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HÖLDER ESTIMATES FOR SOLUTIONS OF UNIFORMLY DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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Abstract

In this paper the author discusses the quasilinear parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left[a_{ij}(x, t, u)\frac{\partial u}{\partial x_j}\right]+b_i(x, t, u)\frac{\partial u}{\partial x_i}+c(x, t, u)$$

Which is uniformly degenerate at $u=0$. Let $u(x, t)$ be a classical solution of the equation satisfying $0<u(x, t)\leqslant M$. Under some assumptions the author establishes the interior estimations of Hölder coefficient of the solution $u(x, t)$ for the equation and the global estimations for Cauchy problems and the first boundary value problems, where Hölder coefficients and exponents are independent of the lower positive bound of $u(x, t)$.

L. A. Caffarelli and A. Friedman[1,2] studied Hölder continuity of the solution of Cauchy problem for the n-dimensional porous medium equation

$$\frac{\partial u}{\partial t}=\triangle(u^m), \quad m>1.$$

They first obtained the estimate for the lower bound of $\frac{\partial u}{\partial t}$ based on the simple form of the equation and then found Hölder estimate for the solution. Therefore, it is not easy to generalize their method to more complicated equations and boundary value problems. In addition, they got only interior estimates for Cauchy problem.

In this paper, applying the method nsed by O. A. Ladyzhenskaya and N. N. Ural'tseva[3] and overcoming difficuties arisen from degeneration, we establish Hölder interior estimates for solutions of uniformly degenerate parabolic equations and global estimates for Cauchy problems and the first boundary value problems.

In § 1 we state the hypotheses about the equation and in § 2 we introduce the generalized $\mathscr{B}_2$ class as it was done in [3]. We give the preliminary lemmas on the generalized $\mathscr{B}_2$ class in § 3. Finally, we obtain Hölder estimates for solutions in § 4.

§ 1

Let $\mathbf{R}^n$ be an n-Euclidean space, Ω an open domain of $\mathbf{R}^n$ and $\partial\Omega$ the boundary

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of Ω. Let Q_T be the $(n+1)$-dimensional domain $\Omega\times(0, T]$ and $\Gamma=\bar{Q}_T\setminus Q_T$. In Q_T we consider the quasilinear parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a_{ij}(x, t, u)\frac{\partial u}{\partial x_j}\right)+b_i(x, t, u)\frac{\partial u}{\partial x_i}+c(x, t, u)u. \tag{1.1}$$

We shall assume that $u(x, t)$ is a classical solution of equation (1.1) such that $0<u(x, t)\leqslant M$ and then discuss its Hölder estimate. Equation (1.1) will be degenerate when $u=0$ by the conditions below. Since the nonnegative weak solutions of equation (1.1) is often approximated by positive classical solutions, Hölder continuity of weak solutions will refer to the problem mentioned above.

Suppose that the coefficients of equation (1.1) satisfy the following conditions:

(i) For any $\xi\in\mathbf{R}^n$, $(x, t)\in Q_T$, $0<u<\infty$,

$$\nu(|u|)|\xi|^2\leqslant a_{ij}(x, t, u)\xi_i\xi_j\leqslant\Lambda\nu(|u|)|\xi|^2, \tag{1.2}$$

where Λ is a constant and $\nu(s)$ is a function which has the following properties:

(a) $\nu(s)\in C[0, \infty)$,

$$\nu(0)=0 \text{ and } \nu(s)>0 \text{ if } s>0. \tag{1.3}$$

(b) Denote $\varphi(u)=\int_0^u\nu(s)\,ds$. There exist $\delta>0$ and $m>1$ such that for $0<u\leqslant\delta$ we have

$$1\leqslant\frac{\varphi'(u)u}{\varphi(u)}\leqslant m. \tag{1.4}$$

(ii) If $(x, t)\in Q_T$, $0<u\leqslant M$, then

$$\frac{1}{\nu(u)}\sum_{i=1}^n b_i^2(x, t, u)+\sum_{i=1}^n\left|\frac{\partial b_i(x, t, u)}{\partial x_i}\right|+|c(x, t, u)|\leqslant\Lambda. \tag{1.5}$$

We shall not describe the conditions about the smoothness of the coefficients in detail. In fact, Hölder estimates for solutions obtained in this paper will depend only on the constants n, M, Λ, δ, m and T. In order to find boundary estimates for the first boundary value problems, we have to give an additional condition about the boundary $\partial\Omega$ of the domain Ω:

(iii) There exist $a_0>0$ and $\theta_0\in(0, 1)$ such that for any n-dimensional ball $K(\rho)$ with its centre on $\partial\Omega$ and radius ρ, we have

$$\operatorname{mes}\{K(\rho)\cap\Omega\}\leqslant(1-\theta_0)\operatorname{mes}K(\rho) \tag{1.6}$$

if $\rho\leqslant a_0$, where mes $\{\cdot\}$ is the measure of a set in $\mathbf{R}^n$.

Remark. Using the transform $u=e^{\alpha t}v$, we may, without loss of generality, suppose

$$c(x, t, u)\leqslant 0.$$

Hence, we can change equation (1.1) into the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a_{ij}(x, t, u)\frac{\partial u}{\partial x_j}+\right)b_i(x, t, u)\frac{\partial u}{\partial x_j}+\tilde{c}(x, t, u) \tag{1.1$'$}$$

with the condition

$$-\Lambda \leqslant \tilde{c}(x, t, u) \leqslant 0. \tag{1.7}$$

Let $w=\varphi(u)=\int_0^u \nu(s)ds$ and its inverse be

$$u=\Phi(w). \tag{1.8}$$

By virtue of condition (1.4) we can prove that for $0<w_1<w_2\leqslant\varphi(\delta)$

$$\frac{1}{m}\leqslant\frac{\Phi'(w_1)}{\Phi'(w_2)}\leqslant m\left(\frac{w_2}{w_1}\right)^{1-\frac{1}{m}}. \tag{1.4)'}$$

In fact, condition (1.4) implies

$$\frac{1}{mw}\leqslant\frac{\Phi'(w)}{\Phi(w)}\leqslant\frac{1}{w} \quad \text{if} \quad 0<w\leqslant\varphi(\delta). \tag{1.9}$$

For $0<w_1<w_2\leqslant\varphi(\delta)$, integrating (1.9) from w_1 to w_2, we have

$$\left(\frac{w_2}{w_1}\right)^{\frac{1}{m}}\leqslant\frac{\Phi(w_2)}{\Phi(w_1)}\leqslant\frac{w_2}{w_1}. \tag{1.10}$$

Using inequalities (1.9) and (1.10) we can obtain (1.4)′ without any difficulty. We may suppose that (1.4)′ is satisfied for $0<w_1<w_2\leqslant\varphi(M)$ if we change the constant m properly.

As a matter of fact, in this paper we may use inequality (1.4)′ instead of condition (1.4).

§ 2

Let $(x^0, t^0)\in Q_T$ and $K(\rho)$ be a ball in $\mathbf{R}^n$ with its centre at x_0 and radius ρ. For $0\leqslant t\leqslant T$ denote

$$A_{k,\rho}(t)=\{x\in K(\rho)\cap\Omega\,|\,w(x, t)>k\},$$
$$B_{k,\rho}(t)=\{x\in K(\rho)\cap\Omega\,|\,w(x, t)<k\}, \tag{2.1}$$

where

$$w(x, t)=\varphi(u(x, t))=\int_0^{u(x,t)}\nu(s)ds$$

and $u(x, t)$ is a solution of equation (1.1)′.

Lemma 1. *Suppose that the coefficients of equation (1.1)′ satisfy conditions (1.2), (1.3), (1.4), (1.5) and (1.7). Let $u(x, t)$ be a classical solution of equation (1.1)′ satisfying $0<u(x, t)\leqslant M$ and $\zeta(x)$ be a cut-off function in $K(\rho)$. We have*

(i) *if* $k\geqslant\max\limits_{x\in K(\rho)\cap\partial\Omega} w(x, t)$, *then*

$$\frac{\partial}{\partial t}\left[e^{-\gamma t}\int_{A_{k,\rho}(t)}\zeta^2\chi_k(w-k)dx\right]+\frac{1}{2}e^{-\gamma t}\int_{A_{k,\rho}(t)}\zeta^2|\nabla w|^2dx\leqslant\gamma\int_{A_{k,\rho}(t)}|\nabla\zeta|^2(w-k)^2dx \tag{2.2}$$

(ii) *if* $k\leqslant\min\limits_{x\in K(\rho)\cap\partial\Omega} w(x, t)$, *then*

$$\frac{\partial}{\partial t}\left[e^{-\gamma t}\int_{B_{k,\rho}(t)}\zeta^2\tilde{\chi}_k(k-w)dx\right]+\frac{1}{2}e^{-\gamma t}\int_{B_{k,\rho}(t)}\zeta^2|\nabla w|^2dx$$
$$\leqslant\gamma\left[\int_{B_{k,\rho}(t)}|\nabla\zeta|^2(w-k)^2dx+\operatorname{mes}B_{k,\rho}(t)\right], \tag{2.3}$$

where $\gamma=\gamma(n, \Lambda, M)$, ∇ *is the gradient operator with respect to* x *and*

$$\chi_k(s)=\int_0^s \Phi'(k+\tau)\tau d\tau, \quad \tilde{\chi}_k(s)=\int_0^s \Phi'(k-\tau)\tau\, d\tau. \tag{2.4}$$

