$ respectively. And from Lemma 5, for given $D>0$, the set $\{u_\varepsilon(x,t)\}$ of vector valued functions is uniformly bounded for ε in the functional space $L_\infty((0,\ T);\ C^{(0,s)}(-D,\ D))$, where $s=(2+2\mu)^{-1}$. It can be seen that the set $\{u_\varepsilon(x,\ t)\}$ of vector valued functions is compact in $L_p((0,\ T);\ C^{(0,r)}(-D,\ D))$, where $1\leqslant p<\infty$, $0<r<s$[10]. A subsequence, denoted by $\{u_{\varepsilon_i}(x,\ t)\}$, can be selected from $\{u_\varepsilon(x,\ t)\}$, such that when $i\to\infty$, $\varepsilon_i\to0$, $u_{\varepsilon_i}(x,\ t)$ converges to $u(x,\ t)$ in $L_p((0,\ T);\ C^{(0,r)}(-D,\ D))$. Then, in Q_T, $u_{\varepsilon_i}(x,\ t)$ converges almost everywhere to $u(x,\ t)$. Hence in Q_T, $\operatorname{grad}\varphi_{\varepsilon_i}(u_{\varepsilon_i}(x,\ t))$ is also almost everywhere convergent to $\operatorname{grad}\varphi(u(x,\ t))$. Since $\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(x,\ t))$ is bounded in $L_\infty((0,\ T);\ W_2^{(1)}(-D,\ D))$ uniformly for ε, $\{\operatorname{grad}\varphi_{\varepsilon_i}(u_{\varepsilon_i}(x,\ t))\}$ is weakly convergent to $\operatorname{grad}\varphi(u(x,\ t))$ in $L_p((0,\ T);\ L_2(-D,\ D))$ and $\{(\operatorname{grad}\varphi_{\varepsilon_i}(u_{\varepsilon_i}(x,\ t)))_x\}$ is weakly

convergent to $(\operatorname{grad}\varphi(u(x,t)))_x$ in $L_p((0,T);\ L_2(-D,D))$, where $1\leqslant p<\infty$. Therefore, as $i\to\infty$ and $\varepsilon_i\to 0$, the limit of integral relation (39) is just integral relation (27). That is, $u(x,t)$ is the weak solution of periodic boundary problem (2) for nonlinear degenerate system (1).

The above obtained weak solution $u(x,t)$ belongs to functional space $L_p(0,T);\ C^{(0,r)}(-D,D))$, and there is $(\operatorname{grad}\varphi(u))_x\in L_p((0,T);\ L_2(-D,D))$, where $1\leqslant p<\infty$, $0<r<s$. From uniform boundedness of $\{u_\varepsilon(x,t)\}$ in $L_\infty((0,T);\ C^{(0,s)}(-D,D))$, it can be seen that the norm of $u(x,t)$ in $L_p((0,T);\ C^{(0,r)}(-D,D))$ is uniformly bounded for $1\leqslant p<\infty$, $0<r<t$. Hence $u(x,t)\in L_\infty((0,T);\ C^{(0,s)}(-D,D))$. Moreover, on account of the uniform boundedness for ε of $v_{\varepsilon x}(x,t)$ in $L_\infty((0,T);\ L_2(-D,D))$, it can be obtained that the norm of $(\operatorname{grad}\varphi(u))_x$ in $L_p((0,T);\ L_2(-D,D))$ is uniformly bounded for $1<p\leqslant\infty$. Hence $(\operatorname{grad}\varphi(u))_x\in L_\infty((0,T);\ L_2(-D,D))$.

Theorem 4. *Suppose that the function $\varphi(u)$ satisfies conditions (I), (II), (III) and $u_0(x)\in W_2^{(1)}(-D,D)$ is the initial vector valued function, periodic with period $2D$. Then for periodic boundary problem (2) of nonlinear degenerate system (1), there exists at least one weak solution $u(x,t)\in L_\infty((0,T);\ C^{(0,s)}(-D,D))$ and $\operatorname{grad}\varphi(u(x,t))\in L_\infty((0,T);\ W_2^{(1)}(-D,D))$, where $s=(2+2\mu)^{-1}$.*

§ 6

Now we are going to consider the uniqueness of the weak solution $u(x,t)$ of periodic boundary problem (2) for nonlinear degenerate system (1).

