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RINGS WITH INVLOUTION WHOSE SYMMETRIC ELEMENTS ARE G-INVERTIBLE

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Abstract

This paper investigates structure theorems for some types of rings with involution in which for everv symmetric element s there exists a symmetric element t such that $st=ts$, $s=s^2t$.

§ 1. Introduction

Let R be a ring with involtion $*$, and let $S(R)$ be the set of symmetric elements of R with respect to $*$,

$$S(R)=\{x\in R \mid x=x^*\}.$$

Sometimes we will simply write S instead of $S(R)$.

The structures of rings with involution have been investigated when S satisfies cartain conditions, e. g. every non-zero symmetric element is invertible[6], is not nilpotent[5] or is periodic[5]; S has Von Neumann regularity[9]; S satisfies a polynomial identity[1] or generalized polynomial identity[11]; etc.

On the other hand, the concept of G-inverse of a complex matrix, which plays an important role in modern matrix theory, has been generalized to rings with involution[4].

Definition. *Let A be a subset of a ring R with involution $*$. An element $x\in A$ is said to be G-invertible in A if there exists an element $y\in A$ such that*

$$xyx=x, \quad yxy=y, \qquad \text{(I)}$$
$$(xy)^*=xy, \ (yx)^*=yx.$$

A ring R is said to be a G-ring if every element of R is G-invertible in R. R is a GS-ring if every element of $S(R)$ is G-invertible in $S(R)$.

A division ring with involution is a G-ring. The ring of all $n\times n$ matrices over the complex field with transposed conjugation as the involution is also a G-ring.

In this paper we study the structures of some types of GS-rings.

Using the same techniques as in matrix theory, it has been shown that for each element x in a G-ring R there exists only one element y satisfying (I). y is called the

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G-inverse of x, and an operation $+$ is defined on R by setting $x^+=y$. It is easy to see $(x^+)^*=(x^*)^+$.

Recall that a ring R with involution $*$ is called $*$-regular if $xx^*=0$ implies $x=0$ for any $x\in R$. In a $*$-regular ring R, for every elements $x, y, z\in R$, $yxx^*=zxx^*$ implies $yx=zx$ (cancellation law).

§ 2. GS-rings

Note that a symmetric element s is G-invertible in S if and only if there exists $t\in S$ such that

$$sts=s,\ tst=t,\ st=ts. \tag{II}$$

Lemma 1. *In a GS-ring R there are no non-zero nilpotent elements in $S(R)$.*

Proof Let $s\in S$, and let $t\in S$ satisfying (II), then

$$s=s^2t=s^3t^2=\cdots=s^nt^{n-1},\quad \text{for every integer}\quad n\geqslant 2,$$

so $s\neq 0$ implies $s^n\neq 0$ for every positive integer n.

Corollary. *In a GS-ring R, $xx^*=0$ implies $xx^*=0$ for every $x\in R$. In fact, if $xx^*=0$ then $(x^*x)^2=0$, so $x^*x=0$ by Lemma 1.*

Lemma 2. *In a $*$-regular ring R, if s is a symmetric element which is G-invertible in R, then s is G-invertible in S.*

Proof If $y\in R$ satisfies (I) with s, then $sy=(sy)^*=y^*s$ and $ys=(ys)^*=sy^*$, so $s=sys=y^*s^2$ and $s=(sys)^*=sy^*s=ys^2$. Therefore $(y-y^*)s^2=0$, so

$$0=(y-y^*)s\ s(y^*-y)=[(y-y^*)s][(y-y^*)s]^*.$$

which implies $(y-y^*)s=0$, since R is $*$-regular. Thus $ys=y^*s$ and $sy=sy^*$. So we have $y=ysy=y^*sy^*=(ysy)^*=y^*$.