Proof Let $w^+=\max\{0, w\}$. Multiplying equation (1.1)′ by $\zeta^2(x)\,(w-k)^+$ and integrating it over Ω and taking notice of (1.7), we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)dx+\int_{A_{k,\rho}(t)} \zeta^2 a_{ij}(x, t, u)\frac{\partial w}{\partial x_i}\frac{\partial u}{\partial x_j}dx\\ &\leqslant -2\int_{A_{k,\rho}(t)} \zeta(w-k)a_{ij}\frac{\partial \zeta}{\partial x_i}\frac{\partial u}{\partial x_j}dx\\ &\quad+\int_{A_{k,\rho}(t)} \zeta^2(w-k)b_i(x, t, u)\frac{\partial u}{\partial x_i}dx.\end{aligned} \tag{2.5}$$

Now set

$$h_i(x, t, s)=\int_0^s b_i(x, t, \Phi(k+\tau))\Phi'(k+\tau)\tau\, d\tau. \tag{2.6}$$

By condition (1.5), it follows tnat

$$\begin{aligned}|h_i(x, t, s)|^2&\leqslant\int_0^s b_i^2(x, t, \Phi(k+\tau))\Phi'(k+\tau)\tau\, d\tau\cdot\int_0^s \Phi'(k+\tau)\tau\, d\tau\\ &\leqslant\frac{1}{2}\Lambda s^2\chi_k(s).\end{aligned} \tag{2.7}$$

By virtue of (1.2), (1.5) and (2.6), inequality (2.5) yields

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)dx+\int_{A_{k,\rho}(t)} \zeta^2|\nabla w|^2dx\\ &\leqslant 2\Lambda\int_{A_{k,\rho}(t)} \zeta|\nabla\zeta||\nabla w|(w-k)\, dx+\int_{A_{k,\rho}(t)} \zeta^2\frac{\partial h_i(x, t, w-k)}{\partial x_i}dx\\ &\quad-2\int_{A_{k,\rho}(t)} \zeta^2\left[\int_0^{w-k}\frac{\partial b_i(x, t, \Phi(k+\tau))}{\partial x_i}\Phi'(k+\tau)\tau\, d\tau\right]dx.\end{aligned} \tag{2.8}$$

Integrating by parts the second term on the right side and using (1.5), (2.7), we find

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)dx+\int_{A_{k,\rho}(t)} \zeta^2|\nabla w|^2dx\\ &\leqslant 2\int_{A_{k,\rho}(t)} \Lambda\zeta|\nabla\zeta||\nabla w|(w-k)dx+\sqrt{2\Lambda}\int_{A_{k,\rho}(t)} \zeta|\nabla\zeta|(w-k)\chi_k^{\frac{1}{2}}(w-k)dx\\ &\quad+\Lambda\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)dx.\end{aligned}$$

By Schwarz inequality, it follows that

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)\, dx+\frac{1}{2}\int_{A_{k,\rho}(t)} \zeta^2|\nabla w|^2dx\\ &\leqslant\gamma\left[\int_{A_{k,\rho}(t)} |\nabla\zeta|^2(w-k)^2dx+\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)\, dx\right],\end{aligned}$$

which implies (2.2). Inequality (2.3) can be proved in the same way but the term $\tilde{c}(x, t, u)$ yields an additional term mes $B_{k,\rho}(t)$.

We shall call the family of all the functions satisfying (2.2) and (2.3) the generalized $\mathscr{B}_2(Q_T, M, m, \gamma)$ class (cf. [3]).

§ 3

In this section we shall discuss the properties of the generalized $\mathscr{B}_2$ class. We apply the method used in [3], but there are many important differences in view of the degeneration.

Lemma 3.1. *For any $u\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$, we have*

$$\int_{A_0}|u|^2dx\leqslant C(\operatorname{mes}A_0)^{\frac{2}{n}}\int_{A_0}|\nabla u|^2dx, \tag{3.1}$$

where $A_0=\{x\in\Omega\,|\,u(x)>0\}$ and $C=C(n)$.

Lemma 3.2. *For any $u\in W_m^{(1)}(K(\rho))$, $m>1$, we have*

$$(\lambda-k)\operatorname{mes}A_{\lambda,\rho}^{1-\frac{1}{n}}\leqslant\frac{\beta\rho^n}{\operatorname{mes}(K(\rho)\setminus A_{k,\rho})}\int_{A_{k,\rho}\setminus A_{\lambda,\rho}}|\nabla u|\,dx, \tag{3.2}$$

where $\lambda>k$, $\beta=\beta(n)$ and $A_{k,\rho}=\{x\in K(\rho)\,|\,u(x)>k\}$.

These two lemmas can be found in [3].

The functions $\chi_k(s)$ and $\tilde{\chi}_k(s)$ given in (2.4) for nondegenerate equations have the properties: $\chi_k(s)\sim s^2$ and $\tilde{\chi}_k(s)\sim s^2$. Now they do not have these properties again due to the degeneration. However, we have the following lemma.

Lemma 3.3. (i) *For $\mu>k\geqslant\frac{\mu}{2}>0$, $H\leqslant\mu-k$, $0<\beta<1$, we have*

$$H^2/2m\leqslant\chi_k(H)/\Phi'(\mu)\leqslant mH^2, \tag{3.3}$$

$$\chi_k(H)/\chi_k(\beta H)\leqslant1+m^2(1-\beta^2)/\beta^2. \tag{3.4}$$

(ii) *For $k>H>0$, $0<\beta<1$, we have*

$$H^2/2m\leqslant\tilde{\chi}_k(H)/\Phi'(k)\leqslant m^2H^2, \tag{3.5}$$

$$\tilde{\chi}_k(H)/\tilde{\chi}_k(\beta H)\leqslant1+\max\{4m^3(1-\beta)^{\frac{1}{m}},\ 2m^2(1-\beta^2)/\beta^2\}. \tag{3.6}$$

Proof Inequality (3.3) is obvious if we take notice of expression (2.4) of $\chi_k(s)$ and condition (1.4)′.

Now we prove (3.4). In fact

$$\frac{\chi_k(H)}{\chi_k(\beta H)}-1=\frac{\int_{\beta H}^{H}s\Phi'(k+s)/\Phi'(k+\beta H)ds}{\int_0^{\beta H}s\Phi'(k+s)/\Phi'(k+\beta H)ds}.$$

By inequality (1.4)′, it follows that

$$\frac{\chi_k(H)}{\chi_k(\beta H)}-1\leqslant\frac{m^2\int_{\beta H}^{H}s\,ds}{\int_0^{\beta H}s\,ds}=\frac{m^2(1-\beta^2)}{\beta^2},$$

which is required.

The first inequality in (3.5) is easily obtained by means of (2.4) and (1.4)′. As for the second part of (3.5), since

$$\frac{\tilde{\chi}_k(H)}{\Phi'(k)}=\int_0^H\frac{\Phi'(k-s)}{\Phi'(k)}\,s\,ds\leqslant m\int_0^H\left(\frac{k}{k-s}\right)^{1-\frac{1}{m}}s\,ds,$$

and the integrand is monotonic with respect to k, we have

$$\frac{\tilde{\chi}_k(H)}{\Phi'(k)}\leqslant m\int_0^H\left(\frac{H}{H-s}\right)^{1-\frac{1}{m}}s\,ds\leqslant m^2H^2.$$

Now we pass to (3.6). Like the proof of (3.4), we observe

$$\frac{\tilde{\chi}_k(H)}{\tilde{\chi}_k(\beta H)}-1=\frac{\int_{\beta H}^H s\Phi'(k-s)/\Phi'(k-\beta H)\,ds}{\int_0^{\beta H}s\Phi'(k-s)/\Phi'(k-\beta H)\,ds}\leqslant\frac{m^2\int_{\beta H}^H(k-s)^{\frac{1}{m}-1}s\,ds}{\int_0^{\beta H}(k-s)^{\frac{1}{m}-1}s\,ds}.$$

If $H<k\leqslant 2H$, then

$$\frac{\tilde{\chi}_k(H)}{\tilde{\chi}_k(\beta H)}-1\leqslant\frac{m^2\int_{\beta H}^H(H-s)^{\frac{1}{m}-1}s\,ds}{\int_0^{\beta H}(2H)^{\frac{1}{m}-1}s\,ds}\leqslant 4m^3(1-\beta)^{\frac{1}{m}};$$

and if $k>2H$, then

$$\frac{\tilde{\chi}_k(H)}{\tilde{\chi}_k(\beta H)}-1\leqslant\frac{m^2\int_{\beta H}^H(k-H)^{\frac{1}{m}-1}s\,ds}{\int_0^{\beta H}k^{\frac{1}{m}-1}s\,ds}\leqslant\frac{2m^2(1-\beta^2)}{\beta^2}.$$

Since $k>H$, these inequalities imply (3.6). The proof is complete.

