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UNBOUNDED SOLUTIONS OF CONSERVATIVE OSCILLATORS UNDER ROUGHLY PERIODIC PERTURBATIONS

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Abstract

This note is concerned with the equation

$$\frac{d^2x}{dt^2}+g(x)=p(t), \tag{1}$$

where $g(x)$ is a continuously differentiable function of $x\in\mathbf{R}$, $xg(x)>0$ whenever $x\neq 0$, and $g(x)/x$ tends to ∞ as $|x|\to\infty$. Let $p(t)$ be a bounded function of $t\in\mathbf{R}$. Define its norm by $\|p\|=\sup_{t\in\mathbf{R}}|p(t)|$.

The study of this note leads to the following conclusion which improves a result due to J. E. Littlewood.

For any given small constants $\alpha>0$ and $\varepsilon>0$, there is a continuous and roughly periodic (with respect to $\Omega(\alpha)$) function $p(t)$ with $\|p\|<\varepsilon$ such that the corresponding equation (1) has at least one unbounded solution.

§ 1. Introduction

In this note we are concerned with the equation

$$\frac{d^2x}{dt^2}+g(x)=p(t), \tag{1.1}$$

where $g(x)$ is, for simplicity, a continuously differentiable function of $x\in\mathbf{R}$, $xg(x)>0$ whenever $x\neq 0$, and $g(x)/x$ increases to ∞ as $|x|\to\infty$. Let $p(t)$ be a bounded function of $t\in\mathbf{R}$. Define its norm by

$$\|p\|=\sup_{t\in\mathbf{R}}|p(t)|.$$

For any constant $\alpha\geqslant 0$, consider a family of intervals

$$I_n(\alpha)=\{t\in\mathbf{R}:|t-n\pi|<\alpha\},\quad (n\in\mathbf{Z}),$$

and a closed set

$$\Omega(\alpha)=\mathbf{R}\setminus\bigcup_{n\in\mathbf{Z}}I_n(\alpha).$$

Note that $\Omega(0)=\mathbf{R}$.

A function $p(t)$ is said to be roughly periodic (with respect to $\Omega(\alpha)$) if $p(t+2\pi)=p(t)$ for all $t\in\Omega(\alpha)$. A function $p(t)$ is said to be roughly periodic in limit (with

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respect to $\Omega(\alpha)$) if there is a constant $\tau=\tau(\alpha)>0$ such that $p(t+2\pi)=p(t)$ for all $t\in\Omega(\alpha)$ whenever $t\geqslant\tau$. It is clear that if $p(t)$ is roughly periodic, then it is roughly periodic in limit; but the converse is not true in general.

In 1965, J. E. Littlewood[1] proved that for any given positive constant α, there is a sectionally continuous function $p(t)$ with $\|p\|\leqslant 1$, which is roughly periodic in limit(with respect to $\Omega(\alpha)$), such that the above-stated equation (1.1) admits at least one unbounded solution. However, G. Morris[2] proved the boundedness of solutions of (1.1) in a very special case of that $g(x)=2x^3$ and $p(t)$ in2π-periodic Therefore, it is O.K. O.K. to decide whether all solutions of (1.1) are bounded if $p(t)$ is a continuous periodic function with small norm (cf. [3]), or even if $p(t)$ is a continuous and roughly periodic function with small norm.

The study of this note leads to the following conclusion.

For any given small constants $\alpha>0$ and $\varepsilon>0$, there is a continuous and roughly periodic (with respect to $\Omega(\alpha)$) function $p(t)$, $\|p\|<\varepsilon$, such that the corresponding equation (1.1) has at least one unbounded solution.

