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THE DECISION PROBLEMS ABOUT THE PERIODIC SOLUTIONS OF THE DOMINO PROBLEMS

LIN YUCAI (林毓材)
(Yunnan Normal University)

Abstract

In this paper, by reducing the Post Corresponding Problem to (1) the problem of deciding whether or not the origin-constrained domino problem has periodic solutions and (2) the problem of deciding whether or not the unrestricted domino problem has periodic solutions, it is obtained that the above two decision problems are both unsolvable.

The domino problems were raised for the purpose of studying the AEA case of the first-order predicate sentences. The unsolvabilities of the origin-constrained domino problem, the diagonal-constrained domino problem, and the unrestricted domino problem were shown in 1960—1966, and along with this the unsolvabilities of the cases E&AEA and AEA were obtained[1,2,3].

Now we shift to the discussion on decision problems about the periodic solutions of the domino problems. By reductions from the Post Corresponding Problem to

(1) the problem of deciding whether or not the origin-constrained domino problem has any periodic solutions, and

(2) the problem of deciding whether or not the unrestricted domino problem has any periodic solutions,

we have proved that the two decision problems are both unsolvable.

A domino D is a quadruple $D=(a, b, c, d)$ with $a, b, c, d\in\mathscr{C}$, a infinite set of colors. In other words, a domino $D=(a, b, c, d)$ is a square with its edges colored by colors a, b, c and d as its bottom-color, left-color, top-color and right-color respectively. A set $P=\{D_1, \cdots, D_m\}$ of distinct dominoes is called a set of domino types.

For any given set P of domino types, if there exists a mapping A: $N\times N\to P$ from the set $N\times N$ of all ordered pairs of nonnegative integers into P such that, for any $\langle x, y\rangle\in N\times N$,

$$a(A(\langle x, y+1\rangle))=c(A(\langle x, y\rangle)),$$

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$$b(A(\langle x+1,\ y\rangle))=d(A(\langle x,\ y\rangle)),$$

where $a(D)$, $b(D)$, $c(D)$ and $d(D)$ are the first, second, third and fourth members of D respectively, then A is called a *solution* of P, and P is called *solvable*. In other words, for any given set P of domino types, if there exists a covering of the whole first quatrant of the infinite plane with dominoes of these types in P such that all corners fall on the lattice points and any two adjoining edges have the same color, then the covering is called a solution of P and P is called solvable.

The problem of deciding, for any given set P of domino types, whether or not P has a solution is called the *unrestricted domino problem*.

The problem of deciding, for any given set P of domino types and a domino type $C\in P$, whether or not P has a solution with origin occupied by a domino of type C is called the *origin-constrained domino problem*.

For any given set P of domino types, if for some nonnegative integers m, n, there exists a mapping B: $\{0,\ 1,\ \cdots,\ m\}\times\{0,\ 1,\ \cdots,\ n\}\to P$ such that

$$a(B(\langle x,\ y+1\rangle))=c(B(\langle x,\ y\rangle)),\quad 0\leqslant x\leqslant m,\ 0\leqslant y<n,$$
$$b(B(\langle x+1,\ y\rangle))=d(B(\langle x,\ y\rangle)),\quad 0\leqslant x<m,\ 0\leqslant y\leqslant n,$$

and

$$a(B(\langle 0,\ 0\rangle))=a(B(\langle 1,\ 0\rangle))=\cdots=a(B(\langle m,\ 0\rangle))=c(B(\langle 0,\ n\rangle))$$
$$=c(B(\langle 1,\ n\rangle))=\cdots=c(B(\langle m,\ n\rangle)),$$
$$b(B(\langle 0,\ 0\rangle))=b(B(\langle 0,\ 1\rangle))=\cdots=b(B(\langle 0,\ n\rangle))=d(B(\langle m,\ 0\rangle))$$
$$=d(B(\langle m,\ 1\rangle))=\cdots=d(B(\langle m,\ n\rangle)),$$

then B is called a *ring surface* of P, and say that P has a *periodic solution*. In other words, for any given set P of domino types, if there exists a rectangle consisting of $(m+1)\times(n+1)$ dominoes of these types in P such that two adjoining edges have the same color, and the bottom-colors of the dominoes in the bottom-string of dominoes and the top-colors of the dominoes in the top-string of dominoes are all the same, and the left-colors of the dominoes in the left-string of dominoes and the right-colors of the dominoes in the right-string of dominoes are all the same, then the rectangle is called a ring surface of P, and say that P has a periodic solution.