For any fixed $\rho\in(0,1]$, we shall consider the domain

$$Q_\rho=\{(x,t)\,|\,|x-x^0|<\rho,\ t^0-aB\rho^2<t<t^0\},\tag{3.7}$$

where $B=\Phi'(\rho^\varepsilon)$, a is a constant defined in Lemma 3.4 and ε is any constant in (0, 1]. Denote

$$\varkappa_n=\text{mes}\{K(1)\},\tag{3.8}$$

$$\mu=\max_{Q_\rho\cap\Omega}\{w(x,t)\},\ \tilde{\mu}=\min_{Q_\rho\cap\Omega}\{w(x,t)\},\ \omega=\mu-\tilde{\mu},\tag{3.9}$$

$$\bar{\sigma}_0=\sqrt[n]{\frac{1}{2}},\quad \bar{\rho}_3=\bar{\sigma}_0\rho,\quad \bar{\rho}_2=\frac{1+2\bar{\sigma}_0}{3}\rho,\quad \bar{\rho}_1=\frac{2+\bar{\sigma}_0}{3}\rho,\tag{3.10}$$

$$A=\frac{1}{m^2}\Phi'(\mu),\quad B_k=\frac{1}{m}\Phi'(k).\tag{3.11}$$

In this section, we shall suppose that

$$\mu\geqslant 2\rho^\varepsilon\tag{3.12}$$

and so by condition (1.4)'

$$A\leqslant B_k\leqslant B=\Phi'(\rho^\varepsilon)\quad\text{if}\quad \rho^\varepsilon\leqslant k\leqslant\mu.$$

In the following lemmas we shall omit to specify the dependence of constants on the parameters of $\mathscr{B}_2$ class.

Lemma 3.4. *There exist constants β, a, $b\in(0,1)$ such that*

(i) if

$$k\geqslant\max\left\{\frac{\mu}{2},\ \max_{Q_\rho\cap\Gamma}w(x,t)\right\},\ H=\mu-k>0,\tag{3.13}$$

$$\operatorname{mes} A_{k,\bar{\rho}_1}(t^0-aA\rho^2)\leqslant\frac{1}{2}\varkappa_n\bar{\rho}_1^n,$$

then for $t\in[t^0-aA\rho^2,\ t^0]$

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus A_{k+\beta H,\bar{\rho}_1}(t)]\geqslant b\varkappa_n\bar{\rho}^n; \tag{3.14}$$

(ii) *if*

$$k\leqslant\min_{Q_\rho\cap\Gamma}\{w(x,\ t)\},\qquad H=k-\tilde{\mu}\geqslant\rho^\varepsilon,$$

$$\operatorname{mes} B_{k,\bar{\rho}_1}(t^0-aB_k\rho^2)\leqslant\frac{1}{2}\varkappa_n\bar{\rho}_1^n, \tag{3.15}$$

then for $t\in[t^0-aB_k\rho^2,\ t^0]$

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus B_{k-\beta H,\bar{\rho}_1}(t)]\geqslant b\varkappa_n\bar{\rho}_1^n. \tag{3.16}$$

Proof For $0<\sigma<1$ let

$$\zeta(x;\ \rho,\ \rho-\sigma\rho)=\begin{cases}1 & |x-x^0|<\rho-\sigma\rho,\\ \dfrac{\rho-|x-x^0|}{\sigma\rho} & \rho-\sigma\rho\leqslant|x-x^0|\leqslant\rho,\\ 0 & |x-x^0|\geqslant\rho.\end{cases} \tag{3.17}$$

We prove the first part of the lemma. Integrating inequality (2.2) with $\zeta(x)=\zeta(x;\ \bar{\rho}_1,\ \bar{\rho}_1-\sigma\bar{\rho}_1)$ with respect to t from $t^0-aA\rho^2$ to t, we obtain

$$e^{-\gamma t}\int_{A_{k,\bar{\rho}_1}(t)}\zeta^2\chi_k(w-k)\,dx \leqslant e^{-\gamma(t^0-aA\rho^2)}\int_{A_{k,\bar{\rho}_1}(t^0-aA\rho^2)}\zeta^2\chi_k(w-k)\,dx+\frac{\gamma aA\rho^2H^2}{(\sigma\bar{\rho}_1)^2}\varkappa_n\bar{\rho}_1^n.$$

By the condition (3.13), it follows that for $t\in[t^0-aA\rho^2,\ t^0]$

$$\int_{A_{k,\rho}(t)}\zeta^2\chi_k(w-k)\,dx\leqslant e^{\gamma aA\rho^2}\chi_k(H)\cdot\frac{1}{2}\varkappa_n\bar{\rho}_1^n+\frac{2\gamma ae^{\gamma T}AH^2}{\sigma^2}\varkappa_n\bar{\rho}_1^n.$$

On the other hand

$$\int_{A_{k,\bar{\rho}_1}(t)}\zeta^2\chi_k(w-k)\,dx\geqslant\int_{A_{k+\beta H,\bar{\rho}_1-\sigma\bar{\rho}_1}(t)}\chi_k(w-k)\,dx\geqslant\chi_k(\beta H)\operatorname{mes} A_{k+\beta H,\bar{\rho}_1-\sigma\bar{\rho}_1}(t).$$

Hence, for $t\in[t^0-aA\rho^2,\ t^0]$,

$$\operatorname{mes} A_{k+\beta H,\bar{\rho}_1-\sigma\bar{\rho}_1}(t)\leqslant\frac{\chi_k(H)}{\chi_k(\beta H)}e^{a\gamma A\rho^2}\cdot\frac{1}{2}\varkappa_n\bar{\rho}_1^n+\frac{2a\gamma e^{\gamma T}}{\sigma^2}\cdot\frac{AH^2}{\chi_k(\beta H)}\varkappa_n\bar{\rho}_1^n.$$

By virtue of (3.3) and (3.4), it follows that

$$\operatorname{mes} A_{k+\beta H,\bar{\rho}_1-\sigma\bar{\rho}_1}\leqslant\left\{\frac{1}{2}\left(1+\frac{m^2(1-\beta^2)}{\beta^2}\right)e^{a\gamma A\rho^2}+\frac{4a\gamma e^{\gamma T}}{m\beta^2\sigma^2}\right]\varkappa_n\bar{\rho}_1^n.$$

We may select $\beta=\beta(m)\in(0,\ 1)$ such that

$$\frac{1}{2}\left(1+m^2(1-\beta^2)/\beta^2\right)\leqslant\frac{3}{4}.$$

Noting that $\mu\geqslant2\rho^\varepsilon$ and $A=\frac{1}{m^2}\Phi'(\mu)$, we may take a, b_1 and σ_0(depend only on n, m, γ, T) so small that

$$\frac{1}{2}(1+m^2(1-\beta^2)/\beta^2)e^{a\gamma A\rho^2}<(1-b_1)(1-\sigma_0)^n,$$

and then take a so small that

$$\frac{1}{2}(1+m^2(1-\beta^2)/\beta^2)e^{a\gamma A\rho^2}+\frac{4a\gamma e^{\gamma T}}{m\beta^2\sigma_0^2}\leqslant(1-b_1)(1-\sigma_0)^n.$$

Thus, for $t\in[t^0-aA\rho^2,\ t^0]$,

$$\operatorname{mes}A_{k+\beta H,\bar\rho_1-\sigma_0\bar\rho_1}(t)\leqslant(1-b_1)(1-\sigma_0)^n\varkappa_n\bar\rho_1^n$$

and

$$\begin{aligned}\operatorname{mes}(K(\bar\rho_1)-A_{k+\beta H,\bar\rho_1}(t))&\geqslant\operatorname{mes}(K(\bar\rho_1-\sigma_0\bar\rho_1)-A_{k+\beta H,\bar\rho_1-\sigma_0\bar\rho_1}(t))\\&\geqslant(1-\sigma_0)^n\varkappa_n\bar\rho_1^n-(1-b_1)(1-\sigma_0)^n\varkappa_n\bar\rho_1^n\\&=b_1(1-\sigma_0)^n\varkappa_n\bar\rho_1^n\end{aligned}$$

as claimed if we let $b=b_1(1-\sigma_0)^n$. The second part of the lemma can be proved in the same way.

Remark. From the proof we can find that the first part of the lemma still holds for any μ satisfying

$$\mu\geqslant\max_{\substack{x\in K(\rho)\\ t\in[t^0-\frac{a}{m^2}\Phi'(\mu)\rho^2,\,t^0]}}\{w(x,\ t)\},\quad \mu\geqslant2\rho^\varepsilon,\quad k\geqslant\frac{\mu}{2}$$

instead of μ defined by (3.9). The following lemmas about the properties of $A_{k,\rho}(t)$ will do the same.

Lemme 3.5. *Suppose that $Q_\rho\subset Q_T$, For any $\theta_1>0$ there exists $s=s(\theta_1)>0$ such that*

(i) *if*

$$k\geqslant\frac{\mu}{2},\quad H=\mu-k>0,\tag{3.18}$$

$$\operatorname{mes}A_{k,\bar\rho_1}(t^0-aA\rho^2)\leqslant\frac{1}{2}\varkappa_n\bar\rho_1^n,$$

then

$$\int_{t^0-aA\rho^2}^{t^0}\operatorname{mes}A_{\mu-\frac{H}{2^{s+1}},\bar\rho_1}(t)\,dt\leqslant\theta_1A\bar\rho_1^{n+2};\tag{3.19}$$

(ii) *if*

$$\max_{t\in[t^0-aB\rho^2,\,t^0-aA\rho^2]}\operatorname{mes}B_{\tilde\mu+\frac{\omega}{2},\bar\rho_1}(t)\leqslant\frac{1}{2}\varkappa_n\bar\rho_1^n,\tag{3.20}$$

then

$$\omega\leqslant2^{s+2}\rho^\varepsilon\tag{3.21}$$

or

$$\int_{t^0-aB_{k_{s+2}}\rho^2}^{t^0}B_{k_{s+1},\bar\rho_1}(t)\,dt\leqslant\theta_1B_{k_{s+2}}\bar\rho_1^{n+2},\tag{3.22}$$

where $k_s=\tilde\mu+\omega/2^s$, $B_{k_s}=\Phi'(k_s)/m$.

