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EXISTENCE CONDITIONS OF A SPECIAL TYPE OF LIAPUNOV FUNCTIONAL FOR TWO-DIMENSIONAL DELAY SYSTEM

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Abstract

In this paper, the author considers the two-dimensional delay systems

$$\dot{x}(t)=Ax(t)+Bx(t-r),\ A,\ B\in R^{2\times 2},\ x\in R^2,\ r=\text{const.}\geqslant 0, \tag{*}$$

and gives the necessary and sufficient conditions under which where exists a simple type of positive definite Liapunov functional

$$V(\varphi)\overset{\text{def.}}{=\!=}\varphi'(0)T\varphi(0)+\int_{-r}^{0}\varphi'(\theta)E\varphi(\theta)d\theta,$$

and $\alpha(s)$ (where T, E are positive definite 2×2 matrices, $\varphi\in C([-r,\ 0],\ R^n)$, "$'$" stands for transpose, $\alpha(s)$ is continuous and $\alpha(0)=0$, $\alpha(s)>0$, $s>0$.) such that $\dot{V}_{(*)}(\varphi)\leqslant-\alpha(|\varphi(0)|)$.

§ 1. Introduction

Consider the delay systems

$$\dot{x}(t)=Ax(t)+Bx(t-r), \tag{1.1}$$

where A, B are $n\times n$ constant matrices and $r=\text{const.}\geqslant 0$. J. K. Hale in [1] discussed the stability of the zero solutions of (1.1) by the functional

$$V(\varphi)=\varphi'(0)T\varphi(0)+\int_{-r}^{0}\varphi'(\theta)E\varphi(\theta)d\theta,$$

where $\varphi\in C([-r,\ 0],\ R^n)$, T, E are $n\times n$ symmetric matrices. Obviously we have

$$\dot{V}_{(1.1)}(\varphi)=-(\varphi'(0),\ \varphi'(-r))S\begin{pmatrix}\varphi(0)\\ \varphi(-r)\end{pmatrix},$$

where

$$S=\begin{pmatrix}-(TA+A'T)-E & -TB\\ -B'T & E\end{pmatrix}. \tag{1.2}$$

Particularly, if T, E are positive definite and S is also positive, then Liapunov's Theorem implies that the zero solution of (1.1) is asymptotically stable for any $r\geqslant 0$. Naturally, one will raise such a question: What are the necessary and sufficient conditions for the existence of the positive definite matrices T, E and S? (This question was first mentioned in [1]) Huang Zheng-xun and Lin Siao-biao

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pointed out lately that, if there are $T>0$, $E>0$ such that $S>0$, it isn't necessary that there is an estimate of the rate of the selutions to zero which does not depend on the delay for (1.1); in addition, there are matrices A, B, such that the zero solution of (1.1) is asymptotically stable for any delay, but there do not exist $T>0$, $E>0$ such that $S>0$ (see[2]).) We know, for $n=1$, i. e. the scalar equation

$$\dot{x}(t)=ax(t)+bx(t-r),$$

the necessary and sufficient conditions are $a<0$, $|a|>|b|$. But when $n>1$, the above problem will become complex. In this paper, we will discuss this problem for $n=2$, and will give a necessary and sufficient condition, that is

Theorem 1. *Suppose A, B are 2×2 matrices, then there are positive definite matrices T, E such that S in (1.2) is positive definite if and only if one of the following conditions holds:*

(i) *A is stable, $|B|>0$ and*

$$w(A,\ B)\overset{\text{def.}}{=\!=}\sup\{\operatorname{Re}\lambda\ (1+r):|A+Be^{-\lambda r}-\lambda I|=0,\ r\geqslant 0\}<0.$$

(ii) *A is stable, $|B|\leqslant 0$, $|A+B|>0$, $|A-B|>0$ and*

$$f(A,\ B)\overset{\text{def.}}{=\!=}(\operatorname{tr} A)^2-(\operatorname{tr} B)^2+2[(|A+B|\cdot|A-B|)^{\frac{1}{2}}-|A|+|B|]>0.$$

HereA is said to be stable if all the eigenvalues of A have negative real parts, and

$$|A|=\det(A).$$

(We will point out the differences between (i) and (ii) later)

