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A NOTE ON THE RESONANCE CASE FOR ASYMPTOTICALLY LINEAR WAVE EQUATIONS

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Abstract

First, the authors drop some convex and concave conditions on function g, which are needed for Theorems 1 and 3 in [1], by making use of a better integral estimate. Secondly, the authors consider two other resonance cases. In particular, the case $g'(\infty)=0$ is discussed.

§ 1. Introduction

In [1] we consider the existence of nontrivial periodic solutions of the following wave equation

$$
\text{(I)} \quad \begin{cases} u_{tt}-u_{xx}+g(x,\ t,\ u)=0, \\ u(0,\ t)=u(\pi,\ t)=0, \\ u(x,\ t+2\pi)=u(x,\ t), \end{cases}
$$

where $(x,\ t)\in\Omega=\{0<x<\pi,\ 0<t<2\pi\}$.

Let A be the selfadjoint extension of the operator $\square=\dfrac{\partial^2}{\partial t^2}-\dfrac{\partial^2}{\partial x^2}$ determined by (I). Its distinguishing eigenvalues are denoted by $\{\lambda_i\}$, and their multiplicity by $M(\lambda_i)$ and the corresponding eigenvector subspaces by F_i, for $i\in\mathbb{Z}$, where $\cdots<\lambda_{-i}<\lambda_{-i+1}<\cdots<\lambda_0=0<\lambda_1<\cdots<\lambda_i<\cdots$, and $M(\lambda_i)$ is an even integer. We write $g(x,\ t,\ \xi)$ as

$$
g(x,\ t,\ \xi)=b\xi+g_1(x,\ t,\ \xi)
$$

and set

$$
G(x,\ t,\ \xi)=\int_0^\xi g(x,\ t,\ \eta)\,d\eta \quad G_1(x,\ t,\ \xi)=\int_0^\xi g(x,\ t,\ \eta)\,d\eta.
$$

Assumption $[g]$. The function $g(x,\ t,\ \xi)$ is strictly increasing and continuously differentiable in ξ, for $(x,\ t,\ \xi)\in\bar{\Omega}\times(\mathbb{R}^1\backslash\{0\})'$ and satisfies the following conditions:

(g_∞) conditions at infinity.

There is a constants $b=g'(\infty)$ such that

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$$\lim_{|\xi|\to\infty} g(x, t, \xi)/\xi = b \in (0, +\infty)$$

uniformly in $(x, t)\in\bar{\Omega}$. And

(a_∞) $$\inf_{(x,t)\in\bar{\Omega}} g'_{1,\xi}(x, t, \xi) > -b.$$

In case $b=-\lambda_{-p}$, for some finite positive integer p, we further assume that

(b_∞) $\sup_{(x,t,\xi)\in\bar{\Omega}\times\mathbf{R}^1} |g_1(x, t, \xi)|\leqslant M$, for some $M>0$;

$(c_\infty)^\pm$ $G_1(x, t, \xi)\to\pm\infty$, as $|\xi|\to\infty$, uniformly in $(x, t)\in\bar{\Omega}$.

(g_0) conditions at zero.

(i) $g(x, t, 0)=0$ and

$$\lim_{|\xi|\to 0} g(x, t, \xi)/\xi=+\infty, \text{ uniformly in } (x, t)\in\bar{\Omega}.$$

(ii) In a neighbourhood of zero, we have

(a_0) $g(x, t, \xi_2)-g(x, t, \xi_1)\leqslant g'_\xi(x, t, (\xi_1+\xi_2)/2)(\xi_2-\xi_1)$ for $\xi_2>\xi_1\geqslant 0$ or $\xi_1<\xi_2\leqslant 0$;

(b_0) $g(x, t, \theta\xi_1+(1-\theta)\xi_2)\geqslant\theta g(x, t, \xi_1)+(1-\theta)g(x, t, \xi_2)$ for $\xi_2, \xi_1\geqslant 0$, $\theta\in[0, 1]$, and $(x, t)\in\bar{\Omega}$; the converse inequality holds for $\xi_2, \xi_1\leqslant 0$;

namely, $g(x, t, \xi)$ is concave in $\xi\in\mathbf{R}^+$ and convex in $\xi\in\mathbf{R}^-$.