Proof We prove the first part of the lemma. By conditions (3.18) and Lemma 3.4, it follows that

$$\operatorname{mes}[K(\bar\rho_1)\backslash A_{k+\beta H,\bar\rho_1}(t)]\geqslant b\varkappa_n\bar\rho_1^n\quad\text{for}\quad t\in[t^0-aA\rho^2,\ t^0].\tag{3.23}$$

Taking r_0 such that

$$1-\beta\geqslant\frac{1}{2^{r_0}}$$

and denoting $k_l=\mu-\dfrac{H}{2^l}$, one obtains for $t\in[t^0-aA\rho^2,\ t^0]$

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus A_{k_l,\bar{\rho}_1}(t)]\geqslant b\varkappa_n\bar{\rho}_1^n \quad \text{if} \quad l\geqslant r_0. \tag{3.24}$$

Using Lemma 3.2, we have

$$(k_{l+1}-k_l)\operatorname{mes}^{1-\frac{1}{n}}A_{k_{l+1},\bar{\rho}_1}(t)\leqslant\frac{\beta\bar{\rho}_1^n}{\operatorname{mes}[K(\bar{\rho}_1)\setminus A_{k_l,\bar{\rho}_1}(t)]}\int_{D_l(t)}|\nabla w|\,dx,$$

where

$$D_l(t)=A_{k_l,\bar{\rho}_1}(t)\setminus A_{k_{l+1},\bar{\rho}_1}(t).$$

In virtue of (3.24), it follows that for $t\in[t^0-aA\rho^2, t^0]$,

$$\frac{H}{2^{l+1}}\operatorname{mes}A_{k_{l+1},\bar{\rho}_1}(t)\leqslant\frac{\beta}{b\varkappa_n^{1-\frac{1}{n}}}\bar{\rho}_1\Big(\int_{D_l(t)}|\nabla w|^2dx\Big)^{\frac{1}{2}}(\operatorname{mes}D_l(t))^{\frac{1}{2}}.$$

Integrating this inequality from $t^0-aA\rho^2$ to t^0 and applying Schwarz inequality, one obtains

$$\frac{H^2}{2^{2(l+1)}}\Big[\int_{t^0-aA\rho^2}^{t^0}\operatorname{mes}A_{k_{l+1},\bar{\rho}_1}(t)\,dt\Big]^2 \leqslant\Big(\frac{\beta}{b\varkappa_n^{1-\frac{1}{n}}}\Big)^2\bar{\rho}_1^2\int_{t^0-aA\rho^2}^{t^0}\int_{A_{k_l,\bar{\rho}_1}(t)}|\nabla w|^2dx\,dt\cdot\int_{t^0-aA\rho^2}^{t^0}\operatorname{mes}D_l(t)\,dt. \tag{3.25}$$

Integrating inequality (2.2) with $\zeta(x)=\zeta(x;\rho,\bar{\rho}_1)$, we find

$$\begin{aligned}\frac{1}{2}e^{-\gamma T}&\int_{t^0-aA\rho^2}^{t^0}\int_{A_{k_l},\bar{\rho}_1(t)}|\nabla w|^2dxdt\\ &\leqslant\int_{A_{k_l},\rho(t^0-aA\rho^2)}\zeta^2\chi_{k_l}(w-k_l)\,dx+\frac{9a\gamma A}{(1-\bar{\sigma}_0)^2}\frac{H^2}{2^{2l}}\varkappa_n\rho^n\\ &\leqslant\Big[\chi_{k_l}\Big(\frac{H}{2^l}\Big)+\frac{9a\gamma}{(1-\bar{\sigma}_0)^2}\cdot\frac{AH^2}{2^{2l}}\Big]\varkappa_n\rho^n.\end{aligned}$$

By estimate (3.3), it follows that

$$\int_{t^0-aA\rho^2}^{t^0}\int_{A_{k_l},\bar{\rho}_1(t)}|\nabla w|^2dxdt\leqslant 2e^{\gamma T}\Big[2m^3+\frac{9a\gamma}{(1-\bar{\sigma}_0)^2}\Big]\frac{AH^2}{2^{2l}}\varkappa_n\rho^n.$$

Substituting it into (3.25), we have

$$\Big(\int_{t^0-aA\rho^2}^{t^0}\operatorname{mes}A_{k_{l+1},\bar{\rho}_1}(t)\,dt\Big)^2\leqslant C_1A\rho^{n+2}\int_{t^0-aA\rho^2}^{t^0}\operatorname{mes}D_l(t)\,dt,$$

where $C_1=C_1(n, m, \gamma, T)$. Summing it from r_0 to s with respect to l, we find

$$(s-r_0+1)\Big[\int_{t^0-aA\rho^2}^{t^0}\operatorname{mes}A_{k_{s+1},\bar{\rho}_1}(t)\,dt\Big]^2\leqslant C_1\varkappa_n aA^2\rho^{2n+4}.$$

Taking s such that

$$\sqrt{\frac{C_1\varkappa_n a}{s-r_0+1}}\leqslant\theta_1\bar{\sigma}_0^{n+2},$$

we can obtain (3.19).

The proof of the second part of the lemma is similar. If (3.21) fails, then $\omega\geqslant 2^{s+2}\rho^\varepsilon$. We shall show that (3.23) holds at this time. By hypothesis (3.20) and the second part of Lemma 3.4, it follows that for $t\in[t^0-aB\rho^2, t^0]$

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus B_{\tilde{\mu}+\frac{\omega}{2}(1-\beta),\bar{\rho}_1}(t)]\geqslant b\varkappa_n\bar{\rho}_1^n.$$

Select r_0 such that for $t\in[t^0-aB\rho^2, t^0]$,

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus B_{k_l,\bar{\rho}_1}(t)]\geqslant b\varkappa_n\bar{\rho}_1^n \quad \text{if} \quad l\geqslant r_0,$$

where $k_l=\tilde{\mu}+\frac{\omega}{2^l}$. It is similar to (3.25) that for $s\geqslant l\geqslant r_0$,

$$\frac{\omega^2}{2^{2l+2}}\int_{t^0-aB_{k_{s+2}}\rho^2}^{t^0}\operatorname{mes}B_{k_l,\bar{\rho}_1}(t)\,dt$$

$$\leqslant\left(\frac{\beta}{b\varkappa_n^{1-\frac{1}{n}}}\right)^2\bar{\rho}_1^2\int_{t^0-aB_{k_{s+2}}\rho^2}^{t^0}\operatorname{mes}\tilde{D}_l(t)\,dt\cdot\int_{t^0-aB_{k_{s+2}}\rho^2}^{t^0}\int_{B_{k_l},\bar{\rho}_1(t)}|\nabla w|^2dx\,dt, \tag{3.26}$$

where $\tilde{D}_l(t)=B_{k_l,\bar{\rho}_1}(t)\setminus B_{k_{l+1},\bar{\rho}_1}(t)$. The rest of the proof will be analogous to the previous proof for (3.19).

Lemma 3.5'. *Suppose that*

$$\omega_1=\operatorname{osc}\{w;\ \Gamma_\rho\}\leqslant L\rho^{\varepsilon_1}, \tag{3.27}$$

where $\varepsilon_1>0$ and $\Gamma_\rho=\Gamma\cap Q_\rho$.

(A) *If $K(\bar{\rho}_1)$ satisfies*

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus(K(\bar{\rho}_1)\cap\Omega)]\geqslant b_1\bar{\rho}_1^n, \tag{3.28}$$

where b_1 is a positive constant, then for any $\theta_1>0$ there exists $s=s(\theta_1)>0$ such that we have one of the following:

$$\text{(i)}\quad \omega=\operatorname{osc}\{w,\ Q_\rho\cap Q_T\}\leqslant 2^{s+2}\rho^\varepsilon \tag{3.29}$$

or

$$\text{(ii)}\quad \int_{t^0-aA\rho^2}^{t^0}\operatorname{mes}A_{\mu-\frac{\omega}{2^{s+1}},\bar{\rho}_1}(t)\,dt\leqslant\theta_1A\bar{\rho}_1^{n+2} \tag{3.30}$$

or

$$\text{(iii)}\quad \int_{t^0-aB_{k_{s+2}}\rho^2}^{t^0}\operatorname{mes}B_{\mu+\frac{\omega}{2^{s+1}},\bar{\rho}_1}(t)\,dt\leqslant\theta_1B_{k_{s+2}}\bar{\rho}_1^{n+2} \tag{3.31}$$

where $k_s=\mu+\omega/2^s$, s is any number in $(0,\ \varepsilon_1]$.

(B) *If*

$$t^0-aA\rho^2\leqslant 0,$$

then instead of (3.30) and (3.31), we set respectively

(ii)′ $\operatorname{mes}A_{\mu-\frac{\omega}{4},\bar{\rho}_1}(0)=0$,

$$\int_0^{t^0}\operatorname{mes}A_{\mu-\frac{\omega}{2^{s+1}},\bar{\rho}_1}(t)\,dt\leqslant\theta_1t^0\bar{\rho}_1^n, \tag{3.30$'$}$$

(iii)′ $\operatorname{mes}B_{\tilde{\mu}+\frac{\omega}{4},\bar{\rho}_1}(0)=0$,

$$\int_0^{t^0}\operatorname{mes}B_{\tilde{\mu}+\frac{\omega}{2^{s+1}},\bar{\rho}_1}(t)\,dt\leqslant\theta_1t^0\bar{\rho}_1^n, \tag{3.31$'$}$$

and we have the same conclusion.