Let $u_1(x,t)$ and $u_2(x,t)$ be two different weak solutions of problem (1), (2). Then their difference $u_2(x,t)-u_1(x,t)$ satisfies the integral relation

$$\iint_{Q_T}[\psi_t(u_2-u_1)-\psi_x((\operatorname{grad}\varphi(u_2))_x-(\operatorname{grad}\varphi(u_1))_x)]\,dx\,dt=0,$$

where $\psi(x,t)\in\Psi$ is any test function. Now we take the vector valued test function

$$\tilde{\psi}(x,t)=\int_T^t[\operatorname{grad}\varphi(u_2(x,\zeta))-\operatorname{grad}\varphi(u_1(x,\zeta))]\,d\zeta. \tag{40}$$

It can obviously be seen that all of its components are the test functions belonging to Ψ. The equality

$$\iint_{Q_T}[(\tilde{\psi}_t,\ u_2-u_1)-(\tilde{\psi}_x,\ (\operatorname{grad}\varphi(u_2))_x-(\operatorname{grad}\varphi(u_1))_x)]\,dx\,dt=0 \tag{41}$$

holds. Substituting expression (40) of $\tilde{\psi}$ in (41), the first term of thus obtained relation is

$$\iint_{Q_T}(\tilde{\psi}_t,\ u_2-u_1)\,dx\,dt=\iint_{Q_T}(\operatorname{grad}\varphi(u_2)-\operatorname{grad}\varphi(u_1),\ u_2-u_1)\,dx\,dt$$
$$=\iint_{Q_T}\left(\int_0^1(H(\tau u_2+(1-\tau)u_1)(u_2-u_1),\ (u_2-u_1))\,d\tau\right)dx\,dt.$$

The second term is

$$\iint\limits_{Q_T}(\tilde{\psi}_x, (\operatorname{grad}\varphi(u_2))_x-(\operatorname{grad}\varphi(u_1))_x)dx\,dt$$

$$=\iint\limits_{Q_T}\Big(\int_T^t[(\operatorname{grad}\varphi(u_2(x,\zeta)))_x-(\operatorname{grad}\varphi(u_1(x,\zeta)))_x]d\zeta,$$

$$(\operatorname{grad}\varphi(u_2(x,t)))_x-(\operatorname{grad}(\varphi(u_1(x,t)))_x\Big)dx\,dt$$

$$=\frac{1}{2}\iint\limits_{Q_T}\frac{d}{dt}\left|\int_T^t[(\operatorname{grad}\varphi(u_2(x,\zeta)))_x-(\operatorname{grad}\varphi(u_1(x,\zeta)))_x]d\zeta\right|^2dx\,dt$$

$$=-\frac{1}{2}\int_{-D}^{D}\left|\int_0^T[(\operatorname{grad}\varphi(u_2(x,\zeta)))_x-(\operatorname{grad}\varphi(u_1(x,\zeta)))_x]\,d\zeta\right|^2dx$$

$$\leqslant 0.$$

Then (41) becomes

$$\iint\limits_{Q_T}\Big[\int_0^1(H(\tau u_2+(1-\tau)u_1)(u_2-u_1),(u_2-u_1))d\tau\Big]dx\,dt\leqslant 0$$

or

$$\iint\limits_{Q_T}\sigma(\tilde{u})\,|u_2-u_1|^2dx\,dt=0,$$

where $\tilde{u}$ is the intermediate value on the joining segment of u_1 and u_2 in $\mathbf{R}_u^N$. This equality shows that the measure of the set in Q_T of points which makes

$$\sigma(\tilde{u}(x,t))\,|u_2(x,t)-u_1(x,t)|^2\neq 0$$

must equal to zero, or the measure of the set of points for which $u_2(x,t)\neq u_1(x,t)$ can only be zero, too. Hence there is $u_2(x,t)=u_1(x,t)$ almost everywhere in Q_T.

Theorem 5. *The weak solution $u(x,t)$ of periodic boundary problem (2) for nonlinear degenerate system (1) is unique.*

Corollary. *The generalized solution $u(x,t)\in Z(Q_T)$ obtained in Theorem 1 for periodic boundary problem (2) of nonlinear non-degenerate system (1) is unique.*

§ 7

In tnis section, we want to consider the existence theorem of the weak vector valued solution for the initial value problem

$$u(x,0)=u_0(x),\quad x\in\mathbf{R}\tag{3}$$

of the nonlinear degenerate parabolic system

$$u_t=(\operatorname{grad}\varphi(u))_{xx}\tag{1}$$

in domain $Q_T^*=\{x\in\mathbf{R},\ 0\leqslant t\leqslant T\}$.