§ 2. Definition and properties of strong stable domain D_A

Let $R^{2\times 2}$ be the spaco of the 2×2 real matrices with norm $\|A\|=\sum_{i,j=1}^{2}|a_{ij}|$ for $A=(a_{ij})$. We denote $T>0$ if T is a symmetric and positive definite matrix. For A, B, T, $E\in R^{2\times 2}$, where T, E are symmetric, we define

$$S(A,\ B,\ T,\ E)=\begin{pmatrix}-(TA+A'T)-E & -TB\\ -B'T & E\end{pmatrix}.$$

Definition 1. *For any stable matrix $A\in R^{2\times 2}$,* the set *$D_A=\{B:B\in R^{2\times 2}$, there are symmetric matrices T, $E\in R^{2\times 2}$ such that $S(A,\ B,\ T,\ E)>0\}$ is said to be the strong stable domain associated with A.*

Obviously, if A is stable and $S(A,\ B,\ T,\ E)>0$,then there must be $T>0$, $E>0$. Therefore if $B\in D_A$, the zero solution of (1.1) is asymptotically stable for any $r\geqslant 0$. Now we will give some properties of D_A.

Lemma 1. *For any stable matrix $A\in R^{2\times 2}$, D_A is a nonempty, connected and open set of $R^{2\times 2}$.*

Proof It is clear that D_A is nonempty and open. To prove D_A is connected, suppose $B\in D_A$, By the definition, there are symmetric matrices T, E such that

$S(A, B, T, E)>0$. We can check that for any t, $0\leqslant t\leqslant 1$, $S(A, tB, T, E)>0$. Thus, the segment $\{tB: 0\leqslant t\leqslant 1\}\subset D_A$. Therefore D_A is connected.

Lemma 2. *Suppose A, B are $n\times n$ matrices and A is stable. If there are v, $\theta\in R$ such that $|A+Be^{-i\theta}-ivI|=0$, then for any $u\in R$ any $\varepsilon>0$, there are $r_0\in R^+$ and λ_0 such that*

(1) $$|A+e^u Be^{-\lambda_0 r_0}-\lambda_0 I|=0,$$

(2) $$|\operatorname{Re}\lambda_0(1+r_0)-u|<\varepsilon.$$

Proof Suppose $v\neq 0$. For $u\in R$, let $F(\lambda)=|A+e^u Be^{-\lambda-i\theta}-ivI|$,

$$H(\lambda, k)=|A+e^u Be^{-\lambda-i\theta}-(|v|\lambda/(2k\pi+iv)+i|v|)I|,$$

where k is a positive integer. By the hypothesis, $F(\lambda)$ has the zero point $\lambda_1=u+i(\theta-v)$ and $F(\lambda)\not\equiv 0$ since A is stable. Moreover $F(\lambda)$, $H(\lambda, k)$ are analytic functions and $H(\lambda, k)\to F(\lambda)$ as $k\to\infty$ is uniform for λ in any compact set of $\mathbf{C}$ (the complex field). Then, using Rouche's Theorem we can choose k_0 sufficiently large and λ_2 such that

$$|v|(|u|+\varepsilon/2)/(2k_0\pi+|v|)<\varepsilon/2,\quad |\lambda_2-\lambda_1|<\varepsilon/2,\quad H(\lambda_2, k_0)=0. \tag{2.1}$$

Now letting $r_0=(2k_0\pi+|v|)/|v|$, $\lambda_0=\lambda_2/r_0+iv$, we have

$$\begin{aligned}|A+e^u Be^{-\lambda_0 r_0}-\lambda_0 I|&=|A+e^u Be^{-\lambda_2-i(2k_0\pi/|v|+1)v}-(|v|\lambda_2/(2k_0\pi+|v|)+iv)I|\\&=|A+e^u Be^{-\lambda_2-i\theta}-(|v|\lambda_2/(2k_0\pi+|v|)+iv)I|\\&=H(\lambda_2, k_0)=0,\end{aligned}$$

and

$$\begin{aligned}|\operatorname{Re}\lambda_0(1+r_0)-u|&\leqslant|\operatorname{Re}\lambda_2/r_0|+|\operatorname{Re}(\lambda_2-\lambda_1)|\\&<(|u|+\varepsilon/2)|v|/(2k_0\pi+|v|)+\varepsilon/2<\varepsilon.\end{aligned}$$

The lemma holds for $v\neq 0$. If $v=0$, letting $H(\lambda, r)=|A+e^u Be^{-\lambda}-\lambda/r|$, $r>0$, similarly we can prove that the lemma also holds.