Proof Take $r_0\geqslant 2$ such that $2^{r_0}\geqslant 4L$. If $\omega\geqslant 2^{s+2}\rho^\varepsilon$, then

$$\omega\geqslant 2^{r_0}\rho^\varepsilon\geqslant 4L\rho^\varepsilon\geqslant 4\omega_1\quad\text{if}\quad s\geqslant r_0.$$

Hence, the range of $w(x,t)$ on Γ_ρ superimposes at most on one of the intervals $\left[\tilde{\mu},\ \tilde{\mu}+\frac{\omega}{4}\right]$ and $\left[\mu-\frac{\omega}{4},\ \mu\right]$. We shall show that (3.30) is true if it superimposes on $\left[\tilde{\mu},\ \tilde{\mu}+\frac{\omega}{4}\right]$ and that (3.31) is true in another case.

Now let the first case come up. At this time take $k=\mu-\frac{\omega}{2^{r_0}}$. It is clear that

$$k\geqslant\max\left\{\frac{\mu}{2},\ \max_{\Gamma_\rho} w(x,\ t)\right\},$$

$$\operatorname{mes}(K(\bar\rho_1)\backslash A_{k,\bar\rho_1}(t))\geqslant b_1\bar\rho_1^n \quad\text{for}\quad t\in[t^0-aA\rho^2,\ t^0].$$

The proof of (3.30) will be similar to that of (3.19).

If $t^0-aA\rho^2<0$, we have

$$\operatorname{mes} A_{\mu-\frac{\omega}{4},\bar\rho_1}(0)=0 \tag{3.32}$$

because the range of $w(x,\ t)$ on Γ_ρ does not superimpose on $\left[\mu-\frac{\omega}{4},\ \mu\right]$ at this time. By means of the method used in Lemma 3.4 and Lemma 3.5, we can obtain (3.30)′.

The rest of the lemma will be similar to Lemma 3.5.

Lemma 3.6. *For any $\theta_2>0$ there exists $\theta_1>0$ such that*

(i) *If*

$$k\geqslant\max\left\{\frac{\mu}{2},\ \max_{Q_\rho\cap\Gamma} w(x,\ t)\right\},\quad H=\mu-k>0,$$
$$\int_{t^0-aA\rho^2}^{t^0}\operatorname{mes} A_{k,\bar\rho_1}(t)\,dt\leqslant\theta_1 A\bar\rho_1^{n+2}, \tag{3.33}$$

then for $t\in\left[t^0-\frac{1}{4}aA\rho^2,\ t^0\right]$,

$$\operatorname{mes} A_{k+\frac{H}{2},\bar\rho_2}(t)\leqslant\theta_2\bar\rho_2^n. \tag{3.34}$$

Moreover, if $\operatorname{mes} A_{k,\bar\rho_1}(t^0-aA\rho^2)=0$, *then* (3.34) *holds in* $[t^0-aA\rho^2,\ t^0]$.

(ii) *If*

$$k\leqslant\min_{Q_\rho\cap\Gamma} w(x,\ t),\quad H=k-\tilde\mu\geqslant\rho^\varepsilon,$$
$$\int_{t^0-aB_{k-H^2}\rho^2}^{t^0}\operatorname{mes} B_{k,\bar\rho_1}(t)\,dt\leqslant\theta_1 B_{k-\frac{H}{2}}\bar\rho_1^{n+2}, \tag{3.35}$$

then for $t\in\left[t^0-\frac{1}{4}aB_{k-\frac{H}{2}}\rho^2,\ t^0\right]$,

$$\operatorname{mes} B_{k-\frac{H}{2},\bar\rho_2}(t)\leqslant\theta_1\bar\rho_2^n. \tag{3.36}$$

Moreover, if $\operatorname{mes} B_{k,\bar\rho_1}(t^0-aB_{k-\frac{H}{2}}\rho^2)=0$, *then* (3.36) *holds in* $[t^0-aB_{k-\frac{H}{2}}\rho^2,\ t^0]$.

Proof We prove the second part of the lemma as an example. Integrating (2.3) with $\zeta(x)=\zeta(x;\ \bar\rho_1,\ \bar\rho_2)$ with respect to t from τ to t for $t^0\geqslant t>\tau\geqslant t^0-a\rho^2B_{k-\frac{H}{2}}$, we find

$$e^{-\gamma t}\tilde\chi_k\left(\frac{H}{2}\right)\operatorname{mes} B_{k-\frac{H}{2},\bar\rho_2}(t)$$
$$\leqslant e^{-\gamma t}\tilde\chi_k(H)\operatorname{mes} B_{k,\bar\rho_1}(\tau)+\gamma\left[\frac{H}{(\bar\rho_1-\bar\rho_2)^2}+1\right]\int_\tau^t\operatorname{mes}B_{k,\bar\rho_1}(t)\,dt. \tag{3.37}$$

Since

$$\int_{t^0-aB_{k-\frac{H}{2}}\rho^2}^{t^0-\frac{1}{4}aB_{k-\frac{H}{2}}\rho^2}\operatorname{mes} B_{k,\bar\rho_1}(t)\,dt\leqslant\theta_1 B_{k-\frac{H}{2}}\bar\rho_1^{n+2},$$

there exists $\tau\in\left[t^0-aB_{k-\frac{H}{2}}\rho^2,\ t^0-\frac{1}{4}aB_{k-\frac{H}{2}}\rho^2\right]$ such that

$$\operatorname{mes}B_{k,\bar\rho_1}(\tau)\leqslant\frac{4}{3a}\theta_1\bar\rho_1^n.$$

Substituting it into (3.37) we obtain for $t\in\left[t^0-\frac{a}{4}B_{k-\frac{H}{2}}\rho^2,t^0\right]$,

$$\operatorname{mes}B_{k-\frac{H}{2},\bar\rho_2}(t)\leqslant e^{\gamma T}\left\{\frac{\tilde\chi_k(H)}{\tilde\chi_k\left(\frac{H}{2}\right)}\frac{4}{3a}+\frac{\gamma B_{k-\frac{H}{2}}}{\tilde\chi_k\left(\frac{H}{2}\right)}\left(\frac{9H^2}{(1-\bar\sigma_0)^2}+\rho^2\right)\right\}\theta_1\bar\rho_1^n.$$

By Lemma 3.3, one has

$$\tilde\chi_k(H)\Big/\tilde\chi_k\left(\frac{H}{2}\right)\leqslant1+4m^3,\quad\tilde\chi_k\left(\frac{H}{2}\right)\Big/B_{k-\frac{H}{2}}\geqslant H^2/8m.$$

Therefore, there exists θ_1 so small that (3.36) holds.

If mes $B_{k,\bar\rho_1}(t^0-aB_{k-\frac{H}{2}}\rho^2)=0$, it suffices to take $\tau=t^0-aB_{k-\frac{H}{2}}\rho^2$ in the above-mentioned argument.

Lemma 3.7. *There exists $\theta_2>0$ such that*

(i) *If*

$$k\geqslant\max\left\{\frac{\mu}{2},\ \max_{\Gamma_\rho}w(x,t)\right\},\quad H=\mu-k>0,\qquad(3.38)$$

$$\max_{t\in[t^0-aA\rho^2,t^0]}\operatorname{mes}A_{k,\bar\rho_2}(t)\leqslant\theta_2\bar\rho_2^n,$$

then for $t\in\left[t^0-\frac{1}{4}aA\rho^2,\ t^0\right]$,

$$\operatorname{mes}A_{k+\frac{H}{2},\bar\rho_3}(t)=0.\qquad(3.39)$$

Moreover, if mes $A_{k,\bar\rho_2}(t^0-aA\rho^2)=0$, *then* (3.39) *holds for* $t\in[t^0-aA\rho^2,\ t^0]$.

(ii) *If*

$$k\leqslant\min_{\Gamma_\rho}w(x,t),\quad H=k-\tilde\mu\geqslant\rho^s,\qquad(3.40)$$

$$\max_{t\in[t^0-aB_k\rho^2,t^0]}\operatorname{mes}B_{k,\bar\rho_2}(t)\leqslant\theta_2\bar\rho_2^n,$$

then for $t\in\left[t^0-\frac{1}{4}aB_k\rho^2,\ t^0\right]$,

$$\operatorname{mes}B_{k-\frac{H}{2},\bar\rho_3}(t)=0.\qquad(3.41)$$

Moreover, if mes $B_{k,\bar\rho_2}(t^0-aB_k\rho^2)=0$ *then* (3.41) *holds for* $t\in[t^0-aB_k\rho^2,\ t^0]$.