Definition 2. *The vector valued function $u(x,t)$ is called the vector valued weak solution of initial value problem (3) for nonlinear degenerate parabolic system (1) in Q_T^*, if $u(x,t)$ satisfies the following conditions:*

(1) *The vector valued function* $u(x, t)$ *belongs to the functional space* $L_\infty((0, T); L_2(\mathbf{R}))$.

(2) $\operatorname{grad}\varphi(u(x, t))$ *belongs to* $L_\infty((0, T); W_2^{(1)}(\mathbf{R}))$.

(3) *For any test function contained in* $H^1(Q_T^*)$, *the integral relation*

$$\iint_{Q_T^*} [\psi_t u - \psi_x (\operatorname{grad}\varphi(u))_x] dx\, dt + \int_{-\infty}^{\infty} \psi(x, 0) u_0(x) dx = 0 \tag{42}$$

holds, where $\psi(x, T) \equiv 0$ $(x \in \mathbf{R})$ *and* $\operatorname{supp}\psi(x, t) < \infty$.

Now we take the vector valued solutions $u_\varepsilon(x, t)$ of the periodic boundary problem for non-degenerate system (28) or (29) as the approximations of the vector valued weak solution $u(x, t)$ of the initial value problem (3) for the degenerate system (1).

Let us construct a sequence $\{D_\varepsilon\}$ such that $D_\varepsilon \to \infty$ as $\varepsilon \to 0$, and $\varepsilon D_\varepsilon$ is kept to be bounded or $\varepsilon D_\varepsilon \to 0$. Suppose $\varepsilon D_\varepsilon \leqslant 1$. For every $\varepsilon > 0$, we define a vector valued function $u_{0\varepsilon}(x)$ such that $u_{0\varepsilon}(x) = u_0(x)$, as $x \in [-(D_\varepsilon - 1), (D_\varepsilon - 1)]$, and the estimation relations

$$\begin{aligned} &\|u_{0\varepsilon}\|_{W_2^{(1)}(D_\varepsilon, D_\varepsilon)} \leqslant A\|u_0\|_{W_2^{(1)}(\mathbf{R})} + B, \\ &\|\varphi(u_{0\varepsilon})\|_{L_1(-D_\varepsilon, D_\varepsilon)} \leqslant A\|\varphi(u_0)\|_{L_1(\mathbf{R})} + B \end{aligned} \tag{43}$$

take place, where $A \geqslant 1$, $B \geqslant 0$. From Theorem 1, we know the periodic boundary problem

$$\begin{aligned} &u(x + 2D_\varepsilon, t) = u(x, t), \\ &u(x, 0) = u_{0\varepsilon}(x), \quad x \in [-D_\varepsilon, D_\varepsilon], \end{aligned} \tag{44}$$

for non-degenerate system (28) or (29) has at least one vector valued generalized solution $u_\varepsilon(x, t) \in Z(Q_T^\varepsilon)$, where $\varepsilon > 0$, and $Q_T^{(\varepsilon)} = \{-D_\varepsilon \leqslant x \leqslant D_\varepsilon, 0 \leqslant t \leqslant T\}$.

Using the similar method, we get the following lemmas.

Lemma 6. *Suppose that the function* $\varphi(u)$ *satisfies conditions* (I), (II) *and* (III), *and suppose that the initial vector valued function* $u_0(x) \in W_2^{(1)}(\mathbf{R})$ *and* $\varphi(u_0(x)) \in L_1(\mathbf{R})$. *For the vector valued generalized solutions* $\{u_\varepsilon(x,t)\}$, $\varepsilon > 0$ *of periodic boundary problem* (44) *of approximate non-degenerate system* (28) *or* (29), *the following estimation*

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \|u_\varepsilon(\cdot, t)\|^2_{L_2(-D_\varepsilon, D_\varepsilon)} + \sup_{0 \leqslant t \leqslant T} \|(\operatorname{grad}\varphi_\varepsilon(u_\varepsilon(\cdot, t)))_x\|^2_{L_2(-D_\varepsilon, D_\varepsilon)} \\ &+ \sup_{0 \leqslant t \leqslant T} \|\varphi_\varepsilon(u_\varepsilon(\cdot, t))\|_{L_1(-D_\varepsilon, D_\varepsilon)} + \sup_{0 \leqslant t \leqslant T} \|u_{\varepsilon t}(\cdot, t)\|^2_{H^{-1}(-D_\varepsilon, D_\varepsilon)} \leqslant K_{12} \end{aligned} \tag{45}$$

holds, where K_{12} *is independent of* $\varepsilon > 0$ *and* $T > 0$.