§ 2. Auxiliary equations

We first consider an auxiliary equation

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=-g(x)+\frac{1}{2}E, \tag{2.1}$$

where E is a positive constant. Let

$$G(x)=2\int_0^x g(x)dx.$$

Then a first integral of (2, 1) can be derived in the following form

$$y^2+[G(x)-Ex]=G(c)-Ec, \tag{2.2}$$

where c is an arbitrary constant. Since

$$\frac{1}{x^2}[G(x)-Ex]\to\infty \quad \text{as } |x|\to\infty,$$

the equation (2, 2) defines a simple closed orbit $\Gamma_1(c)$ whenever $|c|$ is large enough. It can easily be seen that $\Gamma_1(c)$ surrounds the origin of (x, y)-plane and is symmetric with respect to x-axis. Let $\tau_1(c)$ denote the least period of $\Gamma_1(c)$. Then we arrive at

$$\tau_1(c)=2\int_{-a}^{c}\frac{dx}{\sqrt{[G(c)-Ec]-[G(x)-Ex]}},$$

where the constant $a>0$ is determined by

$$G(-a)+Ea=G(c)-Ec, \tag{2.3}$$

or in other words, $\Gamma_1(c)$ intersects the x-axis at the points $(c, 0)$ and $(-a, 0)$. As in [1], we can prove

$$\tau_1(c)=o(1) \quad \text{as } c\to\infty. \tag{2.4}$$

Let $\Gamma_1(c)$ intersect the y-axis at the points $(0, b)$ and $(0, -b)$. Then we have

$$b=\sqrt{G(c)-Ec}, \tag{2.5}$$

which increases to ∞ as $c\to\infty$.

Denote the motion of (2.1) satisfying the initial condition $(x(\xi), y(\xi))=(-a, 0)$ by

$$x=x_1(t, \xi, c),\ y=y_1(t, \xi, c), \tag{2.6}$$

for $t\in\left[\xi, \xi+\left(k+\frac{1}{2}\right)\tau_1(c)\right]$. Geometrically, the motion (2.6) moves clockwise along the orbit $\Gamma_1(c)$ from the point $(-a, 0)$ to the point $(c, 0)$ for $\left(k+\frac{1}{2}\right)$ turns.

We next consider another auxiliary equation

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-g(x)-\frac{1}{2}E. \tag{2.7}$$

In the same way as before, we get a first integral

$$y^2+[G(x)+Ex]=G(c)+Ec, \tag{2.8}$$

which defines a cycle $\Gamma_2(c)$ of (2, 7) whenever c is large enough. The cycle $\Gamma_2(c)$ has a similar property as that of $\Gamma_1(c)$, and its least period is defined by

$$\tau_2(c)=2\int_{-e}^{c}\frac{dx}{\sqrt{[G(c)+Ec]-[G(x)+Ex]}},$$

where the constant $e>0$ is determined by

$$G(-e)-Ee=G(c)+Ec. \tag{2.9}$$

This means that $\Gamma_2(c)$ intersects the x-axis at the points $(-e, 0)$ and $(c, 0)$. Moreover, we have

$$\tau_2(c)=o(1) \quad \text{as } c\to\infty. \tag{2.10}$$

Let $\Gamma_2(c)$ intersect the y-axis at the points $(0, d)$ and $(0, -d)$. Then we get

$$d=\sqrt{G(c)+Ec}, \tag{2.11}$$

which increases to ∞ as $c\to\infty$.

Denote the motion of (2, 7) satisfying the initial condition $(x(\eta), y(\eta))=(c, 0)$ by

$$x=x_2(t, \eta, c),\ y=y_2(t, \eta, c). \tag{2.12}$$

for $t\in\left[\eta, \eta+\left(l+\frac{1}{2}\right)\tau_2(c)\right]$. Then the motion (2.12) moves clockwise along the orbit $\Gamma_2(c)$ from the point $(c, 0)$ to the point $(-e, 0)$ for $\left(l+\frac{1}{2}\right)$ turns.