Proof We still prove only the second part of the lemma. Let

$$k_h=k-\frac{H}{2}+\frac{H}{2^{h+1}},\quad t_h=t^0-\frac{1}{4}aB_k\rho^2-\frac{3}{2^{h+2}}aB_k\rho^2,$$

$$\rho_h=\bar\rho_3+\frac{(\bar\rho_2-\bar\rho_3)}{2^h},\quad\mu_h=\max_{t\in[t_h,t^0]}\operatorname{mes}B_{k_h,\rho_h}(t),\qquad(3.42)$$

$$\zeta_h(x)=\zeta(x;\ \rho_h,\ \rho_{h+1}),\quad I_h(t)=e^{-\gamma t}\int_{B_{k_h,\rho_h}(t)}\tilde\chi_{k_h}(k_h-w)\zeta_h^2\,dx.$$

Since $k_h\geqslant\frac{k}{2}$, by Lemma 3.3 it follows that

$$\tilde{\chi}_{k_h}(k_h-w)\leqslant m^2\Phi'(k_h)(k_h-w)^2\leqslant 2m^4B_k(k_h-w)^2.$$

Hence

$$I_h(t)\leqslant 2m^4 B_k\int_{B_{k_h,\rho_h(t)}}(k_h-w)^2\zeta_h^2\,dx. \tag{3.43}$$

In virtue of Lemma 3.1, it follows that

$$I_h(t)\leqslant CB_k\mu_h^{\frac{2}{n}}\Big[\int_{B_{k_h,\rho_h(t)}}\zeta_h^2|\nabla w|^2dx+\frac{H^2}{(\rho_h-\rho_{h+1})^2}\mu_h\Big], \tag{3.44}$$

where $C=C(n, m)$. Inequality (2.3) implies that

$$I_h'(t)+\frac{1}{2}e^{-\gamma t}\int_{B_{k_h,\rho_h(t)}}|\nabla w|^2\zeta_h^2\,dx\leqslant\gamma\Big[\frac{H^2}{(\rho_h-\rho_{h+1})^2}+1\Big]\mu_h. \tag{3.45}$$

For any fixed $t\in[t_{h+1}, t^0]$ there are three cases:

(a) If $I_h'(t)\geqslant 0$, we obtain from (3.45)

$$\int_{B_{k_h,\rho_h(t)}}|\nabla w|^2\zeta_h^2\,dx\leqslant 2\gamma e^{\gamma T}\Big[\frac{H^2}{(\rho_h-\rho_{h+1})^2}+1\Big]\mu_h.$$

Substituting it into (3.44) we find

$$I_h(t)\leqslant CB_k\mu_h^{\frac{2}{n}+1}\Big[(2\gamma e^{\gamma T}+1)\frac{H^2}{(\rho_h-\rho_{h+1})^2}+2\gamma e^{\gamma T}\Big]. \tag{3.46}$$

(b) If $I_h'(t)<0$ and there exists $\tau\in[t_h, t]$ such that $I_h'(\tau)=0$, then we may select τ such that $I_h'(s)<0$ for $s\in(\tau, t]$ and so $I_h(t)\leqslant I_h(\tau)$. $I_h(\tau)$ has the estimate (3.46) and so does $I_h(t)$.

(c) If $I_h'(\tau)<0$ for any $\tau\in[t_h, t]$, it follows from (3.45) that

$$\frac{1}{2}e^{-\gamma t}\int_{t_h}^{t}\int_{B_{k_h,\rho_h(t)}}|\nabla w|^2\zeta_h^2dx\,dt\leqslant I_h(t_h)+\gamma(t-t_h)\Big[\frac{H^2}{(\rho_h-\rho_{h+1})^2}+1\Big]\mu_h.$$

Integrating (3.44) from t_h to t and using the previous inequality, one finds

$$\int_{t_h}^{t}I_h(\tau)d\tau\leqslant CB_k\mu_h^{\frac{2}{n}+1}\Big[2e^{\gamma T}I_h(t_h)+(2\gamma e^{\gamma T}+1)\frac{H^2(t-t_h)\mu_h}{(\rho_h-\rho_{h+1})^2}+2\gamma e^{\gamma T}(t-t_h)\mu_h\Big],$$

In virtue of the decrease of $I_h(\tau)$ in $[t_h, t]$ and inequality (3.43), it follows that for $t\in[t_{h+1}, t^0]$,

$$I_h(t)\leqslant CB_k\mu_h^{\frac{2}{n}+1}\Big[\frac{4m^3e^{\gamma T}B_kH^2}{t_{h+1}-t_h}+\frac{(2\gamma e^{\gamma T}+1)H^2}{(\rho_h-\rho_{h+1})^2}+2\gamma e^{\gamma T}\Big]. \tag{3.47}$$

Thus, no matter which case it is, one has (3.47) for $t\in[t_{h+1}, t^0]$.

On the other hand, applying Lemma 3.3 and inequality (1.4)′ we have

$$\begin{aligned}I_h(t)&\geqslant\tilde{\chi}_{k_h}(k_h-k_{h+1})\,\mathrm{mes}\,B_{k_{h+1},\rho_{h+1}}(t)\\&\geqslant\frac{1}{4m}B_k(k_h-k_{h+1})^2\,\mathrm{mes}\,B_{k_{h+1},\rho_{h+1}}(t).\end{aligned}$$

Combining this with (3.47) we find

$$\mu_{h+1}\leqslant 4mC\mu_h^{\frac{2}{n}+1}2^{2(h+1)}\Big[\frac{4m^4e^{\gamma T}B_k}{(t_{h+1}-t_h)}+\frac{2\gamma e^{\gamma T}+1}{(\rho_h-\rho_{h+1})^2}+\frac{2\gamma e^{\gamma T}}{H^2}\Big].$$

By definition (3.42) of t_h and ρ_h it follows that

$$\mu_{h+1}\leqslant C_12^{4h}\mu_h^{\frac{2}{n}+1}/\rho^2,$$

where $C_1=C_1(n, m, \gamma, T)$. Setting $y_h=\mu_h/\rho^n$, we obtain

$$y_{h+1}\leqslant C_1 2^{4h} y_h^{\frac{2}{n}+1}. \tag{3.48}$$

Hypothesis (3.40) implies that

$$y_0\leqslant\theta_2.$$

We shall prove by induction that

$$y_h\leqslant\theta_2 2^{-2nh} \quad (h=0, 1, 2, \cdots). \tag{3.49}$$

In fact

$$y_{h+1}\leqslant C_1 2^{4h} y_h^{\frac{2}{n}+1}\leqslant C_1 2^{4h}(\theta_2 2^{-2nh})^{\frac{2}{n}+1}\leqslant C_1\theta_2^{\frac{2}{n}+1}2^{-2nh}.$$

If we take θ_2 satisfying

$$\theta_2^{\frac{2}{n}}\leqslant 2^{-2n}/C_1,$$

the induction argument will be valid. Thus (3.49) holds for any natural number h. Setting $h\to\infty$ in (3.49) we obtain (3.41).

If $\operatorname{mes} B_{k,\rho}(t^0-aB_k\rho^2)=0$, it suffices to take $t_h=t^0-aB_k\rho^2$ $(h=0, 1, 2, \cdots)$ in the previous argument.

Lemma 3.8. *Suppose that $Q_\rho\subset Q_T$. Then there exists $s>0$ such that*

(i) *if* $k\geqslant\frac{\mu}{2}$, $H=\mu-k>0$,

$$\operatorname{mes} A_{k,\bar\rho_1}(t^0-aA\rho^2)\leqslant\frac{1}{2}\varkappa_n\rho^n, \tag{3.50}$$

then for $t\in\left[t^0-\frac{1}{16}aA\rho^2, t^0\right]$,

$$\operatorname{mes} A_{\mu-\frac{H}{2^{s+3}},\bar\rho_s}(t)=0; \tag{3.51}$$

(ii) *if*

$$\max_{t\in[t^0-aB\rho^2, t^0-aA\rho^2]}\operatorname{mes} B_{\tilde\mu+\frac{\omega}{2},\bar\rho_1}(t)\leqslant\frac{1}{2}\varkappa_n\bar\rho_1^n, \tag{3.52}$$

then

$$\omega\leqslant 2^{s+2}\rho^s \tag{3.53}$$

or

$$\operatorname{osc}\{w; \tilde Q_{\frac{\rho}{4}}\}\leqslant\left(1-\frac{1}{2^{s+3}}\right)\operatorname{osc}\{w; Q_\rho\}, \tag{3.54}$$

where

$$\tilde Q_{\frac{\rho}{4}}=\left\{(x, t)\,|\,x\in K\left(\frac{\rho}{4}\right),\ t^0-aA\left(\frac{\rho}{4}\right)^2<t<t^0\right\}. \tag{3.55}$$

Proof We first determine the constant θ_2 by Lemma 3.7 and then θ_1 by Lemma 3.6 and finally s by Lemma 3.5. One can derive (3.51) from these lemmas without difficulty. As for the second part of the lemma, in the same way one can obtain

$$\omega\leqslant 2^{s+2}\rho^s$$

or

$$\operatorname{mes} B_{\tilde\mu+\frac{\omega}{2^{s+3}},\bar\rho_s}(t)=0 \quad \text{for} \quad t\in\left[t^0-aB_{k_{s+2}}\left(\frac{\rho}{4}\right)^2, t^0\right],$$

where $B_{k_s}=\frac{1}{m}\Phi'(k_s)$, $k_s=\tilde\mu+\frac{\omega}{2^s}$. For the second possibility it is clear that

$$\operatorname{osc}\{w;\ \tilde{Q}_{\frac{\rho}{4}}\} \leqslant \mu - \left(\tilde{\mu} + \frac{\omega}{2^{s+3}}\right) \leqslant \left(1 - \frac{1}{2^{s+3}}\right)\omega,$$

which is required.