Lemma 3. *In a GS-ring R, for each $s\in S$, there exists only one element $t\in S$ satisfying (II).*

Proof If there exist $t_1, t_2\in S$ such that

$$t_1st_1=t_1,\ st_1s=s,\ st_1=t_1s;$$
$$t_2st_2=t_2,\ st_2s=s,\ st_2=t_2s.$$

then $s=t_1s^2=t_2s^2$, $[(t_1-t_2)s]^2=0$ and $(t_1-t_2)s=0$ by Lemma 1. But $t_1=t_1^2s$, $t_2=t_2^2s$, so

$$t_1-t_2=t_1^2s-t_2^2s=t_1^2s-t_2(t_1s)=(t_1-t_2)t_1s=(t_1-t_2)st_1=0.$$

Lemma 4. *A ring R is a GS-ring if and only if for every $s\in S$ there exists an element $t\in S$ satisfying*

$$sts=s,\ st=ts. \tag{III}$$

Proof Suppose s and t satisfy (III). Let $t_1=tst$, then

$$st_1s=ststs=s,\quad t_1st_1=tststst=tst=t_1,$$
$$st_1=stst=tsts=t_1s,$$

so s is G-invertible in S.

Theorem 1. *If a GS-ring R is prime, then it is either a $*$-regular ring or a ring of 2×2 matrices over a field.*

Proof Since R is prime in which no non-zero element of $S(R)$ is nilpotent, by [5, p. 73], either $xx^*=0$ implies $x=0$ for any $x\in R$, hence R is $*$-regular, or $S(R)\subseteq Z(R)$ the center of R.

In the second case, for any element $s\in S$, $s\neq 0$, there exists an element $t\in S$ satisfying (II). So for any $x\in R$,

$$xs=xsts,\ (x-xst)s=0.$$

But $s\in Z(R)$, R is prime, so s can not be a zero-divizor. Thus $x=xst$, for all $x\in R$, therefore st is the identity of ring R and s is invertible. By [5, p. 62], R is either a division ring, of course, it is a $*$-regular ring, or a ring of 2×2 matrices over a field, relative to the symplectic involution.

We will see in next section that in the first case, R just is a G-ring.

§ 3. G-rings

Theorem 2. *A ring R with involution $*$ is a G-ring if and only if R is a $*$-regular GS-ring.*

Proof Let R be a G-ring, $x\in R$. If $xx^*=0$, then

$$0=xx^*(x^+)^*=x(x^+x)^*=xx^+x=x,$$

so R is $*$-regular. By Lemma 2, all the symmetric elements in R are G-invertible in S, so R is a GS-ring.

Conversely, let R be a $*$-regular GS-ring, and let $x\in R$. Then $xx^*\in S$, so there exists $t\in S$ such that

$$xx^*txx^*=xx^*,\qquad txx^*t=t,$$
$$xx^*t=txx^*.$$

Let $y=x^*t$. Then $xyxx^*=xx^*$ implies $(xyx-x)(xyx-x)^*=0$ and $xyx=x$; $txx^*t=t$ implies $yxy=y$; $xx^*t=txx^*$ implies $xy=(xy)^*$, and $(yx)^*=yx$ is obvious.

Lemma 5. *Every two sided ideal I in a G-ring R with involution $*$ is a $*$-ideal, and I is a G-ring relative to $*$.*

Proof Let $a\in I$, then $a^+=a^+aa^+\in I$ and

$$a^*=(aa^+a)^*=a^*(aa^+)^*=a^*aa^+\in I.$$

Theorem 3. *Let σ be a homomorphism of a G-ring R with involution $*$ onto a ring R'. Then $*$ induces an involution on R', R' is a G-ring relative to this involution, and*

$$\sigma^+(a)=\sigma(a^+),$$

for all $a\in R$.

Proof Let $N=\{x\in R\mid \sigma(x)=0\}$, then $N=N^*$, $R/N\cong R'$ and $*$ induces an involution $\sigma^*(a)=\sigma(a^*)$ in R'.