If let $\varepsilon\to 0$ in Lemma 2, we can obtain the following

Coollary 1. *Suppose the hypothesis of Lemma 2 holds. Then for any $u\in R$, we have $w(A, e^u B)=\sup\{\operatorname{Re}\lambda(1+r): |A+e^u Be^{-\lambda r}-\lambda I|=0,\ r\geqslant 0\}\geqslant u$.*

Lemma 3. *Suppose $B\in D_A$, Then $w(A, B)<0$.*

Proof For any $B\in D_A$, by the definition of D_A, we have $w(A, B)\leqslant 0$. So it is only required to prove $w(A, B)\neq 0$. If $w(A, B)=0$, there would be sequences

$$\{\lambda_n=u_n+iv_n\},\ \{r_n\},\ u_n,\ v_n\in R,\ r_n\geqslant 0$$

such that

$$|A+Be^{-(u_n+iv_n)r_n}-(u_n+iv_n)I|=0,\ n=1, 2, \cdots, \tag{2.2}$$

and $u_n(1+r_n)\to 0$ as $n\to\infty$. Obviously $\{\tilde\lambda_n=e^{-iv_n r_n}\}$ is bounded and $u_n\to 0$, $u_n r_n\to 0$, then $\{v_n\}$ is also bounded. Therefore there are subsequences of $\{v_n\}$ and $\{\tilde\lambda_n\}$ which are labeled in the same way and $v\in R$ and $\tilde\lambda$, such that $v_n\to v$, $\tilde\lambda_n\to\tilde\lambda$ as $n\to\infty$. Obviously, $|\tilde\lambda|=1$, so there exists $\theta\in R$ such that $\tilde\lambda=e^{-i\theta}$. In (2.2), letting $n\to\infty$, we obtain

$$|A+Be^{-i\theta}-ivI|=0.$$

Using Corollary 1, for any $t>1$ we have $w(A, tB)=w(A, e^{\ln t}B)\geqslant \ln t>0$. But, on the other hand, since D_A is open and $B\in D_A$, we can choose $t_0>1$ such th at $t_0B\in D_A$. So we have $w(A, t_0B)\leqslant 0$. This is a contradiction and the lemma is proved.

Lemma 4. *Suppose $B\in D_A$. Then the following conclutions hold:*

(i) $|A+B|>0$, $|A-B|>0$, (ii) $f(A, B)>0$ ($f(A, B)$ *is defined as Theorem* 1).

Proof By $B\in D_A$, there are $T>0$, $E>0$ such that $S(A, B, T, E)>0$. This implies

$$-[T(A\pm B)+(A\pm B)'T]>0. \tag{2.3}$$

Appling Liapunov's Theorem, we know that both $A+B$ and $A-B$ are stable. There fore $|A+B|>0$, $|A-B|>0$. (i) holds. Moreover, (2.3) implies $-(TA+A'T)>0$, Therefore, there is a nonsingular matrix P such that

$$-P(TA+A'T)P'=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad P(TB+B'T)P'=\begin{pmatrix}c_1 & 0\\ 0 & c_2\end{pmatrix}, \tag{2.4}$$

and $|c_j|<1$, $j=1, 2$. Using (2.4), we can suppose

$$-2PTAP'=\begin{pmatrix}1 & k\\ -k & 1\end{pmatrix},\quad 2PTBP'=\begin{pmatrix}c_1 & s\\ -s & c_2\end{pmatrix}.$$

Let $T_0=(2PTP')^{-1}=\begin{pmatrix}t_{11} & t_{12}\\ t_{12} & t_{22}\end{pmatrix}$, $A_0=(P^{-1})'AP'$, $B_0=(P^{-1})'BP'$. Obviously

$$t_{11}t_{22}\geqslant |T_0|>0.$$

$$A_0=-T_0\begin{pmatrix}1 & k\\ -k & 1\end{pmatrix},\quad B_0=T_0\begin{pmatrix}c_1 & s\\ -s & c_2\end{pmatrix}. \tag{2.5}$$

Using (2.5), we obtain

$$\begin{gathered}(\operatorname{tr} A_0)^2-(\operatorname{tr} B_0)^2=(t_{11}+t_{22})^2-(t_{11}c_1-t_{22}c_2)^2>2|T_0|(1-c_1c_2),\\ (|A_0+B_0|\cdot|A_0-B_0|)^{\frac{1}{2}}>|T|_0\cdot|k^2-s^2|,\\ |A_0|-|B_0|=|T_0|(1-c_1c_2+k^2-s^2).\end{gathered} \tag{2.6}$$

Clearly $f(A, B)=f(QAQ^{-1}, QBQ^{-1})$ for any nonsingular matrix Q, Therefore by (2.6) we have $f(A, B)=f(A_0, B_0)>0$. (ii) also holds. The lemma is proved.