Lemma 7. *Under the conditions of Lemma* 6, *the vector valued generalized solutions* $\{u_\varepsilon(x, t)\}$ *of periodic boundary problem* (44) *for approximate non-degenerate* *system* (28) *or* (29). *have the estimation*

$$\|\operatorname{grad}\varphi_s(u_s)\|_{L_\infty(Q_T^{(s)})}\leqslant K_{13}, \tag{46}$$

where K_{13} *is independent of* $s>0$.

Lemma 8. *Under the conditions of Lemma* 6, *the set* $\{u_s(x, t)\}$ *of the solutions of approximate problem* (28), (44) *or* (29), (44) *has the estimation*

$$\sup_{0\leqslant t\leqslant T}\|u_s(\cdot, t)\|_{C^{(0,s)}(-D_s,D_s)}\leqslant K_{14}, \tag{47}$$

where K_{14} *is independent of* $\varepsilon>0$ *and* $s=(2+2\mu)^{-1}$.

For the generalized solution $u_s(x, t)$ of non-degenerate problem (28), (44) or (29), (44), the integral relation

$$\iint_{Q_T^{(s)}}[\psi_t u_s-\psi_x(\operatorname{grad}\varphi_s(u_s))_x]\,dx\,dt+\int_{-D_s}^{D_s}\psi(x, 0)u_{0s}(x)=0 \tag{48}$$

holds, where $\psi(x, t)$ is any given test function belonging to $H^1(Q_T^*)$, i. e., $\psi(x, T)\equiv 0$ $(x\in\mathbf{R})$ and $\operatorname{supp}\psi(x, t)<\infty$, and εtis taken to be sufficiently small or D_s is taken to be sufficiently large.

Since $\{u_s(x, t)\}$ and $\{u_{st}(x,t)\}$ are uniformly bounded for ε in the norms of functional spaces $L_\infty((0, T); C^{(0,s)}(-D_s, D_s))$ and $L_\infty((0, T); H^{-1}(-D_s, D_s))$ respectively, $\{u_s(x, t)\}$ is compact in $L_p((0, T); C^{(0,r)}(-L, L))$, where L is any given positive constant, $1\leqslant p<\infty$ and $0<r<s$. The subsequence, denoted by $\{u_{s_i}(x, t)\}$, can be selected from $\{u_s(x, t)\}$, such that the subsequence in any given rectangular domain $\tilde{Q}_T=\{-L\leqslant x\leqslant L;\ 0\leqslant t\leqslant T\}$ converges to a vector valued function $u(x, t)\in L_p((0, T); C^{(0,r)}(\mathbf{R}))$ in the sence of the norm of functional space $L_p((0, T); C^{(0,r)}(-L, L))$. Hence $u_{s_i}(x, t)$ converges to $u(x, t)$ almost everywhere in Q_T^*. Then $\operatorname{grad}\varphi_{s_i}(u_{s_i}(x, t))$ also converges to $\operatorname{grad}\varphi(u(x, t))$ almost everywhere in Q_T^*. Also $\operatorname{grad}\varphi_{s_i}(u_{s_i}(x, t))$ and $(\operatorname{grad}\varphi_{s_i}(u_{s_i}(x, t)))_x$ weakly converge to $\operatorname{grad}\varphi(u(x, t))$ and $(\operatorname{grad}\varphi(u(x, t)))_x$ respectively. When $\varepsilon_i\to 0$, $D_{s_i}\to\infty$, the limit of integral relation (48) is naturally integral relation (43). Therefore the limiting vector valued function $u(x, t)$ is just the weak solution of initial value problem (3) for nonlinear degenerate system (1). The weak solution $u(x, t)$ thus obtained is contained in $L_p((0, T); C^{(0,r)}(\mathbf{R}))$, where $1\leqslant p<\infty$, $0<r<s$. According to the estimations in Lemmas 6, 7 and 8, it can be seen that $u(x, t)\in L_\infty((0, T); C^{(0,s)}(\mathbf{R}))$ and $\operatorname{grad}\varphi(u(x, t))\in L_\infty((0, T); W_2^{(1)}(\mathbf{R}))$.