Theorem 1. *The problem of deciding whether or not the origin-constrained domino problem has any periodic solutions is unsolvable.*

Proof Let $S=\{\alpha_1,\ \cdots,\ \alpha_n;\ \beta_1,\ \cdots,\ \beta_n\}$(where α_i and β_i are all words on $\{0,\ 1\}$, and $|\alpha_i|\geqslant 2$ and $|\beta_i|\geqslant 2$, $i=1,\ \cdots,\ n$)be any given Post Corresponding System. Now it is clear that, for any two strings

$$U=\alpha_{i_1}\cdots\alpha_{i_u}=x_{i_11}\cdots x_{i_1r_{i_1}}\cdots x_{i_u1}\cdots x_{i_ur_{i_u}}$$

and

$$V=\beta_{j_1}\cdots\beta_{j_v}=y_{j_11}\cdots y_{j_1s_{j_1}}\cdots y_{j_v1}\cdots y_{j_vs_{j_v}}$$

of α-words and β-words respectively, U and V are corresponding iff the sequences

$i_1, \cdots, i_u$ and $j_1, \cdots, j_v$ are the same; and $U=V$ iff the sequences $x_{i_11}\cdots x_{i_1r_{i_1}}\cdots x_{i_u1}\cdots x_{i_ur_{i_u}}$ and $y_{j_11}\cdots y_{j_1s_{j_1}}\cdots y_{j_v}\cdots y_{j_vs_{j_v}}$ are the same. Hence, for purpose to make sure that whether U and V are corresponding and equal, we may characterize U and V by

$$U=\overset{i_1}{x_{i_11}}x_{i_12}\cdots x_{i_1r_{i_1}}\cdots \overset{i_u}{x_{i_u1}}x_{i_u2}\cdots x_{i_ur_{i_u}}$$

and

$$V=\overset{j_1}{y_{j_11}}y_{j_12}\cdots y_{j_1s_{j_1}}\cdots \overset{j_v}{y_{j_v1}}y_{j_v2}\cdots y_{j_vs_{j_v}}$$

of the strings of 0's and 1's with indexes. With this representations it is clear that U and V are corresponding and equal iff V can be obtained from U by a finite number of applications of the following operation R on U: remove some indexes of $i_2, \cdots, i_u$ a bit left, remove some other indexes of $i_2, \cdots, i_u$ a bit right, and keep the remaining indexes fixed in any way with no two indexes in the same position and keeping the original order of indexes.

With this in mind we choose a set P_S of domino types which consists of the following domino types:

1. $LB=(\bigstar, \bigstar, *, +)$, $L=(*, \bigstar, *, \pm)$, $LT=(*, \bigstar, \bigstar, -)$,
   $RB=(\bigstar, +, \div, \bigstar)$, $R=(\div, \cdot, \div, \bigstar)$, $RT=(\div, =, \bigstar, \bigstar)$;
2. $0=(0, \cdot, 0, \cdot)$, $1=(1, \cdot, 1, \cdot)$;
3. the following s_j+1 domino types for each $\beta_j=y_{j1}\cdots y_{jl}\cdots y_{js_j}$:

$$B_{j1}=((j, y_{j1}), -, \bigstar, (b, j, 1)),$$
$$B_{jl}=(y_{jl}, (b, j, l-1), \bigstar, (b, j, l)),\ l=2, \cdots, s_j-1,$$
$$B_{js_j}=(y_{js_j}, (b, j, s_j, -1), \bigstar, -),$$
$$B'_{js_j}=(y_{js_j}, (b, j, s_j-1), \bigstar, =);$$