About the relation between A and the boundary points of D_A, we have

Lemma 5. *Suppose $B\in\partial D_A$ (the set of all the boundary points of D_A). If $|B|>0$, then $w(A, B)=0$. If $|B|\leqslant 0$, then one of the following equalities must be valid:*

(i) $|A+B|=0$, (ii) $|A-B|=0$. (iii) $f(A, B)=0$.

(The proof of this lemma is tedious. We will give the proof in IV).

§ 3. Proof of Theorem 1

Now we will prove Theorem 1 by using the above lemmas and corollary.

First, if there are $T>0$, $E>0$ such that $S(A, B, T, E)>0$, we can deduce that A is stable. Therefore, $B\in D_A$ and Lemmas 3, 4 imply that the conditions in Theorem 1 are necessary. We are going to show that the conditions are also sufficient.

a. Suppose Condition (i) in Theorem 1 holds.

If the theorem were not true, i. e. $B \notin D_A$, considering the set $\{tB;\ 0 \leqslant t \leqslant 1\}$, since $tB \in D_A$ for $t=0$ and $tB \notin D_A$ for $t=1$, there would be $t_0 \in (0, 1]$ such that $t_0 B \in \partial D_A$. By Lemma 5 we have $w(A, t_0 B)=0$. Following the proof of Lemma 3, there exist v, $\theta \in R$ such that $|A+t_0 Be^{-i\theta}-ivI|=0$. Using Corollary 1, we have

$$w(A, B) \geqslant \ln(1/t_0) \geqslant 0.$$

This contradicts the condition (i). Then $B \in D_A$ and there are $T>0$, $E>0$ such that $S(A, B, T, E)>0$.

b. Suppose Condition (ii) in Theorem 1 holds.

We will show that $B \in D_A$ still holds. In fact, if $B \notin D_A$, similarly there would be $t_0 \in (0, 1]$ such that $t_0 B \in \partial D_A$. Obviously, $|t_0 B| = t_0^2 |B| \leqslant 0$, Hence, for A and $t_0 B$, one of Equalities (i), (ii) and (iii) in Lemma 5 must be valid. On the other hahd, if Condition (ii) in Theorem 1 holds, we can easily check tha t $|A+t_0 B|>0$, $|A-t_0 B|>0$ and $f(A, t_0 B)>0$ for any $t_0 \in [0, 1]$. This is a contradiction and the proof is completed.

Remark 1. Theorem 1 and Lemma 4 imply that A is stable and $|B|>0$, then $w(A, B)<0 \Leftrightarrow B \in D_A \Rightarrow |A+B|>0$, $|A-B|>0$, $f(A, B)>0$. But $|A+B|>0$,

$$|A-B|>0,\ f(A, B)>0 \not\Rightarrow w(A, B)<0.$$

For example, letting $A=\begin{pmatrix}-1 & 2\\ -2 & -1\end{pmatrix}$, $B=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$, we have $|A+B|>0$, $|A-B|>0$, $f(A, B)>0$. But $A+B$ is not stable, So $w(A, B)>0$.

Remark 2. If A is stable and $|B|<0$, then $|A+B|>0$, $|A-B|>0$,

$$f(A, B)>0 \Leftrightarrow B \in D_A \Rightarrow w(A, B)<0.$$

But $w(A, B)<0 \not\Rightarrow B \in D_A$. For example, letting $A=\begin{pmatrix}-1 & 2\\ -2 & -1\end{pmatrix}$, $B=\begin{pmatrix}b & 0\\ 0 & -b\end{pmatrix}$ where $1<|b|<2$, we can check that $w(A, B)<0$ (use Theorem 1 in [3], p. 168, and the fact $|A+Be^{-i\theta}-ivI| \neq 0$, θ, $v \in R$). But $f(A, B)<0$, So $B \notin D_A$.

Remark 2 shows that A being stable and $w(A, B)<0$ can not ensure the existence of the positive definite mareices T, E and $S(A, B, T, E)$ if $|B|<0$. This is the difference between $|B|>0$ and $|B|<0$.

Remark 3. If $|B|=0$, then $w(A, B)<0 \Leftrightarrow A$ is stable, $|A+B|>0$, $|A-B|>0$, $f(A, B)>0 \Leftrightarrow B \in D_A$.

§ 4. Proof of Lemma 5

Here we will prove Lemma 5. For convenience of the proof, we denote $T \geqslant 0$ if T is symmetric and positive se mi-definite.