Theorem 6. *Suppose that the function satisfies conditions* (I), (II), (III) *and suppose that the initial vector valued function* $u_0(x)\in W_2^{(1)}(\mathbf{R})$ *and* $\varphi(u_0(x))\in L_1(\mathbf{R})$. *For initial value problem* (3) *of nonlinear degenerate system* (1), *there exists at least one weak vector valued solution* $u(x, t)\in L_\infty((0, T); C^{(0,s)}(\mathbf{R}))$ *and* $\operatorname{grad}\varphi(u(x, t))\in L_\infty((0, T); W_2^{(1)}(\mathbf{R}))$, *where* $s=(2+2\mu)^{-1}$.

§ 8

Now we turn to the uniqueness problem of the weak solution $u(x, t)$ for degenerate initial value problem (1), (3).

Let $u_1(x, t)$ and $u_2(x, t)$ be the two different weak solutions of initial value problem (3) for nonlinear degenerate system (1). Their difference satisfies the integral relation

$$\iint_{Q_T^*} [\psi_t(u_2-u_1)-\psi_x((\operatorname{grad}\varphi(u_2))_x-(\operatorname{grad}\varphi(u_1))_x)]dx\,dt=0,$$

where $\psi(x, t)$ is any test function.

Let us take the vector valued test function

$$\tilde{\psi}(x, t)=\alpha_n(x)\int_T^t [\operatorname{grad}\varphi(u_2(x, \zeta))-\operatorname{grad}\varphi(u_1(x, \zeta))]d\zeta, \tag{49}$$

where $\alpha_n(x)$ has uniformly bounded for n continuous derivative in $x\in\mathbf{R}$; $\alpha_n(x)=0$, as $|x|\leqslant n-1$; $\alpha_n(x)=1$ as $|x|\geqslant n$; and $0\leqslant\alpha_n(x)\leqslant 1$, as $n-1\leqslant|x|\leqslant n$. It is obvious that $\tilde{\psi}(x, T)\equiv 0(x\in\mathbf{R})$, $\operatorname{supp}\tilde{\psi}(x, t)<\infty$ and $\tilde{\psi}(x, t)$ has the generalized derivatives

$$\tilde{\psi}_t(x, t)=\alpha_n(x)[\operatorname{grad}\varphi(u_2(x, t))-\operatorname{grad}\varphi(u_1(x, t))],$$

$$\tilde{\psi}_x(x, t)=\alpha_n'(x)\int_T^t [\operatorname{grad}\varphi(u_2(x, \zeta))-\operatorname{grad}\varphi(u_1(x, \zeta))]d\zeta$$
$$+\alpha_n(x)\int_T^t [(\operatorname{grad}\varphi(u_2(x, \zeta)))_x-(\operatorname{grad}\varphi(u_1(x, t)))_x]d\zeta.$$

Substituting these expressions into the integral relation

$$\iint_{Q_T^*} [(\tilde{\psi}_t, u_2-u_1)-(\tilde{\psi}_x, (\operatorname{grad}\varphi(u_2))_x-(\operatorname{grad}\varphi(u_1))_x)]dx\,dt=0,$$

we get

$$\iint_{Q_T^*} \alpha_n(x)(\operatorname{grad}\varphi(u_2)-\operatorname{grad}\varphi(u_1), u_2-u_1)dx\,dt$$
$$-\iint_{Q_T^*} \alpha_n(x)\Big(\int_T^t [(\operatorname{grad}\varphi(u_2(x, \zeta)))_x-(\operatorname{grad}\varphi(u_1(x, \zeta)))_x]d\zeta,$$
$$(\operatorname{grad}\varphi(u_2(x, t)))_x-(\operatorname{grad}\varphi(u_1(x, t)))_x\Big)dx\,dt$$
$$=\iint_{Q_T} \alpha_n'(x)\Big(\int_T^t [\operatorname{grad}\varphi(u_2(x, \zeta))-\operatorname{grad}\varphi(u_1(x, \zeta))]d\zeta,$$
$$(\operatorname{grad}\varphi(u_2(x, t)))_x-(\operatorname{grad}\varphi(u_1(x, t)))_x\Big)dx\,dt.$$