The main result of [1] is

Theorem(*) *In addition to assumption* [g], *if the function $g(x, t, \xi)$ is odd in ξ, then problem (I) has infinitely many periodic solutions, which are on different orbits.*

It is improved and extended in this paper. In section 2, we point out that conditions (a_0), (b_0) can be droped out of [g]. Thus the result is parallel to the work of K. Thews[6]. In section 3 we deal with some resonance cases which are not treated in [1]. In particular, the case $b=0$ is discussed.

§ 2. Improvement of Theorem(*)

Theorem 2.1. *Theorem(*) still holds without conditions (a_0), (b_0) in its assumptions.*

For simplicity of expressions, we do all the arguments in form $g=g(t)$. As in [1, 5], we reduce the problem (I) into the variational problem

$$I(u)=\frac{1}{2}\langle Ku, u\rangle+\iint_\Omega H(u)\,dx\,dt \tag{1}$$

in real Hilbert space $L^2(\Omega)$, where $K=A^{-1}$ defined on the range $R(A)$ of operator A and $H(t)=\int_0^t h(s)ds$, and $h(s)$ is the inverse function of g having the form

$$h(s)=as+h_1(s),$$

where $a=1/b$, and $h_1(s)=-1/bg_1(h_1(s))$. Set $H_1(t)=\int_0^t h_1(s)ds$. Then it is easily seen that the following conditions are satisfied:

(h_∞) $\lim\limits_{|t|\to\infty} h(t)/t=a=1/b\in(0, \infty)$;

as $a=-\mu_{-p}=-1/\lambda_{-p}$, we have

$(b_\infty)'$ $|h_1(t)|\leqslant M$;

$(c_\infty)'^{\pm}$ $H_1(t)\to\mp\infty$, as $|t|\to\infty$;

(b_0) $h(0)=0$, and $h'(0)=\lim\limits_{|t|\to 0} h(t)/t=0$.

Note that conditions (a_0), (b_0) were used only in varifying (P. S) condition for the functional I. Hence it suffices to verify (P. S) condition under the assumptions of Theorem 2.1.

Set $a=-\mu_{-k}$. Let N be the kernel of the operator K, which is a finite dimensional space; and N^- be the orthogonal summation of subspaces $F_{-1}, \cdots, F_{-k}$; and N^+ the orthogonal complement of $N\oplus N^-$ in space $R(A)$. For any $u\in R(A)$, we set

$$u=u^0+u^++u^-=u^0+u', \quad \text{and} \quad u'=u^++u^-.$$

Suppose that the sequence $\{u_n\}\in R(A)$ is such that

$$|I(u_n)|\leqslant M \quad \text{and} \quad I'(u_n)=Ku_n+Ph(u_n)\to 0,$$

where P is the projector of H on $R(A)$. We now show that there is a convergent subsequence of $\{u_n\}$. The proof consists of 5 steps.

Claim 1[1]. *$\{u_n\}$ is a bounded sequence.*

Then there is a subsequence (still denoted by u_n) weakly convergent to u in H:

$$u_n\rightharpoonup u. \tag{3}$$

Claim 2. *$\int_Q H(u_n)dxdt\to\int_Q H(u)dx\,dt$, for all $Q\subset\Omega$.*

As $H(u)$ is a convex function of u, we get

$$H(u_n)-H(u)\geqslant h(u)(u_n-u).$$

Integrating it on $Q\subset\Omega$, we have

$$\underline{\lim}\int_Q H(u_n)\geqslant\int_Q H(u) \tag{4}$$

by $u_n\rightharpoonup u$.

Set $Ku_n+Ph(u_n)=\varepsilon_n$. We have

$$\int_\Omega(H(u)-H(u_n))\geqslant\int_\Omega h(u_n)(u-u_n)=\int_\Omega(-Ku_n+\varepsilon_n)(u-u_n).$$

By means of the compactness of operator K[2], Ku_n strongly converges to u. Hence we get

$$\int_\Omega H(u)\geqslant\overline{\lim}\int_\Omega H(u_n).$$

By virtue of $H(u)\geqslant 0$, we have

$$\int_Q H(u)+\int_{\Omega/Q}H(u)\geqslant\overline{\lim}\Big(\int_Q H(u_n)+\int_{\Omega/Q}H(u_n)\Big)$$

and
$$\int_Q H(u)+\overline{\lim}\int_{\Omega/Q} H(u_n)\geqslant\overline{\lim}\int_Q H(u_n)+\overline{\lim}\int_{\Omega/Q} H(u_n).$$
Hence
$$\int_Q H(u)\geqslant\overline{\lim}\int_Q H(u_n). \tag{5}$$
The inequalities (4) and (5) give (3).