Lemma 6. *If $B\in D_A$, then for any nonsingular matrix $P\in R^{2\times 2}$, $P^{-1}BP\in D_{P^{-1}AP}$.*

Lemma 7. *Suppose $B\in\partial D_A$. Then there are $T_0\geqslant 0$, $T_0\neq 0$, $E_0\geqslant 0$, such that $S(A, B, T_0, E_0)\geqslant 0$, but $S(A, B, T_0, E_0)$ is not positive definite.*

Lemma 8. *Suppose $B\in\partial D_A$ and T_0, E_0 are defined as Lemma 7. Then* rank $(S(A, B, T_0, E_0))<3$.

(The proof of this lemma is similar to the following proof of Lemma 10).

Obviously, if $B\in\partial D_A$ and T_0, E_0 are defined as Lemma 7, then one of the following hypotheses holds:

(i) $|T_0|=0$. (ii) $|T_0|\neq 0$ (i. e. $T_0>0$), $|B|\neq 0$. (iii) $|B|=0$.

Now we will discuss the relations between B and A for (i), (ii), (iii) respectively.

Lemma 9. *Suppose $B\in\partial D_A$ and* (i) *holds. Then one of the equalities $|A+B|=0$, $|A-B|=0$ must be valid.*

Proof First, using Lemma 4 we can deduce $|A+B|\geqslant 0$, $|A-B|\geqslant 0$. By $T_0\geqslant 0$, $T_0\neq 0$, $|T_0|=0$, there is a nonsingular matrix P such that $PT_0P'=\begin{pmatrix}1&0\\0&0\end{pmatrix}$. In addition, $S(A, B, T_0, E_0)\geqslant 0$ implies $-(T_0A+A'T_0)\geqslant 0$ and

$$-[T_0(A\pm B)+(A\pm B)'T_0]\geqslant 0.$$

Letting

$$A_0=(P^{-1})'AP'=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{21}\end{pmatrix},\quad B_0=(P^{-1})'BP'=\begin{pmatrix}b_{11}&b_{21}\\b_{12}&b_{22}\end{pmatrix},$$

we have

$$\begin{pmatrix}-2a_{11}\pm 2b_{11}&-a_{11}\pm b_{12}\\-a_{12}\pm b_{12}&0\end{pmatrix}=-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}(A_0\mp B_0)+(A_0\mp B_0)'\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]$$
$$=-P[T_0(A\mp B)+(A\mp B)'T_0]P'\geqslant 0.$$

Hence $a_{12}=b_{12}=0$, $|a_{11}|\geqslant|b_{11}|$, i. e. $A_0=\begin{pmatrix}a_{11}&0\\a_{21}&a_{22}\end{pmatrix}$, $B_0=\begin{pmatrix}b_{11}&0\\b_{21}&b_{22}\end{pmatrix}$. Moreover $|A+B|\geqslant 0$, $|A-B|\geqslant 0$ imply $|a_{22}|\geqslant|b_{22}|$. We assert that one of $|a_{11}|=|b_{11}|$, $|a_{22}|=|b_{22}|$ will hold. Suppose (for contradiction) that $|a_{11}|>|b_{11}|$, $|a_{22}|>|b_{22}|$. Since A_0 is stable, we have $a_{11}<0$, $a_{22}<0$. For $t>0$, letting

$$T(t)=\begin{pmatrix}t&0\\0&1\end{pmatrix},\quad E(t)=\begin{pmatrix}-ta_{11}&0\\0&a_{22}\end{pmatrix},$$

we can easily check that there would be a sufficiently large $t_0>0$ such that

$$S(A_0, B_0, T(t_0), E(t_0))>0.$$

Thus $B_0\in D_{A_0}$ and Lemma 6 implies $B\in D_A$. This contradicts the hypothesis of $B\in\partial D_A$. So $|a_{11}|=|b_{11}|$ or $|a_{22}|=|b_{22}|$. The lemma is proved.

Lemma 10. *Suppose $B\in\partial D_A$ and* (ii) *holds. Then $w(A, B)=0$ if $|B|>0$, and one of the following equalities must be valid if $|B|<0$:*

(1) $|A+B|=0$, (2) $|A-B|=0$, (3) $f(A, B)=0$.

We will prove this lemma with the following several steps:

Prove (a) $\operatorname{rank}(S(A, B, T_0, E_0))=2$ and $-(T_0A+A'T_0)-E_0>0$, $E_0>0$.

Obviously $|T_0B|\neq 0$, so $\operatorname{rank}(S(A, B, T_0, E_0))\geqslant 2$. Using Lemma 8 we have $\operatorname{rank}(S(A, B, T_0, E_0))=2$. In addition, $|T_0B|\neq 0$ and $S(A, B, T_0, E_0)\geqslant 0$ imply $-(T_0A+A'T_0)-E_0>0$, $E_0>0$. (a) holds.