Let $a'\in R'$ and $a'=\sigma(a)$. Since R is a G-ring so there exists $a^+\in R$ satisfying (I). Hence

$$\sigma(a)\sigma(a^+)\sigma(a)=\sigma(aa^+a)=\sigma(a),$$
$$\sigma(a^+)\sigma(a)\sigma(a^+)=\sigma(a^+aa^+)=\sigma(a^+),$$
$$[\sigma(a^+)\sigma(a)]^*=\sigma^*(a^+a)=\sigma[(a^+a)^*]=\sigma(a^+a)=\sigma(a^+)\sigma(a),$$
$$[\sigma(a)\sigma(a^+)]^*=\sigma^*(aa^+)=\sigma[(aa^+)^*]=\sigma(aa^+)=\sigma(a)\sigma(a^+).$$

therefore R' is a G-ring relative to its induced involution $*$ and $\sigma^+(a)=\sigma(a^+)$ for all $a\in R$.

§ 4. Strong 2-torsion free GS-rings

Definition. *A ring R with involution $*$ is called strong 2-torsion free if for every $*$-ideal I in R, $2x\in I$ impliesx $x\in I$ for any $x\in R$.*

Lemma 6. *If R is a strong 2-torsion free GS-ring, then every $*$-homomorphic image R' of R is also a GS-ring.*

Proof Suppose $\sigma: R\to R'$, $R/I\cong R'$, where I is a $*$-ideal in R. For every symmetric element $x'=\sigma(x)\in S(R')$, $\sigma^*(x)=\sigma(x)$ means $2\sigma(x)=\sigma(x+x^*)$. R is a GS-ring, so there exists an element $t\in S(R)$ satisfying (II) with $x+x^*$. Therefore $(x+x^*)t(x+x^*)=x+x^*$, $2\sigma(x)\sigma(t)2\sigma(x)=2\sigma(x)$, and $2(x\cdot 2t\cdot x-x)\in I$. By the strong 2-torsion free property, $x\cdot 2t\cdot x=x$, so $\sigma(x)\sigma(2t)\sigma(x)=\sigma(x)$. Also $\sigma(t)\cdot 2\sigma(x)\cdot\sigma(t)=\sigma(t)$, so $\sigma(2t)\sigma(x)\sigma(2t)=\sigma(2t)$, and

$$\sigma(x)\cdot\sigma(2t)=2\sigma(x)\sigma(t)=\sigma[(x+x^*)t]=\sigma[t(x+x^*)]=\sigma(2t)\sigma(x).$$

Thus $\sigma(2t)$ satisfies (II), with x', and R' is a GS-ring.

Now we can prove our main theorem.

Theorem 4. *If R is a strong 2-torsion free GS-ring with Jacobson radical J, then $J^3=\{0\}$ and R/J is a subdirect sum of $*$-primitive rings R_α, $\alpha\in\Omega$, some Ω, where each R_α is either a primitive G-ring or a ring of 2×2 matrices over a field or $R_\alpha=P_\alpha\oplus P_\alpha^*$, where P_α is a division ring.*

Proof For any $x\in R$, if $x\in S(R)$, then there exists an element $y\in S(R)$ such that $xyx=x$, since R is GS-ring. If furthermore $x\in J$, then there exists an element $z\in R$ such that $xy+z=zxy$, therefore $xyx+zx=zxyx$, $xyx=x=0$. So $S(R)\cap J=\{0\}$. But, for every $x\in J$, we knew $x+x^*\in J\cap S(R)$, which means $x^*=-x$ for every $x\in J$, hence $J^3=\{0\}$.

Let $\bar{R}=R/J$. $\bar{R}$ is semi-primitive, so is a subdirect sum of primitive rings R'_α, $\alpha\in\Omega$, some Ω. $R'_\alpha\cong\bar{R}/P_\alpha$, $\bigcap_{\alpha\in\Omega}P_\alpha=\{0\}$, where $\{P_\alpha\}$ is the set of all primitive ideals of $\bar{R}$.