§ 3. An auxiliary motion

For any given constant $\alpha>0\left(\alpha>\frac{\pi}{8}\right)$, it follows from (2.4) and (2.10) that there exists a constant $c^*>0$, such that

$$\tau_1(c)+\tau_2(c)<\frac{\alpha}{4} \quad \text{whenever } c\geqslant c^*. \tag{3.1}$$

By (2.3), we can assume $a=a_0>0$ is so large that the corresponding constant $c=c_0$ is larger than c^*. Then we set $\xi=0$ and $c=c_0$ in the motion (2.6), and by (3.1), we can choose an integer $k=k_0\geqslant 0$, such that

$$t_1=\left(k_0+\frac{1}{2}\right)\tau_1(c_0)\in I_1\left(\frac{\alpha}{2}\right).$$

In this case, we denote the corresponding motion by

$$x=\varphi_0(t),\ y=\psi_0(t) \quad (0\leqslant t\leqslant t_1),$$

with the corresponding numbers a_0, c_0 and b_0 defined by (2.3) and (2.5).

Next, we put $\eta=t_1$ and $c=c_0$ in the motin (2.12), and by (3.1), we can take an integer $l=l_1\geqslant 0$, such that

$$t_2=t_1+\left(l_1+\frac{1}{2}\right)\tau_2(c_0)\in I_2\left(\frac{\alpha}{2}\right).$$

And in this case, we rewrite (2.12) in the form

$$x=\varphi_1(t),\quad y=\psi_1(t) \quad (t_1\leqslant t\leqslant t_2),$$

with the corresponding numbers c_0, e_0 and d_0 defined by (2.9) and (2.11).

It follows from (2.3) and (2.9) that

$$G(-e_0)-G(-a_0)=E(a_0+e_0+2c_0)>0,$$

which implies that $e_0>a_0$. Therefore, if we put $a=a_1=e_0$ in (2.3), the corresponding constant $c=c_1$ is larger than c^*. Then the above-stated procedure can be repeated once more.

In general, we can successively define a sequence of motions

$$x=\varphi_n(t),\quad y=\psi_n(t),\quad (t_n\leqslant t\leqslant t_{n+1}), \tag{3.2$_n$}$$

with $t_n\in I_n\left(\frac{\alpha}{2}\right)$ $(n\geqslant 0)$. When $n=2j$, $(3.2)_n$ is a solution of the equation (2.1), with the corresponding constants a_j, c_j and b_j, determined by (2.3) and (2.5); when $n=2j+1$, $(3.2)_n$ is a solution of (2.7), with the corresponding constants c_j, e_j and d_j, determined by (2.9) and (2.11), and moreover, $a_{j+1}=e_j>a_j$ $(j=0, 1, 2, \cdots)$.

Since $\varphi_n(t_{n+1})=\varphi_{n+1}(t_{n+1})$ and $\psi_n(t_{n+1})=\psi_{n+1}(t_{n+1})$, we obtain a continuous motion

$$x=\varphi(t),\ y=\psi(t) \quad (t\in\mathbb{R}), \tag{3.3}$$

by combinning $(3.2)_n$ $(n=0, 1, 2, \cdots)$ together and then taking a continuition as follows

$$\varphi(-t)=\varphi(t),\quad \psi(-t)=-\psi(t) \quad (t\geqslant 0).$$

Hence, the motion (3.3) satisfies the equation

$$\frac{dx}{dt}=y,\quad \frac{dy}{dt}=-g(x)+f(t), \tag{3.4}$$

where $f(t)$ is an even function, and is equal to $(-1)^n\frac{E}{2}$ on the interval (t_n, t_{n+1}) $(n=0, 1, 2, \cdots)$.

It may be noted that $\varphi'(t)=\psi(t)$ is continuous in $t\in\mathbb{R}$, but $\psi'(t)(=\psi'(-t))$ is discontinuous at the points $\{\pm t_n\}$, with the limits

$$\psi'(t)=\begin{cases} -g(-a_j)+\frac{1}{2}E, & \text{when } t=t_{2j}+0; \\ -g(c_j)+\frac{1}{2}E, & \text{when } t=t_{2j+1}-0; \\ -g(c_j)-\frac{1}{2}E, & \text{when } t=t_{2j+1}+0; \\ -g(-a_{j+1})-\frac{1}{2}E, & \text{when } t=t_{2j+2}-0, \end{cases}$$

$(j=0, 1, 2, \cdots)$.