(b) There is a nonsingular matrix $P\in R^{2\times 2}$ such that

$$\begin{pmatrix} I & 0 \\ I & P \end{pmatrix} S(A, B, T_0, E_0)\begin{pmatrix} I & I \\ 0 & P' \end{pmatrix}=\begin{pmatrix} -(T_0A+A'T_0)-E_0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad (4.1)$$

i. e.

$$-(T_0A+A'T_0)-E_0-T_0BP'=0,\ T_0BP'=PE_0P'.$$

Since $\operatorname{rank}(S(A, B, T_0, E_0))=2$, there are $P_{11}, P_{12}, P_{21}, P_{22}\in R^{2\times 2}$ such that

$$S(A, B, T_0, E_0)=\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P'_{11} & P'_{21} \\ P'_{12} & P'_{22} \end{pmatrix}=\begin{pmatrix} P_{11}P'_{11} & P_{11}P'_{21} \\ P_{21}P'_{11} & P_{21}P'_{21} \end{pmatrix}.$$

Then $P_{11}P'_{11}=-(T_0A+A'T_0)-E_0$, $P_{21}P'_{21}=E_0$. Using (a) we know both P_{11} and P_{21} are nonsingular. Put $P=-P_{11}P_{21}^{-1}$. P is also nonsingular and

$$\begin{pmatrix} I & 0 \\ I & P \end{pmatrix} S(A, B, T_0, E_0)\begin{pmatrix} I & I \\ 0 & P' \end{pmatrix}=\begin{pmatrix} I & 0 \\ I & P \end{pmatrix}\begin{pmatrix} P_{11}P'_{11} & P_{11}P'_{21} \\ P_{21}P'_{11} & P_{21}P'_{21} \end{pmatrix}\begin{pmatrix} I & I \\ 0 & P' \end{pmatrix}$$

$$=\begin{pmatrix} P_{11}P'_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

(b) also holds.

(c) For any symmetric matrices $T, E\in R^{2\times 2}$, the matrix

$$D(T, E)=-[T(A+BP')+(A+BP')'T]-E+PEP'$$

is not positive definite.

In fact, if there are symmetric matrices T^*, E^* such that $D(T^*, E^*)>0$, then Letting $D_0=-(T^*A+A'T^*)-E^*$, $H=-(T^*A+A'T^*)-E-T^*BP'$, for any t, we have

$$\begin{pmatrix} I & 0 \\ I & P \end{pmatrix} S(A, B, tT_0+T^*, tE_0+E^*)\begin{pmatrix} I & I \\ 0 & P' \end{pmatrix}$$
$$=\begin{pmatrix} t[-(T_0A+A'T_0)-E_0]+D_0 & H' \\ H' & D(T^*, E^*) \end{pmatrix}. \qquad (4.2)$$

Since $-(T_0A+AT_0)-E_0>0$, $D(T^*, E^*)>0$, we can choose $t=t_0>0$ sufficiently large such that the matrix in (4.2) is positive definite. Therefore, for

$$T_1=t_0T_0+T^*,\ E_1=t_0E_0+E^*,$$

we have $S(A, B, T_1, E_1)>0$. This implies $B\in D_A$, contradicting the hypothesis of $B\in\partial D_A$. (c) is proved.

(d) There is a nonsingular matrix $QR^{2\times 2}$ such that

$$Q(A+BP')Q^{-1}=\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix},\quad v\in R, \text{ and } X=(Q^{-1})'PQ'$$

is orthogonal; or

$$Q(A+BP')Q^{-1}=\begin{pmatrix}0 & 0\\ 0 & c\end{pmatrix}\left[\text{or}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right],\quad c\in R,\ c\neq 0,$$

and

$$X=(Q^{-1})'PQ'=\begin{pmatrix}x_{11} & 0\\ x_{21} & x_{22}\end{pmatrix},\quad |x_{11}|=1.$$

First, Using (c), we know $D(T, E)$ is not positive definite for any symmetric T, E. Particularly, if we put $E=0$, then Liapunov's Theorem implies that Jordan Standard Form of $A+BP'$ must be one of the following

$$J_1=\begin{pmatrix}iv & 0\\ 0 & -iv\end{pmatrix},\quad J_2=\begin{pmatrix}0 & 0\\ 0 & c\end{pmatrix},\quad J_3=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

where $v, c\in R$, $c\neq 0$.