Claim 3. *$H(u_n)$ is equi-integral continuous, namely, for any $\varepsilon>0$, there exists $\delta>0$ such that $\int_Q H(u_n)<\varepsilon$ for all n, provided $\mu(Q)<\delta$ for any $Q\subset\Omega$.*

By virtue of the integral continuity of $H(u(x, t))$, there exists a constant $\delta>0$ such that $\int_Q H(u)<\varepsilon$ for any $Q\subset\Omega$ and $\mu(Q)<\delta_n$, where $\mu(Q)$ is the measure of the set Q.

Suppose that the claim is not true. Then, for each $\delta_k=\delta/2^k$, $k=1, 2, \cdots$, there exists a domain $Q_k\subset\Omega$, $\mu(Q_k)\leqslant\delta/2^k$, and function u_{n_k} such that $\int_{Q_k} H(u_{n_k})\geqslant\varepsilon$, where the index n_k tends to infinity. Set $Q=\bigcup_{k=1}^{\infty}Q_k$. We have $\mu(Q)\leqslant\sum_{k=1}^{\infty}\mu(Q_k)\leqslant\delta$ and
$$\int_Q H(u)=\lim_k\int_Q H(u_{n_k})\geqslant\varepsilon, \text{ a contradiction.}$$

Claim 4. $\int_\Omega H(u_n-u)\to 0$, *as* $n\to+\infty$.

Divide Ω into three parts Ω_a, Ω_{1n}, Ω_{2n}, defined as follows:
$$\Omega_a=\{x\,|\,|u|>a\},$$
$$\Omega_{1n}=\{x\,|\,|u_n-u|<\varepsilon,\ |u|\leqslant a\},$$
$$\Omega_{2n}=\{x\,|\,|u_n-u|\geqslant\varepsilon,\ |u|\leqslant a\}.$$

For $\int_\Omega H(u)\geqslant H(a)\cdot\mu(\Omega_a)$, $\mu(\Omega_a)$ becomes sufficiently small when a is large enough. Take Ω_a with $\mu(\Omega_a)\leqslant\delta$ such that $\int_{\Omega_a} H(u_n)\leqslant\varepsilon$ and $\int_{\Omega_a} H(u)\leqslant\varepsilon$. By condition ($h_\infty$), it is easy to see that there exist constants c_1, $c_2>0$ such that $H(2t)\leqslant c_1H(t)+c_2$ for all $t\in R^1$. Furthermore by the convexity and evenness of $H(u)$, we have
$$H(u_n-u)\leqslant\frac{1}{2}[H(2u_n)+H(2u)]\leqslant c_1'[H(u_n)+H(u)]+c_2.$$
Therefore
$$\int_{\Omega_a} H(u_n-u)\leqslant 2c_1'\varepsilon+c_2\mu(\Omega_a). \tag{6}$$
On the other hand, we have
$$H(u_n-u)\leqslant H(\varepsilon), \text{ on domain } \Omega_{1n}.$$
These two inequalities give
$$\int_{\Omega_{1n}} H(u_n-u)\leqslant cH(\varepsilon). \tag{7}$$

Finally we should show that $\mu(\Omega_{2n})\to 0$, as $n\to+\infty$. Therefore we can get the same estimate as we have got on the domain Ω_a, For this end we first show that the inequalities

$$H(u_n)-H(u)\geqslant h(u)(u_n-u)+\gamma \tag{8}$$

hold on the domain Ω_{2n}, where $\gamma>0$ is dependent on s and a, but independent of n.

The strict monotonicity of the function h implies that

$$H(u_n)-H(u)-h(u)(u_n-u)$$
$$\geqslant H(u+s)-H(u)-h(u)s=s\int_u^{u+s}[h(\tau)-h(u)]d\tau\equiv\lambda(u)>0,$$

when $\bar{x}=(x,t)$ is in the domain $\Omega\cap\Omega_{2n}^+$, where $\Omega_{2n}^+=\{\bar{x}\,|\,u_n-u\geqslant s,\ |u|\leqslant a\}$. The function $\lambda(u)$ is continuous in u, which has a positive lower bound λ_+ on the domain $\{|u|\leqslant a\}$. The same argument shows that $H(u_n)-H(u)-h(u)(u_n-u)$ (as a function of u) has a positive lower bound λ_- on the domain $\{|u|\leqslant a\}$. Taking $\gamma=(\lambda_+,\lambda_-)$, we obtain (8).