Denote $Q_T^{(n)}=\{n-1\leqslant|x|\leqslant n, 0\leqslant t\leqslant T\}$. The above equality becomes

$$\iint_{Q_T^*} \alpha_n(x)(H(\tilde{u})(u_2-u_1),\ (u_2-u_1))dx\,dt$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} \alpha_n(x)\left|\int_0^T [(\operatorname{grad}\varphi(u_2(x,\ \zeta)))_x-(\operatorname{grad}\varphi(u_1(x,\ \zeta)))_x]d\zeta\right|^2 dx$$

$$=\iint_{Q_T^{(n)}} \alpha'_n(x)\Big(\int_T^t [\operatorname{grad}\varphi(u_2(x,\ \zeta))-\operatorname{grad}\varphi(u_1(x,\ \zeta))]d\zeta,$$

$$(\operatorname{grad}\varphi(u_2(x,\ t)))_x-(\operatorname{grad}\varphi(u_1(x,\ t)))_x\Big)dx\,dt$$

Since the integrands of two integrals on the left hand side of the equality are non-negative, these two integrals are non-negative. Because $u_1(x,\ t)$ and $u_2(x,\ t)$ are the weak solutions of problem (1), (3), we have $u_1(x,\ t),\ u_2(x,\ t)\in L_\infty((0,\ T);\ L_2(\mathbf{R}))$, $\operatorname{grad}\varphi(u_1(x,\ t))$, $\operatorname{grad}\varphi(u_2(x,\ t))\in L_\infty((0,\ T);\ W_2^{(1)}(\mathbf{R}))$. Therefore the functions

$$(\operatorname{grad}\varphi(u_2(x,\ t))-\operatorname{grad}\varphi(u_1(x,\ t),\ u_2(x,\ t)-u_1(x,\ t)),$$

$$\left|\int_0^T [(\operatorname{grad}\varphi(u_2(x,\ \zeta)))_x-(\operatorname{grad}\varphi(u_1(x,\ \zeta)))_x]d\zeta\right|^2$$

and

$$\Big(\int_T^t [\operatorname{grad}\varphi(u_2(x,\ \zeta))-\operatorname{grad}\varphi(u_1(x,\ \zeta))]d\zeta,\ (\operatorname{grad}\varphi(u_2(x,\ t)))_x$$

$$-(\operatorname{grad}\varphi(u_1(x,\ t)))_x\Big)$$

are all integrable in Q_T^*. So the three integrals in the equality are uniformly bounded for n.

For the integral, denoted by J_n, in the right part of the equality there is an estimation

$$|J_n|\leqslant C\iint_{Q_T^{(n)}} \Big|\Big(\int_T^t [\operatorname{grad}\varphi(u_2(x,\ \zeta))-\operatorname{grad}\varphi(u_1(x,\ \zeta))]d\zeta,$$

$$(\operatorname{grad}\varphi(u_2(x,\ t)))_x-(\operatorname{grad}\varphi(u_1(x,\ t)))_x\Big)\Big|dx\,dt.$$

The right part of the inequality is the integral in $Q_T^{(n)}$ of the integr and which is integrable in Q_T^*, then it can be regarded as the general term of a convergent positive series. When $n\to 0$, the value of this integral tends to zero. Thus $|J_n|\to 0$, as $n\to\infty$. Therefore when $n\to\infty$, the limit of previous equality is

$$\iint_{Q_T^*} (\operatorname{grad}\varphi(u_2)-\operatorname{grad}\varphi(u_1),\ u_2-u_1)dx\,dt=0$$

or

$$\iint_{Q_T^*} (H(\tilde{u})(u_2-u_1),\ (u_2-u_1))dx\,dt=0$$

At the point $u_1\neq u_2$, $\tilde{u}\neq 0$, $(H(\tilde{u})(u_2-u_1),\ (u_2-u_1))\neq 0$. Hence $u_1(x,\ t)=u_2(x,\ t)$ almost everywhere in Q_T^*. The weak solution is unique.

Theorem 7. *Suppose that the function $\varphi(u)$ in nonlinear degenerate parabolic* *system* (1) *satisfies conditions* (I), (II) *and* (III). *The weak solution of initial value problem* (1), (3) *is unique.*

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