(d. 1) Suppose the standard form of $A+BP'$ is J_1, and $v=0$. Obviously it must be $A+BP'=0$. Therefore, using (4.1) we have $E_0=PE_0P'$. Notice that $E_0>0$. There is a nonsingular Q such that $Q'Q=E_0$. Let $X=(Q^{-1})'PQ'$. We have

$$Q(A+BP')Q^{-1}=0$$

and

$$XX'=(Q^{-1})'PE_0P'Q^{-1}=(Q^{-1})'E_0Q^{-1}=I.$$

So X is orthogonal and conclusion (d) holds.

(d. 2) Suppose the standard form of $A+BP'$ is J_1 and $v\neq 0$, i. e., there is a nonsingular matrix Q such that $Q(A+BP')Q^{-1}=\begin{pmatrix}0 & v\\ -v & 0\end{pmatrix}$. To prove that

$$X=(Q^{-1})'PQ'$$

is orthogonal, put

$$XX'=\begin{pmatrix}1+s_1 & k\\ k & 1+s_2\end{pmatrix}.$$

For any symmetric matrices T, E, we have

$$H(T, E)=-\left[T\begin{pmatrix}0 & v\\ -v & 0\end{pmatrix}+\begin{pmatrix}0 & -v\\ v & 0\end{pmatrix}T\right]-E+XEX'$$
$$=(Q^{-1})'D(Q'TQ, Q'EQ)Q^{-1}.$$

Since $D(Q'TQ, Q'EQ)$ is not positive definite for any T, E (by conclusion(c)), so is $H(T, E)$. Notice that $H(-T, -E)=-H(T, E)$ for any $T, E\in R^{2\times 2}$ and

$$H(T, E)\in R^{2\times 2}.$$

Therefore $|H(T, E)|\leqslant 0$ for any symmetric matrices T, E. Particularly if we let $E=I$, $T=\begin{pmatrix}k/v & t_{12}\\ t_{12} & 0\end{pmatrix}$, then

$$|H(T, E)|=\begin{vmatrix}s_1-t_{12}v & 0\\ 0 & s_2+t_{12}v\end{vmatrix}=(s_1-t_{12}v)(s_2+t_{12}v)\leqslant 0$$

for any t_{12}, This implies $s_1=-s_2$. Therefore $|XX'|=1-s_1^2-k^2\leqslant 1$. Similarly, if we put $E=X^{-1}(X^{-1})'$ and suitably choose T, we can deduce $|X^{-1}(X^{-1})'|\leqslant 1$. Hence we have $|XX'|=1$ and $s_1=s_2=k=0$. Thus $XX=I$ and X is orthogonal. (d) also holds.

(d. 3) Suppose the standard form of $A+BP'$ is J_2 (or J_3). Then there is a nonsingular matrix Q such that $Q(A+BP')Q^{-1}=J_2$(or J_3). Using the above method we can prove that $(Q^{-1})'\,PQ'=X$ must have the form

$$X=\begin{pmatrix} x_{11} & 0 \\ x_{21} & x_{22} \end{pmatrix}, \quad |x_{11}|=1.$$

(d) is proved.

Now, using (d) we give the proof of Lemma 10. By (d), we can suppose:

(1) There is a nonsingular matrix Q, such that

$$Q(A+BP')Q^{-1}=\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}, \quad \text{and} \quad X=(Q^{-1})'\,PQ^{-1} \text{ is orthogonal},$$

where P is defined as (b). Let $A_0=QAQ^{-1}$, $B_0=QBQ^{-1}$, then $A_0+B_0X'=\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}$.

By (4.1), we have $|B|\cdot|P|>0$. If $|B|>0$, then $|P|>0$ and so $|X|>0$. Therefore, there is $\theta\in R$ such that $X=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. We have

$$|A+Be^{-i\theta}-ivI|=|A_0+B_0e^{-i\theta}-ivI|=\left|-B_0X'+\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}+B_0e^{-i\theta}-ivI\right|$$

$$=\left|\sin\theta\, B_0\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}+v\begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix}\right|$$

$$=|\sin\theta\, B_0+vI|\cdot\begin{vmatrix} -i & 1 \\ -1 & -i \end{vmatrix}=0.$$