Now, we are going to prove the unboundedness of the auxiliary motion (3.3), that is

Lemma 1. *The functions $\varphi(t)$ and $\psi(t)$ are unbounded on* $\mathbb{R}$.

Proof In suffices to prove the unboundedness of the sequence $\{a_n\}$.

Now, suppose on the contrary that the sequence $\{a_n\}$ is bounded. Then the sequence $\{c_n\}$ is also bounded. Since they are increasing, their limits exist, say

$$A=\lim_{n\to\infty} a_n \quad\text{and}\quad C=\lim_{n\to\infty} c_n.$$

Then

$$A+C\geqslant a_0+c_0>0. \tag{3.5}$$

On the other hand, it follows from (2.3) and (2.9) that

$$G(-a_n)-G(-a_{n+1})+Ea_n+Ea_{n+1}+2Ec_n=0.$$

Setting $n\to\infty$, we conclude that

$$2E(A+C)=0,$$

which contradicts (3.5). We have thus proved the desired conclusion.

§ 4. The main theorem

By means of smoothing the auxiliary motion (3.3), we are trying to look for a continuous function $p(t)$, with the desired property stated in the Section 1, instead of the function $f(t)$ in (3.4). The basic technique consists of the following lemmas.

Consider the equation

$$\frac{dx}{dt}=y, \quad \frac{dy}{dt}=-g(x)+B \quad (s\leqslant t\leqslant r), \tag{4.1}$$

where the constants $B\neq 0$, and $s<r$. Let

$$x=x(t),\ y=y(t) \quad (s\leqslant t\leqslant r) \tag{4.2}$$

be a continuous solution of (4.1), where we mean that $s=s+0$ and $r=r-0$. Assume that

$$x(s)=\sigma,\ x(r)=\bar{\sigma},\ y(s)=0,\ y(r)=0. \tag{4.3}$$

Then we have

$$x'(s)=y(s)=0,$$

and

$$y'(s)=x''(s)=-g(\sigma)+B. \tag{4.4}$$

Lemma 2. *For (4.2), we can find an arbitrarily small positive constant δ, and a pair of continuously differentiable functions*

$$u=u(t),\ v=v(t)\quad (s\leqslant t\leqslant r), \tag{4.5}$$

such that

(i) *the identities*

$$u(t)=x(t),\ v(t)=y(t)$$

hold on the interval $[s_1, r]$ $(s_1=s+\delta)$;

(ii) $u(s)=\sigma$, $u'(s)=v(s)=0$ *and*

$$v'(s)=u''(s)=-g(\sigma), \tag{4.6}$$

(iii) (4.5) *satisfies the equation*

$$\frac{du}{dt}=v,\quad \frac{dv}{dt}=-g(u)+h(t) \tag{4.7}$$

on $[s, r]$, where $h(t)$ is continuous on $[s, r]$, with the properties: (1) $h(s)=0$; (2) $h(t)=B$ *for all $t\in[s_1, r]$;* (3) $|h(t)|\leqslant 18|B|$ *on* $[s, r]$.

Proof First, we define

$$u(t)=x(t),\ v(t)=y(t)\quad \text{on}[s_1, r].$$

Then let

$$v(t)=-\frac{g(\sigma)}{\delta^2}(t-s)(t-s_1)^2+\lambda(t-s)^2(t-s_1)^2 +\frac{(t-s)^2}{\delta^2}\left[y(s_1)+\left(y'(s_1)-\frac{2}{\delta}y(s_1)\right)(t-s_1)\right]$$

on the interval $[s, s_1]$, λ being an indetermined parameter. It can be verified directly that

$$v(s)=0,\ v'(s)=-g(\sigma)$$

and

$$v(s_1)=y(s_1),\ v'(s_1)=y'(s_1),$$

which guarantee the continuous differentiability of $v(t)$ on $[s, r]$. Then we set