Lemma 3.8'. *Suppose that $\partial\Omega$ satisfies the condition (1.6) and $w(x, t)$ belongs to $C^{\varepsilon_1, \varepsilon_1/2}(\Gamma)$. If $\tilde{Q}_{\frac{\rho}{4}} \cap \Gamma \neq \emptyset$, then for any $0 < \varepsilon \leqslant \varepsilon_1$ there exists a constant s such that*

$$\operatorname{osc}\{w;\ \tilde{Q}_{\frac{\rho}{4}} \cap Q_T\} \leqslant 2^{s+2}\rho^{\varepsilon}. \tag{3.56}$$

or

$$\operatorname{osc}\{w;\ \tilde{Q}_{\frac{\rho}{4}} \cap Q_T\} \leqslant \left(1 - \frac{1}{2^{s+3}}\right)\operatorname{osc}\{w;\ Q_\rho \cap Q_T\}. \tag{3.57}$$

Proof $\tilde{Q}_{\frac{\rho}{4}} \cap \Gamma \neq \emptyset$ and condition (1.7) imply that

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus(K(\bar{\rho}_1)\cap\Omega)] \geqslant b_1\bar{\rho}_1^n \quad \text{or} \quad t^0 - aA\rho^2 \leqslant 0.$$

Applying Lemmas 3.5', 3.6 and 3.7, we can obtain what we want.

§ 4

In order to find Hölder estimate for $w(x, t)$, we still need the following lemma.

Lemma 4.1. *For $\rho_0 \leqslant 1$ suppose that $Q_{\rho_0} \subset Q_T$. Let $u(x, t)$ be a classical solution of equation (1.1)' in Q_{ρ_0} satisfying $0 < u(x, t) \leqslant M$. If*

$$\omega_0 \leqslant \hat{C}\rho_0^{\varepsilon}, \quad \mu_0 \geqslant 2\hat{C}\rho_0^{\varepsilon}, \tag{4.1}$$

where $\mu_0 = \max\limits_{Q_{\rho_0}}\{w(x, t)\}$, $\tilde{\mu}_0 = \min\limits_{Q_{\rho_0}}\{w(x, t)\}$, $\omega_0 = \mu_0 - \tilde{\mu}_0$, $w(x, t) = \varphi(u(x, t))$ and ε, $\hat{C}$ is some constants satisfying $0 < \varepsilon \leqslant 1$, $\hat{C} \geqslant 1$, then for any $(x, t) \in Q_{\rho_0}$ we have

$$|w(x, t) - w(x^0, t^0)| \leqslant c_1\hat{C}(|x - x^0| + |t - t^0|)^{\varepsilon_0/2}, \tag{4.2}$$

where c_1, ε_0 depend only on n, m, Λ, T, M and $\varepsilon_0 \leqslant \varepsilon/4$.

Proof Consider the transformation

$$x' = \frac{x' - x_0}{\rho_0}, \quad t' = \frac{t - t^0}{a\rho_0^2\Phi'(\mu_0)}. \tag{4.3}$$

Then the domain Q_{ρ_0} is mapped into

$$Q_1' = \left\{(x', t')\,\middle|\, |x'| \leqslant 1,\ -\frac{\Phi'(\rho_0^{\varepsilon})}{\Phi'(\mu_0)} < t' < 0\right\}. \tag{4.4}$$

Set $v(x', t') = u(x, t)$. Then $v(x', t')$ satisfies the equation

$$\frac{\partial v}{\partial t'} = \frac{\partial}{\partial x_i'}\left(A_{ij}(x', t')\frac{\partial v}{\partial x_j'}\right) + B_i(x', t')\frac{\partial v}{\partial x_i'} + C(x', t') \text{ in } Q_1', \tag{4.5}$$

where

$$A_{ij}(x', t') = a\Phi'(\mu_0)a_{ij}(x, t, v), \quad B_i(x', t') = a\Phi'(\mu_0)\rho_0 b_i(x, t, v),$$
$$C(x', t') = a\Phi'(\mu_0)\rho_0^2\tilde{c}(x, t, v). \tag{4.6}$$

By (4.1), $\tilde{\mu}_0 \geqslant \frac{1}{2}\mu_0$ and so

$$A_{ij}\xi_i\xi_j \geqslant a\Phi'(\mu_0)\nu(v)|\xi|^2 = \frac{a|\xi|^2\Phi'(\mu_0)}{\Phi'(w)} \geqslant \frac{a}{m}\left(\frac{1}{2}\right)^{1-\frac{1}{m}}|\xi|^2,$$

$$A_{ij}\xi_i\xi_j \leqslant \Lambda a\Phi'(\mu_0)\nu(v)|\xi|^2 \leqslant am\Lambda|\xi|^2.$$

$B_i(x', t')$ and $C(x', t')$ are bounded by constants depending on n, m, Λ, M.

In virtue of Nash interior estimates for nondegenerate parabolic equations, there exist C_1 and $\alpha \in (0, 1)$ such that

$$|v(x', t') - v(0, 0)| \leqslant C_1(|x'|^\alpha + |t'|^{\alpha/2})$$

for $(x', t') \in Q'_{\frac{1}{2}} = \left\{(x', t') \mid |x'| \leqslant \frac{1}{2}, -\frac{1}{2m} < t' < 0\right\}$, where α, C_1 depend only on n, m, Λ, M, T. Thus, we have

$$|u(x, t) - u(x^0, t^0)| \leqslant C_1\left(\frac{|x-x^0|^\alpha}{\rho_0^\alpha} + \frac{|t-t^0|^{\alpha/2}}{[a\rho_0^2\Phi'(\mu_0)]^{\alpha/2}}\right). \tag{4.7}$$

in

$$\tilde{Q}_{\frac{\rho_0}{2}} = \left\{(x, t) \mid |x-x^0| \leqslant \frac{\rho_0}{2}, t^0 - \left[\frac{a}{2m}\Phi'(\mu_0)\rho_0^2\right] < t \leqslant t^0\right\},$$

Consequently, if

$$(x, t) \in \Sigma = \left\{(x, t) \mid |x-x^0| \leqslant \frac{\rho_0^2}{4}, t^0 - \left[\frac{a}{2m}\Phi'(\mu_0)\rho_0^2\right]^2 < t \leqslant t^0\right\},$$

one obtains

$$|u(x, t) - u(x^0, t^0)| \leqslant C_1(|x-x^0|^{\frac{\alpha}{2}} + |t-t^0|^{\frac{\alpha}{4}}). \tag{4.8}$$

If $(x, t) \in Q_{\rho_0} \setminus \Sigma$, then by condition (4.1) we have

$$(|x-x^0|^2 + |t-t^0|)^{\frac{1}{4}} \geqslant \min\{\rho_0/2, [a\Phi'(\mu_0)\rho_0^2/2m]^{\frac{1}{2}}\}$$

$$\geqslant \left(\frac{a}{2m^2}\Phi'(M)\right)^{\frac{1}{2}}\rho_0 \geqslant \left(\frac{a}{2m^2}\Phi'(M)\right)^{\frac{1}{2}}[\hat{C}^{-1}\omega]^{\frac{1}{s}}.$$

Hence, for $(x, t) \in Q_{\rho_0} \setminus \Sigma$

$$\omega \leqslant \left(\frac{2m^2}{a\Phi'(M)}\right)^{s/2}\hat{C}[|x-x^0| + |t-t^0|]^{s/4}. \tag{4.9}$$

Equalities (4.8) and (4.9) imply (4.2).

We can now obtain Hölder interior estimate for $w(x, t)$.

Theorem 4.1. *Suppose that the cofficients of equation (1.1)′ satisfy conditions (1.2), (1.3), (1.4), (1.5) and (1.7). Let $u(x, t)$ be a classical solution of equation (1.1)′ satisfying $0 < u(x, t) \leqslant M$. For any $(x^0, t^0) \in Q_T$ denote*

$$\rho_0 = \min\{1, d(x^0), [t^0/am\Phi'(1)]^{\frac{m}{m+1}}\},$$

where $d(x^0) = \operatorname{dist}\{x_0, \partial\Omega\}$. Then, for any $(x, t) \in Q^*_{\rho_0}$ we have

$$|w(x, t) - w(x^0, t^0)| \leqslant C\rho_0^{-1}[|x-x^0|^\alpha + |t-t^0|^{\alpha/2}] \ (\alpha > 0), \tag{4.10}$$

where C and α depend only on n, m, δ, Λ, M and T, and

$$Q^*_{\rho_0} = \left\{(x, t) \mid |x-x^0| < \rho_0, t^0 - \frac{a}{m}\Phi'(1)\rho_0^2 < t \leqslant t^0\right\},$$

Proof Let

$$\hat{C} = \max\left\{(2^{s+8}m^3)^m, \frac{4\varphi(M)}{\rho_0}\right\}, \tag{4.11}$$

$$\eta = 64m^3\hat{C}^{1-\frac{1}{m}}, \tag{4.12}$$

where the constant s is defined in Lemma 3.8, and take ε so small that

$$0<\varepsilon<1, \left(\frac{1}{\eta}\right)^{\varepsilon}\geqslant 1-2^{-[1+(s+3)N_0]}, \tag{4.13}$$

where $N_0=4m^3\hat{C}^{1-\frac{1}{m}}$.