4. the following r_i+1 domino types for each $\alpha_i=x_{i1}\cdots x_{ik}\cdots x_{ir_i}$:

$$A_{i1}=(\bigstar, +, (i, x_{i1}), (a, i, 1)),$$
$$A_{ik}=(\bigstar, (a, i, k-1), x_{ik}, (a, i, k)),\ k=2, \cdots, r_i-1,$$
$$A_{ir_i}=(\bigstar, (a, i, r_i-1), x_{ir_i}, +),$$
$$ix_{i1}=((i, x_{i1}), \pm, (i, x_{i1}), \cdot),$$

5. the following 10 domino types for each $i=1, \cdots, n$:

$$\overline{i0}=((i, 0), \cdot, (i, 0), \cdot), \quad \overline{i1}=((i, 1), \cdot, (i, 1), \cdot),$$
$$\overleftarrow{i0}=(0, \cdot, (i, 0), (i, L)), \quad \overleftarrow{i0R}=((i, 0), (i, L), 0, \cdot),$$
$$\overrightarrow{i0L}=((i, 0), \cdot, 0, (i, R)), \quad \overrightarrow{i0}=(0, (i, R), (i, 0), \cdot),$$
$$\overleftarrow{i1}=(1, \cdot, (i, 1), (i, L)), \quad \overleftarrow{i1R}=((i, 1), (i, L), 1, \cdot),$$
$$\overrightarrow{i1L}=((i, 1), \cdot, 1, (i, R)), \quad \overrightarrow{i1}=(1, (i, R), (i, 1), \cdot).$$

By our choice of P_S we have that

1. For any given string $\alpha_{i_1}\cdots\alpha_{i_u}=x_{i_11}\cdots x_{i_1r_{i_1}}\cdots x_{i_u1}\cdots x_{i_ur_{i_u}}$ of α-words of S, we can obtain a string

$$LB,\ A_{i_1 1},\ \cdots,\ A_{i_1 r_{i_1}},\ \cdots,\ A_{i_u 1},\ \cdots,\ A_{i_u r_{i_u}},\ RB \tag{1}$$

of dominoes of types in P_S beginning with LB, ending with RB, and having neither LB nor RB in the middle by changing every $x_{i_p q_p}(p=1, \cdots, u;\ q_p=1, \cdots, r_{i_p})$ for a domino $A_{i_p q_p}$, and then putting a domino LB on the left and a domino RB on the right. And, conversely, if a string of dominoes of types in P_S begins with LB, ends with RB and has neither LB nor RB in the middle, it must have (1) as its form and can be obtained from a string $\alpha_{i_1}\cdots\alpha_{i_u}$ of α-words of S by the above manner.

2. Similarly, for any given nonempty string $\beta_{j_1}\cdots\beta_{j_v}=y_{j_1 1}\cdots y_{j_1 s_{j_1}}\cdots y_{j_v 1}\cdots y_{j_v s_{j_v}}$ of β-words of S, we can obtain a string

$$LT,\ B_{j_1 1},\ \cdots,\ B_{j_1 s_{j_1}},\ \cdots,\ B_{j_v 1},\ \cdots,\ B_{j_v s_{j_v}-1},\ B'_{j_v s_{j_v}},\ RT \tag{2}$$

of dominoes of types in P_S beginning with LT, ending with RT, and having neither LT nor RT in the middle by changing every $y_{j_p q_p}(p=1, \cdots, v;\ q_p=1, \cdots, s_{j_p}(p\neq v),\ q_v=1, \cdots, s_{j_v}-1)$ for a domino $B_{j_p q_p}$, $y_{j_v s_{j_v}}$ for the domino $B'_{j_v s_{j_v}}$, and then putting a domino LT on the left and a domino RT on the right. And, conversely, if a string of dominoes of types in P_S begins with LT, ends with RT, and has neither LT nor RT in its middle, it must have (2) as its form and can be obtained from a nonempty string $\beta_{j_1}\cdots\beta_{j_v}$ of β-words of S by the above manner. (Notice that the right-color of LT is not the same as the left-color of RT. Hence the string $\beta_{j_1}\cdots\beta_{j_v}$ is not empty.)