$$u(t)=\int_s^t v(t)dt+x(s),\ \text{for } t\in[s, s_1],$$

and

$$u(s_1)=\int_s^{s_1} v(t)dt+x(s)=x(s_1), \tag{4.8}$$

which imply that $u(t)$ is continuously differentiable on $[s, r]$. It follows from (4.8) that

$$\lambda\int_s^{s_1}(t-s)^2(t-s_1)^2dt=(x(s_1)-x(s))+\frac{g(\sigma)}{\delta^2}\int_s^{s_1}(t-s)(t-s_1)^2dt -\frac{1}{\delta^2}\int_s^{s_1}(t-s)^2\left[\left(y(s_1)+\left(y'(s_1)-\frac{2}{\delta}y(s_1)\right)(t-s_1)\right)\right]dt.$$

Then we obtain

$$\lambda=\frac{30}{\delta^5}\Big[(x(s_1)-x(s))+\frac{\delta^2}{12}g(\sigma)-\Big(\frac{\delta}{2}y(s_1)-\frac{\delta^2}{12}y'(s_1)\Big)\Big], \tag{4.9}$$

and the functions $v(t)$ and $u(t)$ are completely defined on $[s, r]$. Hence, what is left now is the proof of (iii).

In what follows ξ always stands for an infinitesimal as $\delta\to 0$, i. e., $\xi=0(1)$ as $\delta\to 0$. Using (4.9), togethes with the following estimates

$$x(s_1)-x(s)=\delta x'(s)+\frac{\delta^2}{2}(x''(s)+\xi)=-\frac{\delta^2}{2}(g(\sigma)-B)+\delta^2\xi,$$

$$-\frac{\delta}{2}y(s_1)=-\frac{\delta^2}{2}(y'(s)+\xi)=\frac{\delta^2}{2}(g(\sigma)-B)+\delta^2\xi,$$

$$\frac{\delta^2}{12}y'(s_1)=-\frac{\delta^2}{12}(g(\sigma)-B)+\delta^2\xi,$$

we obtain

$$\lambda=\frac{5B}{2\delta^3}+\frac{1}{\delta^3}\xi,$$

which implies

$$v(t)=\xi \quad \text{on } [s, s_1].$$

Then we conclude that, for $t\in[s, s_1]$,

$$u(t)=x(s)+\int_s^t v(t)dt=\sigma+\xi$$

and

$$g(u(t))=g(\sigma)+\xi. \tag{4.10}$$

On the other hand, since

$$v'(t)=-\frac{g(\sigma)}{\delta^2}(t-s_1)^2-\frac{2g(\sigma)}{\delta^2}(t-s)(t-s_1)$$
$$+\frac{2(t-s)}{\delta^2}\Big[y(s_1)+\Big(y'(s_1)-\frac{2}{\delta}y(s_1)\Big)(t-s_1)\Big]+\Big(y'(s_1)-\frac{2}{\delta}y(s_1)\Big)\Big(\frac{t-s}{\delta}\Big)^2$$
$$+2\lambda[(t-s)(t-s_1)^2+(t-s)^2(t-s_1)] \quad (s\leqslant t\leqslant s_1),$$

we have

$$v'(t)=-g(\sigma)+4B\Big(\frac{t-s}{\delta}\Big)-3B\Big(\frac{t-s}{\delta}\Big)^2+5B\Big[\Big(\frac{t-s}{\delta}\Big)\Big(\frac{t-s_1}{\delta}\Big)^2$$
$$+\Big(\frac{t-s}{\delta}\Big)^2\Big(\frac{t-s_1}{\delta}\Big)\Big]+\xi \text{ for } t\in[s, s_1]. \tag{4.11}$$

Let

$$h(t)=v'(t)+g(u(t)) \quad \text{on } [s, r].$$

Then $h(t)$ is continuous on $[s, r]$, and $h(t)=B$ for all $t\in[s_1, r]$. Note that

$$h(s)=v'(s)+g(u(s))=-g(\sigma)+g(\sigma)=0.$$

From (4.10) and (4.11), we obtain the desired estimate

$$|h(t)|\leqslant 4|B|+3|B|+10|B|+|B|=18|B|$$

prorided that $|\xi|\leqslant|B|$, or δ is sufficientlysmall. the proof of the lemma is completed.