Integrating (8) on the domain Ω_{2n} and noting that

$$H(u_n)-H(u)-h(u)(u_n-u)\geqslant 0 \text{ for all } u_n,\ u$$

we get

$$\gamma\mu(\Omega_{2n})\leqslant\int_{\Omega_{2n}}[H(u_n)-H(u)-h(u)(u_n-u)]$$
$$\leqslant\int_\Omega[H(u_n)-H(u)-h(u)(u_n-u)].$$

It implies that the right hand side term tends to zero by (3) and $u_n\rightharpoonup u$.

By the same reasoning on the domain Ω_a, we obtain

$$\int_{\Omega_{2n}}H(u_n-u)\leqslant 2c_1's+c\mu(\Omega_{2n}). \tag{9}$$

Thus we have $\int_\Omega H(u_n-u)\to 0$ as $n\to\infty$ from the inequalities (6), (7) and (9).

Claim 5. *u_n tends to u strongly in the space $H=L^2(\Omega)$.*

By condition (h_∞), $H(t)/t^2\to a/2$ as $|t|\to+\infty$. Then there exists constant $c_s>0$, for any $s>0$ such that $u^2\leqslant c_sH(u)+s^2$. Hence

$$\int_\Omega|u_n-u|^2\leqslant c_s\int_\Omega H(u_n-u)+s^2\mu(\Omega).$$

Taking s small and letting $n\to+\infty$, we complete the verification of (P. S) condition.

§ 3. The other kind of resonance case

When the resonance does not occur at infinity i. e., $b\neq-\lambda_{-p}$, where p is any positive integer, Theorem (*) ensures the existence of infinitely many periodic solutions on different orbits. However, when the resonance happens at infinity, it is necessary to have more restriction on the function g_1 and $b\neq 0$. Now we are going to discuss some fifferent type of resonance case which implies the case $b=0$.

Condition (γ). The function $g(x, t, \xi)$ is an odd and strictly increasing function in the veriable ξ, and there are constants $\gamma<3$ and $c>0$ such that

$$|g(x, t, \xi)|\leqslant\gamma\xi+c.$$

Theorem 3.1. *Under the conditions* (γ), $(c_\infty)^\pm$ *and* (g_0) (i), *problem* (I) *has infinitely mamy periodic solutions, which are on different orbits.*

Proof We simply reduce the problem into the case of Theorem (*). Consider the truncated function $g_M(x, t, \xi)$:

$$g_M(x, t, \xi)=\begin{cases}\gamma(\xi-M-1)+g(x, t, M+1), & \xi\geqslant M+1;\\ g(x, t, \xi), & |\xi|\leqslant M;\\ \gamma(\xi+M+1)+g(x, t, -M-1), & \xi\leqslant -M-1;\\ \text{smooth function}, & \text{otherwise}.\end{cases}$$

It is easily seen that the function g_M satisfies all the conditions in Theorem (*) but (a_∞) (now $b=\gamma$). Note that what we really need is the strict monotonicity of g. Applying Theorem (*) to the problem

$$(I^M)\quad\begin{cases}u^M_{tt}-u^M_{xx}+g_M(x, t, u^M)=0,\\ u^M(0, t)=u^M(\pi, t)=0,\\ u^M(x, t+2\pi)=u^M(x, t),\end{cases}$$

we get the existence of infinitely many solutions which are on different orbits.

It is known that there is an L_∞-estimate for the solution u^M of problem (I^M)[4]. It follows that the solution of (I^M) is also the solution of (I) when M is sufficiently large. The proof is finished.

The following example shows that the restriction on the boundedness of function g_1 could be replaced by the other growth condition when the resonance also occurs at infinity.