3. The string (1) of dominoes has ★ as its left-color and right-color; ★, ★, $\cdots$, ★ as its bottom-color and

$$*,\ i_1 x_{i_1 1},\ x_{i_1 2},\ \cdots,\ x_{i_1 r_{i_1}},\ \cdots,\ i_u x_{i_u 1},\ x_{i_u 2},\ \cdots,\ x_{i_u r_{i_u}},\ \div \tag{3}$$

as its top-color. And this top-color (3) characterizes the string $\alpha_{i_1}\cdots\alpha_{i_u}$ of α-words of S.

4. The string (2) of dominoes has ★ as its left-color and right-color; ★, ★, $\cdots$, ★ as its top-color, and

$$*,\ j_1 y_{j_1 1},\ y_{j_1 2},\ \cdots,\ y_{j_1 s_{j_1}},\ \cdots,\ j_v y_{j_v 1},\ y_{j_v 2},\ \cdots,\ y_{j_v s_{j_v}},\ \div \tag{4}$$

as its bottom-color. And this bottom-color characterizes the string $\beta_{j_1}\cdots\beta_{j_v}$ of β-words of S.

5. Both the top-color (3) and the bottom-color (4) have the form

$$*,\ k_1 z_{k_1 1},\ z_{k_1 2},\ \cdots,\ z_{k_1 t_{k_1}},\ \cdots,\ k_w z_{k_w 1},\ z_{k_w 2},\ \cdots,\ z_{k_w t_{k_w}},\ \div \tag{5}$$

where the k's are all in $\{1, 2, \cdots, n\}$ and the z's are all in $\{0, 1\}$.

6. Rewrite (5) as

$$*,\ \overset{k_1}{z_{k_1 1}},\ z_{k_1 2},\ \cdots,\ z_{k_1 t_{k_1}},\ \cdots,\ \overset{k_w}{z_{k_w 1}},\ z_{k_w 2},\ \cdots,\ z_{k_w t_{k_w}},\ \div. \tag{6}$$

Then a string of dominoes of types in P_S can be put upon a string of dominoes which has (6) as its top-color iff it has ★ as its left-color and right-color, and its top-color may be obtained from (6) by an application of operation R, or it has (2) as its form.

Now it is not difficult to see that S has a solution iff P_S has a ring surface with

★ as the color of its periphery, i. e., iff P_S has a origin-constrained periodic solution with the origin occupied by LB. Thus our theorem follows from the unsolvability of Post Corresponding Problem.

Theorem 2. *The problem of deciding whether or not the unrestricted domino problem has any periodic solutions is unsolvable.*

Proof For any given Post Corresponding System $S=\{\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_n\}$ (where α_i and β_i are all words on $\{0, 1\}$, and $|\alpha_i|\geqslant 2$ and $|\beta_i|\geqslant 2$, $i=1, \cdots, n$), we choose a set P'_S of domino types which consists of the following domino types (where $M=\max(|\alpha_1|, \cdots, |\alpha_n|, |\beta_1|, \cdots, |\beta_n|)-1$):

1. $LB=($✦, ★, $*, +)$, $L=(*,$ ✱, $*, \pm)$,

   $LT=(*,$ ★, ★, $-)$, $RB=($✫, $+, \div,$ ✫$)$,

   $R_u=(\div, u, \div,$ ✱$)$, $u=1, \cdots, M$, $RT=(\div, =,$ ✫, ✱$)$;

2. $0_u=(0, u-1, 0, u)$, $1_u=(1, u-1, 1, u)$, $u=1, \cdots, M$;

3. the following s_j+1 domino types for each $\beta_j=y_{j1}\cdots y_{jl}\cdots y_{js_j}$:

   $B_{j1}=((j, y_{j1}), -,$ ★, $(b, j, 1))$,

   $B_{jl}=(y_{jl}, (b, j, l-1),$ ★, $(b, j, l))$, $l=2, \cdots, s_j-1$,

   $B_{js_j}=(y_{js_j}, (b, j, s_j-1),$ ✫, $-)$,

   $B'_{js_j}=(y_{js_j}, (b, j, s_j-1),$ ✫, $=)$;