Note that $\bigcap_{\alpha\in\Omega} P_\alpha = \bigcap_{\alpha\in\Omega} (P_\alpha \cap P_\alpha^*)$, since P_α^* is also a primitive ideal in $\bar{R}$. So the ring $\bar{R}$ is a subdirect sum of rings $R_\alpha \cong \bar{R}/(P_\alpha \cap P_\alpha^*)$, $\alpha\in\Omega$.

For the simplification, we write P instead of P_α.

If $P=P^*$, then $R_\alpha=\bar{R}/P_\alpha$ is primitive, so it is prime. By Lemma 6, R_α is GS-ring. By Theorems 1 and 2, R_α is either a G-ring or a ring of 2×2 matrices over a field.

If $P\neq P^*$, then in the ring R_α there are two ideals $P/(P\cap P^*)$ and $P^*/(P\cap P^*)$ such that $P/(P\cap P^*)\cap P^*/(P\cap P^*)=\{0\}$. So R_α has a $*$-ideal $I=P/(P\cap P^*)+P/(P\cap P^*)$. It is easy to see $S(I)=I\cap S(R)$.

Since R_α is a GS-ring, so for every $s\in S(I)$ there exists an element $t\in S(R_\alpha)$ satisfying (II) with s.

But $t=t^2s\in I\cap S(R_\alpha)=S(I)$, which means I is also a GS-ring.

On the other hand, for every $x\in P/(P\cap P^*)$, $(x, x)\in S(I)$, so there exists an element $t\in S(I)$ satisfying (II), with (x, x). But every symmetric element in I must be of the form (z, z), $z\in P/(P\cap P^*)$. Suppose $t=(y, y)$, $y\in P/(P\cap P^*)$, then

$$(x, x)(y, y)(x, x)=(x, x),$$
$$(y, y)(x, x)(y, y)=(y, y),$$
$$(x, x)(y, y)=(y, y)(x, x),$$

therefore $xyx=x$, $yxy=y$, $xy=yx$. Thus, ring $P/(P\cap P^*)$ is a ξ-ring[10].

Additionally, $P/(P\cap P^*)\cong(P+P^*)/P^*$, $(P+P^*)/P^*$ is a two sided ideal of primitive ring $R_\alpha^*=\bar{R}/P^*$, so $P/(P\cap P^*)$ is also a primitive ring[7, p. 39]. By [10], a primitive ξ-ring is a division ring so $P/(P\cap P^*)$ is a division ring and I is the direct sum of a division ring and its opposite ring.

Let e be the identity of ring $P/(P\cap P^*)$, then for any $x, y\in I$ and $r\in R_\alpha$, we have

$$[r(e+e^*)-r](x+y^*)=rex-rx+re^*y^*-ry^*=0,$$

which means $[r(e+e^*)-r]I=\{0\}$.

Let

$$J=\{u\in R_\alpha \mid uI=0\},$$

then J is a two sided ideal of ring R_α and $JI=\{0\}$. Since ring R_α is $*$-prime, hence it is semi-prime [3]. So $JI=\{0\}$ implies $(IJ)^2=\{0\}$, $IJ=\{0\}$ and $J^*I=\{0\}$, $J^*\subset J$, J is a $*$-ideal. But, $I\neq\{0\}$, so $J=\{0\}$.

In a word, for any $r\in R_\alpha$, $r(e+e^*)-r=0$, $e+e^*$ is the identity of ring R_α, therefore $R_\alpha=I=P/(P\cap P^*)\oplus P^*/(P\cap P^*)$.

Here we would like to give a more complex example of a primitive G-ring, constructed in [2].

Let U be all the countably infinite matrices over K, the real numbers, which have the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

where A is a $n\times n$ matrix and n varies. U has ordinary transpose $*$ as involution.