In the same way, we can prove the following lemma.

Lemma 3. *For (4.5), we can find an arbitrary small positive constant $\bar{\delta}$, and a pair of continuously differentiable functions*

$$u=\bar{u}(t), \quad v=\bar{v}(t) \quad (s\leqslant t\leqslant r), \tag{4.12}$$

such that (i) *the identities $\bar{u}(t)=u(t)$, $\bar{v}(t)=v(t)$ hold on the interval $[s, r_1]$ $(r_1=r-\bar{\delta})$;*

(ii) $\bar{u}(r)=\bar{\sigma}$, $\bar{u}'(r)=\bar{v}(r)=0$ *and* $\bar{v}'(r)=\bar{u}''(r)=-g(\bar{\sigma})$;

(iii) (4.12) *satisfies the equation*

$$\frac{du}{dt}=v, \quad \frac{dv}{dt}=-g(u)+\bar{h}(t) \tag{4.13}$$

on $[s, r]$, *where* $\bar{h}(t)$ *is continuous on* $[s, r]$, *with the properties*: (1) $\bar{h}(r)=0$; (2) $\bar{h}(t)=B$ *for all* $t\in[s_1, r_1]$; (3) $|\bar{h}(t)|\leqslant 18|B|$ *on* $[s, r]$.

Now we are ready to prove the main result of this note.

Theorem. *For any given function* $g(x)\in C^1(\mathbf{R}, \mathbf{R})$ *with the properties mentioned in the first section, and for any given constants* $\alpha>0$ *and* $\varepsilon>0$, *there is a continuous and roughly periodic (with respect to* $\Omega(\alpha)$*) function* $p(t)$, $\|p\|<\varepsilon$, *such that the corresponding equation* (1.1) *admits at least one unbounded solution.*

Proof We start with the auxiliary motion (3.3). Then, applying Lemmas 2 and 3 to (3.3) on the intervals $[s, r]=[t_n, t_{n+1}]$ (or $[-t_{n+1}, -t_n]$), $(n=0, 1, 2, \cdots)$, we get a pair of continuously differentiable functions

$$u=u(t), \quad v=v(t), \tag{4.14}$$

on the interval $[s, r]$, which satisfies the following conditions:

(i) $u(s)=\varphi(s)$, $v(s)=\psi(s)$, $u(r)=\varphi(r)$, $v(r)=\psi(r)$;

(ii) $v'(s)=-g(u(s))$, $v'(r)=-g(u(r))$;

(iii) (4.14) satisfies the equation

$$\frac{du}{dt}=v, \quad \frac{dv}{dt}=-g(u)+p(t) \tag{4.15}$$

on $[s, r]$, where $p(t)$ is continuous on $[s,r]$, $p(s)=p(r)=0$, $|p(t)|\leqslant 9E$ on $[s, r]$, and $p(t)=(-1)^n\frac{E}{2}$ on $[s+\alpha, r-\alpha]$ when $s=t_n$, $r=t_n$, $r=t_{n+1}$ (or $s=-t_{n+1}$, $r=-t_n$).

Hence, it follows that there is a pair of continuously differentiable functions (4.14) defined on $\mathbf{R}$, which is unbounded because of (i), and satisfies the equation (4.15) with a continuous function $p(t)$ on $\mathbf{R}$. Moreover, $p(t)$ is roughly periodic (with respect to $\Omega(\alpha)$), and $\|p\|\leqslant\varepsilon$ whenever $E=\frac{1}{9}\varepsilon$. We have thus proved the desired conclusion.

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