Denote

$$\begin{aligned}\rho_l&=\rho_0/\eta^l \quad (l=0,\ 1,\ 2,\ \cdots),\\ Q_l&=Q_{\rho_l}=\{(x,\ t)\,|\,|x-x^0|<\rho_l,\quad t^0-a\Phi'(\rho_l^\varepsilon)\rho_l^2<t<t^0\},\\ \mu_l&=\max_{Q_l}\{w(x,\ t)\},\quad \tilde{\mu}_l=\min_{Q_l}\{w(x,\ t)\},\\ \omega_l&=\mu_l-\tilde{\mu}_l,\quad l^*=\min\{l\,|\,\mu_l\geqslant 2\hat{C}\rho_l^\varepsilon\}.\end{aligned} \tag{4.14}$$

By (4.11), we have

$$\omega_0\leqslant\hat{C}\rho_0^\varepsilon. \tag{4.15}$$

If $l^*=0$, one immediately obtains (4.10) by Lemma 4.1. Now suppose that $l^*>0$. We shall argue by induction

$$\omega_l\leqslant\hat{C}\rho_l^\varepsilon \quad \text{if} \quad l\leqslant l^*. \tag{4.16}$$

Assume that (4.16) is true for $l<l^*$. If $\mu_l\leqslant 2\rho_l^\varepsilon$, it is easy to show $\omega_{l+1}\leqslant\mu_l\leqslant\hat{C}\rho_{l+1}^\varepsilon$. So letting $\mu_l\geqslant 2\rho_l^\varepsilon$, we can apply the results in § 3. Consider the following two cases:

(i) If

$$\max_{t\in[t^0-aB\rho_l^2,\ t^0-aA\rho_l^2]}\operatorname{mes}B_{\mu_l+\frac{\omega_l}{2},\,\bar{\rho}_{1l}}(t)\leqslant\frac{1}{2}\varkappa_n\bar{\rho}_{1l}^n, \tag{4.17}$$

where $B=\Phi'(\rho_l^\varepsilon)$, $A=\frac{1}{m^2}\Phi'(\mu_l)$, $\bar{\rho}_{1l}=\frac{2+\bar{\sigma}_0}{3}\rho_l$, $\bar{\sigma}_0=\sqrt[n]{\frac{1}{2}}$, by Lemma 3.8, it follows that

$$\omega_l\leqslant 2^{s+2}\rho_l^\varepsilon\leqslant 2^{s+2}\eta^\varepsilon\rho_{l+1}^\varepsilon\leqslant\frac{2^{-6}}{m^3}\hat{C}^{\frac{1}{m}}\cdot(64m^3\hat{C}^{1-\frac{1}{m}})^\varepsilon\rho_{l+1}^\varepsilon\leqslant\hat{C}\rho_{l+1}^\varepsilon \tag{4.18}$$

or

$$\operatorname{osc}\{w;\ \tilde{Q}_{\frac{\rho_0}{4}}\}\leqslant\left(1-\frac{1}{2^{s+3}}\right)\operatorname{osc}\{w,\ Q_{\rho_l}\}\leqslant\left(\frac{1}{\eta}\right)^\varepsilon\hat{C}\rho_l^\varepsilon\leqslant\hat{C}\rho_{l+1}^\varepsilon, \tag{4.19}$$

By the selection (4.12) of η and the condition $l<l^*$ which means $\mu_l<2\hat{C}\rho_l^\varepsilon$, we can show that $Q_{l+1}\subset\tilde{Q}_{\rho_l/4}$. In fact, in virtue of (1.4)′ we have

$$aA\left(\frac{\rho_l}{4}\right)^2\Big/a\Phi'(\rho_{l+1}^\varepsilon)\rho_{l+1}^2\geqslant\frac{\eta^2}{16m^3}\left(\frac{1}{2\hat{C}\eta^\varepsilon}\right)^{1-\frac{1}{m}}\geqslant\frac{\eta^{1+\frac{1}{m}}}{32m^3\hat{C}^{1-\frac{1}{m}}}\geqslant 2,$$

which implies $Q_{l+1}\subset\tilde{Q}_{\rho_l/4}$. Therefore, (4.16) holds by induction in this case.

(ii) If (4.17) fails, then there exists $\tau\in[t^0-aB\rho_l^2,\ t^0-aA\rho_l^2]$ such that

$$\operatorname{mes}A_{\mu-\frac{\omega_l}{2},\,\bar{\rho}_{1l}}(\tau)\leqslant\frac{1}{2}\varkappa_n\bar{\rho}_{1l}^n. \tag{4.20}$$

Divide the interval $[\tau,\ t^0]$ into N equal parts such that

$$\frac{1}{2}aA\rho_l^2\leqslant\Delta t=\frac{t^0-\tau}{N}\leqslant aA\rho_l^2, \tag{4.21}$$

where

$$N\leqslant\frac{2a\Phi'(\rho_l^\varepsilon)\rho_l^2}{aA\rho_l^2}\leqslant 2m^3\left(\frac{2\hat{C}\rho_l^\varepsilon}{\rho_l^\varepsilon}\right)^{1-\frac{1}{m}}\leqslant 4m^3\hat{C}^{1-\frac{1}{m}}=N_0. \tag{4.22}$$

Let $t_p=\tau+(p-1)\Delta t$ $(p=1,\ 2,\ \cdots,\ N)$ and $t_{N+1}=t^0$. By (4.20), we apply Lemma 3.8 to the interval $[t_1,\ t_2]$ and then obtain $\operatorname{mes} A_{\mu-\frac{\omega}{2^{s+4}},\bar\rho_3}(t_2)=0$, where $\bar\rho_3=\bar\sigma_0\rho$, $\bar\sigma_0=\sqrt[n]{\frac{1}{2}}$ and we omit the subscript l.

Thus $\operatorname{mes} A_{\mu-\frac{\omega}{2^{s+4}},\bar\rho_1}(t_2)\leqslant \operatorname{mes} A_{\mu-\frac{\omega}{2^{s+4}},\bar\rho_3}(t_2)+\varkappa_n(\bar\rho_1^n-\bar\rho_3^n)\leqslant\frac{1}{2}\varkappa_n\bar\rho_1^n$.

By induction we can obtain

$$\operatorname{mes} A_{\mu-\frac{\omega}{2^{(s+3)(N-1)+1}},\bar\rho_1}(t_N)\leqslant\frac{1}{2}\varkappa_n\bar\rho_1^n$$

and

$$\operatorname{mes} A_{\mu-\frac{\omega}{2^{(s+3)N+1}},\bar\rho_3}(t)=0\quad\text{for}\quad t\in\left[t^0-\frac{a}{32}A\rho^2,\ t^0\right].\tag{4.23}$$

Hence we have

$$\operatorname{osc}\{w;\ Q_{l+1}\}\leqslant\left(1-\frac{1}{2^{(s+3)N+1}}\right)\operatorname{osc}\{w;\ Q_l\}\leqslant\eta^{-\varepsilon}\hat C\rho_l^\varepsilon\leqslant\hat C\rho_{l+1}^\varepsilon.\tag{4.24}$$

So far we have shown that (4.16) holds for $l\leqslant l^*$. We have $\omega_{l^*}\leqslant\hat C\rho_{l^*}^\varepsilon$ in particular and $\mu_{l^*}\geqslant 2\hat C\rho_{l^*}^\varepsilon$ by the definition of l^*. Applying Lemma 4.1, we complete the proof.

Theorem 4.2. *Suppose that (1.2), (1.3), (1.4), (1.5), (1.6) and (1.7) are satisfied. Let $u(x,\ t)$ be a classical solution of equation (1.1)′ satisfying $0<u\leqslant M$ and belong to $C^{\varepsilon_1,\varepsilon_1/2}(\Gamma)$. Then, for any $(x^0,\ t^0)\in Q_T$ and $(x,\ t)\in Q^*_{\rho_0}\cap Q_T$ $(\rho_0=a_0)$ we have*

$$|w(x,\ ,\ t)-w(x^0,\ t^0)=C[|x-x^0|^\alpha+|t-t^0|^{\alpha/2}],\tag{4.25}$$

where α and C depend only on $n,\ m,\ \Lambda,\ M,\ T,\ \varepsilon_1,\ \delta$ $\|u\|_{C^{\varepsilon_1,\varepsilon_1/2}}(\Gamma)$ and the constants θ_0, a_0 of the condition (1.6).

Proof Let $\rho_0=a_0$, $\hat C$, η, ε be defined as in (4.11), (4.12), (4.13) respectively and, moreover, $\varepsilon\leqslant\varepsilon_1$. If $Q_{\rho_0}\subset Q_T$, then (4.25) is just the result of Theorem 4.1. Now let $Q_{\rho_0}\cap\Gamma\neq\emptyset$, then there exists $l_0\geqslant 1$ such that $Q_{l_0-1}\cap\Gamma\neq\emptyset$, $Q_{l_0}\cap\Gamma=\emptyset$. By Lemma 3.8′, it follows that $\operatorname{osc}\{w,\ Q_l\cap Q_T\}\leqslant\hat C\rho_l^\varepsilon$ if $l\leqslant l_0-1$, and hence

$$\operatorname{osc}\{w,\ Q_{l_0}\cap Q_T\}\leqslant\hat C\rho_{l_0-1}^\varepsilon\leqslant\hat C\eta^\varepsilon\rho_{l_0}^\varepsilon=\tilde C\rho_{l_0}^\varepsilon.\tag{4.26}$$

Using $\tilde C$, ρ_{l_0} instead of $\hat C$, $\rho_0$respectively in the proof of Theorem 4.1, we can obtain (4.25). The proof is complete.

In virtue of inequalities (1.4)′ and (1.10), it is easy to find

$$|\Phi(w_1)-\Phi(w_2)|\leqslant C|w_1-w_2|^{\frac{1}{m}},\tag{4.27}$$

where $C=C(m,\ M)$. We can immediately establish Hölder estimate for $u(x,\ t)$ from that for $w(x,\ t)$.

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