Using Corollary 1, we have $w(A,\ B)\geqslant 0$. In addition, the fact $w(A,\ \bar{B})<0$ for $\bar{B}\in D_A$(Lemma 3) implies $w(A,\ B)\leqslant 0$ for $B\in\partial D_A$. Then we have $w(A,\ B)=0$. The conclusion of Lemma 10 holds for $|B|>0$. If $|B|<0$, then $|X|<0$. Since X is orthogonal, there must be orthogonal matrix X_1 such that $X_1XX_1=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $A_1=X_1A_0X_1'=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B_1=X_1B_0X_1'=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, we have $A_1+B_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}=X_1\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}X_1'=\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}\left[\text{or}\begin{pmatrix} 0 & -v \\ v & 0 \end{pmatrix}\right]$. Therefore $a_{11}=-b_{11}$, $a_{22}=b_{22}$ and

$$(\text{tr}\,A_1)^2-(\text{tr}\,B_1)^2=(b_{11}-b_{22})^2-(b_{11}+b_{22})^2=-4b_{11}b_{22},$$

$$|A_1+B_1|\cdot|A_1-B_1|=\left|B_1\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}+\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}\right|\cdot\left|B\begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}\right| \tag{4.3}$$

$$=v^2(v+2b_{12})(v+2b_{21}),$$

$$|A_1|=\left|-B_1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\begin{pmatrix}0 & v\\ -v & 0\end{pmatrix}\right|=-|B_1|+v(b_{12}+b_{21})+v^2.$$

Using Lemma 4 and the hypothesis of $B\in\partial D_A$, we have

$$|A_1+B_1|\cdot|A_1-B_1|=|A_0+B_0|\cdot|A_0-B_0|=|A+B|\cdot|A-B|\geqslant 0.$$

This implies $(v+2b_{12})(v+2b_{21})\geqslant 0$. Therefor, appling (4.3) we obtain

$$\begin{aligned}f(A,\ B)&=f(A_1,\ B_1)\\ &=-4b_{11}b_{22}+2\{|v|[(v+2b_{12})(v+2b_{21})]^{\frac{1}{2}}-v(b_{12}+b_{21})-v^2+2|B_1|\}\\ &=-\{|v|-[(v+2b_{12})(v+2b_{21})]^{\frac{1}{2}}\}^2\leqslant 0.\end{aligned}$$

$B\in\partial D_A$ implies $f(A,\ B)\geqslant 0$. Thus, $f(A,\ B)=0$. The assertion of Lemma 10 holds.

(2) There are nonsingular matrices Q and X such that

$$A_0+B_0X'=\begin{pmatrix}0 & C_{12}\\ 0 & C_{22}\end{pmatrix},\quad X=\begin{pmatrix}x_{11} & 0\\ x_{21} & x_{22}\end{pmatrix},\quad |x_{11}|=1,$$

where $A_0=QAQ^{-1}$, $B_0=QBQ^{-1}$, $X=(Q^{-1})'PQ^{-1}$.

Obviously, if $x_{11}=1$, then $|A+B|=|A_0+B_0|=0$; if $x_{11}=-1$, then $|A-B|=0$. we have $w(A,\ B)\geqslant 0$. Noticing $B\in\partial D_A$, we have $w(A,\ B)=0$. Lemma 10 is proved.

Using Lemmas 7, 8, 9, 10, we can obtain

Corollary 2. *Suppose $B\in\partial D_A$, $|B|\neq 0$. Then conclutions of Lemma 5 hold.*

Corollary 3. *Suppose $B\in\partial D_A$, $|B|=0$, Then one of the equalities $|A+B|=0$, $|A-B|=0$, and $f(A,\ B)=0$ must holds.*

We will not give the proof Corollary 3 and only point out such a fact. If $|A+B|>0$, $|A-B|>0$ and $f(A,\ B)>0$, then we can choose $t>1$ such that $|A+tB|>0$, $|A-tB|>0$ and $f(A,\ tB)>0$. For $B_0=tB$, obviously theɹe is a sequence $\{B_n\}$, $|B_n|<0$ such that $|A+B_n|>0$, $|A-B_n|>0$, $f(A,\ B_n)>0$ and $B_n\to B_0$ as $n\to\infty$. Using Corollary 2, we have $\{B_n\}\subset D_A$. So there must be $T_0\geqslant 0$, $T_0\neq 0$, $E_0\geqslant 0$, such that $S(A,\ B_0,\ T_0,\ E_0)\geqslant 0$. Repeating the method used in the proof of the above lemmas, we can prove that $hB_0\in D_A$ for any $|h|<1$. Then $B=1/t$, $B_0\in D_A$ since $|1/t|<1$. Appling this fact and Lemma 4, we obtain Corollary 3.

Obviously, Lemma 5 is the combination of Corollary 2 and Corollary 3.

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