Condition (α). There exist constants c_1, c_2, c_3, $c_4>0$ and $0<\alpha<1$ such that

$$c_1\xi^\alpha-c_2\leqslant g_1(x, t, \xi)\leqslant c_3\xi^\alpha+c_4,\quad \forall\xi>0.$$

Theorem 3.2. *Under the assumptions of Theorem* 2.1 *with condition* (b_∞) *being replaced by condition* (α), *the conclusion of Theorem* 2.1 *still holds.*

Proof It suffices to verify (P. S) condition. It is easy to see that there exist constants c'_1, c'_2, c'_3, $c'_4>0$ such that

$$c'_1\eta^\alpha-c'_2\leqslant -h_1(x, t, \eta)\leqslant c'_3\eta^\alpha+c'_4,\quad \forall\eta>0.$$

Suppose that the sequence $\{u_n\}\in R(A)$ has properties

$$|I(u_n)|\leqslant M\quad\text{and}\quad I'(u_n)=Ku_n+au_n+Ph(u_n)\to 0.$$

In order to get the existence of convergent subsequence of $\{u_n\}$, we only need to show the boundedness of $\{u_n\}$. Then the other steps for conclusion will be the same as we did in [1].

Setting $\varepsilon_n=Ku_n+au_n+Ph_1(u_n)$ and making inner product with u_n^+, we get

$$c|u_n^+|_{L^2}^2 \leqslant \langle Ku_n+\lambda u_n,\ u_n^+\rangle \leqslant |\langle \varepsilon_n,\ u_n^+\rangle| + |\langle h_1(u_n),\ u_n^+\rangle|$$
$$\leqslant c|u_n^+|_{L^2} + \int (c+c|u_n|^\alpha)|u_n^+| \leqslant c|u_n^+|_{L^2} + c|u_n|_{L^{2\alpha}}^\alpha \cdot |u_n^+|_{L^2}.$$

Then
$$|u_n^+|_{L^2} \leqslant c+c|u_n|_{L^2}^\alpha.$$

In the same fashion, we obtain
$$|u_n^-|_{L^2} \leqslant c+c|u_n|_{L^2}^\alpha.$$

Thus we have
$$|u_n'|_{L^2} \leqslant c+c|u_n|_{L^2}^\alpha \leqslant c+c|u_n'|_{L^2}^\alpha + c|u_n^0|_{L^2}^\alpha.$$

It follows that
$$|u_n'|_{L^2} \leqslant c+c|u_n^0|_{L^2}^\alpha. \tag{10}$$

On the other hand we have
$$\int |u_n|^{1+\alpha} \leqslant \int [c-ch_1(u_n)]u_n \leqslant c|u_n|_{L^2} - c\int h_1(u_n)u_n^0 - c\int h_1(u_n)u_n'$$
$$\leqslant c|u_n|_{L^2} + c|u_n^0|_{L^2} + c + c|u_n'|_{L^2}^2 \leqslant c+c|u_n^0|_{L^2} + c|u_n^0|_{L^2}^{2\alpha}$$

and
$$|u_n^0|_{L^2}^{1+\alpha} \leqslant c|u_n^0|_{L^{1+\alpha}}^{1+\alpha} \leqslant c(|u_n|_{L^{1+\alpha}}^{1+\alpha} + |u_n'|_{L^2}^{1+\alpha}) \leqslant c+c|u_n^0|_{L^2} + c|u_n^0|_{L^2}^{2\alpha}. \tag{11}$$

Noting that $1+\alpha > 2\alpha$, we obtain $|u_n^0|_{L^2} \leqslant c$. Hence $|u_n'|_{L^2} \leqslant c$ and $|u_n|_{L^2} \leqslant c$.

Remark 1. Condition (α) could be slightly relaxed as follows:

There exist α, β with $0<\beta<\alpha<(1+\beta)/2$ and constants $c_i>0$ such that
$$c_1\xi^\beta - c_2 \leqslant g_1(x,\ t,\ \xi) \leqslant c_3\xi^\alpha + c_4.$$

In this case inequality (11) becomes
$$|u_n^0|_{L^2}^{1+\beta} \leqslant c+c|u_n^0|_{L^2} + c|u_n^0|_{L^2}^{2\alpha}. \tag{11$'$}$$

However we can not verify (P. S) condition when the function g satisfies certain one sided growth condition such as
$$|g_1(x,\ t,\ \xi)| \leqslant c|\xi|^\alpha$$

with $\alpha<1$.

Remark 2. All the theorems are true when the function g is autonom us by means of the S^1-index theory.

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