For each $n\in N$, the natural numbers, let $U_n=U$, $N_n=N$ and form the direct products

$$R=\prod_{n\in N} U_n, \quad P=\prod_{n\in N} N_n.$$

R has an involution $*$ defined componentwise by $f^*(n)=[f(n)]^*$.

Let F be a non-principal untrafilter on N. Define the mapping σ: $R\to P$ by $\sigma(f)(n)=\text{rank } f(n)$ for each $n\in N$, and

$$J=\{f\in R: \exists k\in N, \quad G\in F\in\forall n\ni G, \quad \sigma(f)(n)<k\}.$$

It was shown in [2], that J is a $*$-ideal and R/J is a primitive ring with involution and zero socle in which for each symmetric element there exists an element c such that $s^2c=s$, $sc=cs$.

We prove that R/J is a G-ring.

Note that $\frac{1}{2}\in U$ and $\frac{1}{2}\in R$. Thus if $s^2c=s$, $sc=cs$ then $s^2c^*=s$, $sc^*=c^*s$ and $s^2t=s$, $st=ts$, where $t=(c+c^*)\in S(R/J)$, which means R/J is a GS-ring.

For any element $f\in R$, if $ff^*\in J$, since rank $(ff^*)(n)=\text{rank } f(n)$. So R/J is a $*$-regular ring.

Therefore R/J is a primitive G-ring, by Theorem 2, with zero socle.

§ 5. G-rings whose sets of symmetric elements are commutative

The definition of G-inverse is a generalization of the definition of inverse in an associative ring. But not in every G-ring R the G-inverse property always satisfies

$$aa^+=a^+a, \quad (bc)^+=c^+b^+, \tag{IV}$$

for all a, b, $c\in R$ like the inverse property does in every ring.

For example, let R be the 2×2 matrices ring over real field, a^* be the transposed matrix of a, for every $a\in R$. This is a G-ring.

Let

$$a=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad c=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$a^+=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b^+=b, \quad c^+=\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

Furthermore, we get

$$aa^+=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a^+a=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(bc)^+=\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&0\\1&0\end{pmatrix}\right)^+=\begin{pmatrix}1&0\\0&0\end{pmatrix},$$

$$c^+b^+=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&0\\0&0\end{pmatrix}.$$

$aa^+=a^+a$, $(bc)^+=c^+b^+$.

Lemma 7. *Let R be a G-ring such that for every $x\in R$, $xx^+\in Z(R)$, the center of R. Then for every x, $y\in R$,*

$$xx^+=x^+x,\qquad (xy)^+=y^+x^+,$$

and $xy=yx$ implies $x^+y=yx^+$.

Proof First note if $x\in R$, then $x^+x=x^+(x^+)^+\in Z(R)$. So

$$xx^+=(xx^+)^2=x(x^+x)x^+=x^+x\cdot xx^+=x^+(xx^+)x=x^+x.$$

We can directly check that y^+x^+ satisfies (I) with xy, so $(xy)^+=y^+x^+$ by the uniqueness of G-inverse.

If $xy=yx$, then

$$x^+y=x^+(xx^+)y=x^+y(xx^+)=x^+(xy)x^+=y(x^+x)x^+=yx^+.$$

Theorem 5. *Elements x, y in a G-ring whose set of symmetric elements is commutative satisfy (IV) and $xx^*=x^*x$.*

Proof Let J be the Jacobson radical of R. For any non-zero element $x\in R$, $(xx^+)^2=xx^+\neq 0$, so $x\notin J$. Therefore $J=\{0\}$. R is semi-primitive. But S is commutative, $S=Z(S)$. By [5, p. 232], $S=Z(S)\subset Z(R)$. Thus, $xx^+=(xx^+)^*\in S\subset Z(R)$, for all $x\in R$. By Lemma 7, the elements of R satisfy (IV).

For every $x\in R$, $x+x^*\in S\subset Z(R)$, so $x(x+x^*)=(x+x^*)x$, therefore $xx^*=x^*x$.