4. the following r_i+1 domino types for each $\alpha_i=x_{i1}\cdots x_{ik}\cdots x_{ir_i}$:

   $A_{i1}=($✫, $+, (i, x_{i1}), (a, i, 1))$,

   $A_{ik}=($★, $(a, i, k-1), x_{ik}, (a, i, k))$, $k=2, \cdots, r_i-1$,

   $A_{ir_i}=($✱, $(a, i, r_i-1), x_{ir_i}, +)$,

   $ix_{i1}=((i, x_{i1}), \pm, (i, x_{i1}), 0)$,

5. the following $6M+4$ domino types for each $i=1, \cdots, n$; $u=1, \cdots, M$

   $\overleftarrow{(i0)}_u=((i, 0), u, (i, 0), 0))$, $\qquad \overleftarrow{(i1)}_u=((i, 1), u, (i, 1), 0)$,

   $\overleftarrow{(i0)}_u=(0, u, (i, 0), (i, L))$ $\qquad i0R=((i, 0), (i, L), 0, 0)$,

   $\overrightarrow{(i0L)}_u=((i, 0), u, 0, (i, R))$, $\qquad \overrightarrow{i0}=(0, (i, R), (i, 0), 0)$,

   $\overleftarrow{(i1)}_u=(1, u, (i, 1), (i, L))$, $\qquad i1R=((i, 1), (i, L), 1, 0)$,

   $\overrightarrow{(i1L)}_u=((i, 1), u, 1, (i, R))$, $\qquad \overrightarrow{i1}=(1, (i, R), (i, 1), 0)$.

It is easy to see that P'_S may be obtained from P_S by the following changes of domino types with color∘.

1) Change every domino type with ∘ only as its right-color for a domino type obtained by changing ∘ for color 0.

2) Change everydomino type with ∘ only as its left-color for M domino types obtained by changing . for colors $1, 2, \cdots, M$ respectively.

3) There are $2n+2$ domino types with∘ as their left-color and right-color. They are $0, 1, \overline{i0}, \overline{i1} (i=1, \cdots, n)$. Change each one of domino types 0 and 1 for M domino types obtained by changing the left-color∘ and right-color∘ for colors 0 and 1, 1 and

2, ⋯, $M-1$ and M respectively. Change each one of the domino types $\overline{i0}$ and $\overline{i1}$ for M domino types obtaind by changing the right-color for color 0 and the left-color for colors 1, 2, ⋯, M respectively.

Since S has a solution $\alpha_{i_1}\cdots\alpha_{i_u}=\beta_{j_1}\cdots\beta_{j_v}$ iff

$$V=\beta_{j_1}\cdots\beta_{j_v}=\overbrace{y_{j_11}y_{j_12}\cdots y_{j_1s_{j_1}}}^{j_1}\cdots\overbrace{y_{j_v1}y_{j_v2}\cdots y_{j_vs_{j_v}}}^{j_v}$$

may be obtained from

$$U=\alpha_{i_1}\cdots\alpha_{i_u}=\overbrace{x_{i_11}x_{i_12}\cdots x_{i_1r_{i_1}}}^{i_1}\cdots\overbrace{x_{i_u1}x_{i_u2}\cdots x_{i_ur_{i_u}}}^{i_u}$$

by a finite number of applications of operation R on U such that in the result of each application of R, for any two indexes i_k and i_{k+1} ($k=1$, ⋯, $u-1$), there are at most M 0's and 1's between i_k and i_{k+1}. Hence P_S has a ring surface with ★ as the color of its periphery iff P'_S has a ring surface with ★ as the color of its periphery.

By exhaustion, it is not difficult to show that if P'_S has a ring surface, then the ring surface must has ★ as the color of its periphery. Thus, by Theorem 1, S has a solution iff P'_S has a ring surface, i. e., iff P'_S has an unrestricted periodic solution. Therefore our theorem follows from the unsolvability of the Post Corresponding Problem.

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