Corollary. *Every left (right, two sided) ideal is a $*$-ideal in a G-ring whose symmetric elements commute with each other.*

In fact, let L be a left ideal in R and $x\in L$, we have

$$x^*=(xx^+x)^*=x^*(xx^+)=(x^*x^+)x\in L.$$

The right and two sided ideals cases are easily proved.

Theorem 6. *Every G-ring R whose set of symmetric elements is commutative is a subdirect sum of division rings R_α, $\alpha\in\Omega$, some Ω, where each R_α either has a commutative set of symmetric elements or has* ch $R_\alpha=2$ *and $(xy-yx)^2\in Z(R_\alpha)$ for any x, $y\in R_\alpha$.*

Proof R is a semi-prime ring, hence R is a subdirect sum of prime rings R_α, $\alpha\in\Omega$, some Ω.

By Theorem 3, each R_α is a G-ring and $R_\alpha\cong R/I_\alpha$, where I_α is a $*$-ideal. Let σ be the homomorphism of R onto R_α.

If $u\in R_\alpha$ and $u=\sigma(a)$, $a\in R$, then $u^+=\sigma(a^+)$ by Theorem 3. R is semi-prime and $S(R)$ is commutative, so $S\subset Z(R)$ by [5, p. 232]. But $aa^+\in Z(R)$ means

$$uu^{+}=\sigma(aa^{+})\in Z(R_\alpha).$$

Note that R_α is prime, $(uu^{+})^2=uu^{+}\in Z(R_\alpha)$, so for any $x\in R_\alpha$, $[xuu^{+}-x]uu^{+}=0$. If $u\neq 0$, then $uu^{+}\neq 0$. So $xuu^{+}=x$ for any $x\in R_\alpha$. Therefore, every non-zero element u in R_α must be invertible, R_α is a division ring with involution.

Furthermore, for every $x\in R_\alpha$, suppose $x=\sigma(b)$, $b\in R$, then

$$x+x^*=\sigma(b)+\sigma^*(b)=\sigma(b+b^*)\in\sigma[Z(R)]\subset Z(R_\alpha),$$

$$xx^*=\sigma(b)\sigma^*(b)=\sigma(bb^*)\in Z(R_\alpha).$$

So for every symmetric element u in ring R_α, $u^2\in Z(R_\alpha)$, $2u\in Z(R_\alpha)$.

If $\mathrm{ch}\,R_\alpha\neq 2$, $2u\in Z(R_\alpha)$ implies $u\in Z(R_\alpha)$, for any $u\in S(R_\alpha)$. Therefore R_α has a commutative set of symmetric elements.

If $\mathrm{ch}\,R_\alpha=2$, for every x, $y\in R_\alpha$, $x+x^*\in Z(R_\alpha)$ implies $(x+x^*)y=y(x+x^*)$, $xy+yx=x^*y+yx^*$; $y+y^*\in Z(R_\alpha)$ implies $xy+yx=xy^*+y^*x$, so $xy+yx\in S(R_\alpha)$, $(xy-yx)^2\in Z(R_\alpha)$.

Remark. If the set $S(R)$ of symmetric elements in a division ring R with involution and $\mathrm{ch}R=2$ is commutative, then R itself is commutative.

In fact, suppose $x\in R$, $x\notin Z(R)$, then there exists $y\in R$ such that $xy+xy\neq 0$. By Theorem 6, $xy+yx\in S(R)=Z(S)\subset Z(R)$. So $x(xy+yx)=x\cdot xy-xy\cdot x\in S$ and

$$x(xy+yx)=[x(xy+yx)]^*=(xy+yx)x^*.$$

On the other hand $xy+yx\in Z(R)$, so $x(xy+yx)=(xy+yx)x$, thus

$$(xy+yx)(x-x^*)=0,\qquad x=x^*$$

which is a contradiction since $S(R)\subset Z(R)$.

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