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THE SECTIONS OF ODD UNIVALENT FUNCTIONS

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Abstract

Let S_k be the class of functions $f(z)=z+\sum_{n=1}^{\infty} b_{nk+1}^{(k)} z^{nk+1}$ which are regular and univalent in $|z|<1$ and denote $S_n^{(k)}(z)=z+\sum_{m=1}^{n} b_{mk+1}^{(k)} z^{mk+1}$.

The authers prove that the functions $S_n^{(2)}(z)$ are starlike in $|z|<\frac{1}{\sqrt{3}}$.

§ 1. Intorduction

Let S_k be the class of functions $f(z)=z+\sum_{n=1}^{\infty} b_{nk+1}^{(k)} z^{nk+1}$ which are regular and univalent in $|z|<1$ and denote

$$S_n^{(k)}(z)=z+b_{k+1}^{(k)} z^{k+1}+\cdots+b_{nk+1}^{(k)} z^{nk+1}.$$

Szegö proved that the functions $S_n^{(1)}(z),(n=2,3,)\cdots$ are univalent in $|z|<\frac{1}{4}$. Moreover, Sun Kung[2] showed that $S_n^{(2)}(z)$, $(n=2, 3, \cdots)$ are univalent in $|z|<\frac{1}{\sqrt{3}}$. Recently, the authors have proved that the functions $S_n^{(1)}(z)$ are starlike in $|z|<\frac{1}{4}$. The aim of the present paper is to prove that the functions $S_n^{(2)}(z)$ are starlike in $|z|=\frac{1}{\sqrt{3}}$, i. e., the following

Theorem. *Let* $f_2(z)=z+b_3^{(2)}z^3+\cdots\in S_2$ *and* $S_n^{(2)}(z)=z+b_3^{(2)}z^3+\cdots+b_{2n+1}^{(2)}z^{2n+1}$. *Then all the functions* $S_n^{(2)}(z)$ *are starlike in* $|z|<\frac{1}{\sqrt{3}}$. $\rho_*=\frac{1}{\sqrt{3}}$ *is best possible, as shown by the function* $\frac{z}{1-z^2}$.

The theorem is proved by Hu in the case of $n=2$, and by Pan in the other cases.

§ 2. Lemmas

Lemma 1. *Let* $p(z)=1+p_1z+\cdots$ *be regular and* Re $p(z)>0$ *in* $|z|<1$. *Then*

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$$\left|\frac{p(z)-p(\zeta)}{z-\zeta}\right|^2 \leqslant \frac{4\operatorname{Re}p(z)\operatorname{Re}p(\zeta)}{(1-|z|^2)(1-|\zeta|^2)}. \tag{1}$$

In particular

$$\operatorname{Re}p(z) \geqslant \frac{1-r^2}{2(1+r^2)}(1+|p(z)|^2) \tag{2}$$

where $|z|=r$.

Proof By the known formula

$$p(z)=\int_0^{2\pi}\frac{1+e^{-it}z}{1-e^{-it}z}\,d\tau(t), \quad \tau(2\pi)-\tau(0)=1,$$

we have

$$\left|\frac{p(z)-p(\zeta)}{z-\zeta}\right|^2 = 4\left|\int_0^{2\pi}\frac{e^{it}d\tau(t)}{(1-e^{-it}z)(1-e^{-it}\zeta)}\right|^2$$

$$\leqslant 4\int_0^{2\pi}\frac{d\tau(t)}{|1-e^{-it}z|^2}\int_0^{2\pi}\frac{d\tau(t)}{|1-e^{-it}\zeta|^2}=\frac{4\operatorname{Re}p(z)\operatorname{Re}p(\zeta)}{(1-|z|^2)(1-|\zeta|^2)},$$

so that (1) holds for $|z|<1$ and $|\zeta|<1$.

The inequality (2) is the special case of (1) with $\zeta=0$.

Lemma 2. *If* $f_2\in S_2$, *then*

$$\operatorname{Re}\frac{zf_2'(z)}{f_2(z)} \geqslant \frac{\rho^2-r^4}{2(\rho^2+r^4)}\left(1+\left|z\frac{f_2'(z)}{f_2(z)}\right|^2\right) \tag{3}$$

for $|z|^2=r^2<\rho=\tanh\frac{\pi}{4}=0.65579$.

Proof Since the radius of starlikeness of S is $\rho=\tanh\frac{\pi}{4}$, it follows that if $f\in S$ then $g(z)=\frac{f(\rho z)}{\rho}\in S^*$, i. e. $\operatorname{Re}p(z)=\operatorname{Re}\frac{zg'(z)}{g(z)}>0$. we now turn to S_2. If $f_2\in S_2$, then there exists $f\in S$ such that $f_2(z)=f^{\frac{1}{2}}(z^2)$. We immediately obtain (3) from (2).

Lemma 3. *Let* $f_2\in S_2$ *and* $R_n(z)=\sum_{k=n+1}^{\infty}b_{2k+1}^{(2)}z^{2k+1}$. Then

$$|R_n(z)|\leqslant 1.17\frac{r^{2(n+1)+1}}{1-r^2}, \tag{4}$$

$$|R_n'(z)|\leqslant 1.17\frac{r^{2(n+1)}[2n+3-(2n+1)r^2]}{(1-r^2)^2}. \tag{5}$$

Its proof is easily deduced by the result of Milin $|b_{2k+1}^{(2)}|<1.17$.

§ 3. The proof of the therom for the case $n\neq 2$

(i) The case $n=1$ is simple, because $|b_3^{(2)}|<1$, $z=\frac{1}{\sqrt{3}}e^{i\theta}$ and

$$\operatorname{Re}\frac{zS_1'(z)}{S_1(z)}=\frac{3[3-4\operatorname{Re}[b_3^{(2)}e^{i2\theta}]+|b_3^{(2)}|^2]}{|3+b_3^{(2)}e^{2i\theta}|^2}\geqslant\frac{3(1-|b_3^{(2)}|)(3-|b_3^{(2)}|)}{|3+b_3^{(2)}e^{2i\theta}|}>0.$$

(ii) The case $n\geqslant 3$. Since $S_n(z)=f_2(z)-R_n(z)$,

$$I=\operatorname{Re}\frac{zS_n'(z)}{S_n(z)}=\operatorname{Re}z\frac{f_2'(z)}{f_2(z)}+\operatorname{Re}z\frac{f_2'(z)R_n(z)-f_2(z)R_n'(z)}{f_2(z)(f_2(z)-R_n(z))}$$

$$\geqslant\operatorname{Re}\frac{zf_2'(z)}{f_2(z)}-\frac{\left|\frac{zf_2'(z)}{f_2(z)}\right||R_n(z)|+|z||R_n'(z)||}{|f_2(z)|-|R_n(z)||}. \qquad (6)$$

It is sufficient to prove $I>0$ for $|z|=\frac{1}{\sqrt{3}}$. Denoting $y=\left|\frac{zf_2'(z)}{f_2(z)}\right|$ a simple calculation shows that, by lemma 3

$$\left|R_n\left(\frac{1}{\sqrt{3}}\right)\right|\leqslant 0.0216667r, \qquad (7)$$

$$\left|R_n'\left(\frac{1}{\sqrt{3}}\right)\right|\leqslant 0.216667, \qquad (8)$$

$$\left\|f_2\left(\frac{1}{\sqrt{3}}\right)\right|-\left|R_n\left(\frac{1}{\sqrt{3}}\right)\right\|\geqslant\frac{r}{1+r^2}-1.17\frac{r^9}{1-r^2}=0.72833r, \qquad (9)$$

where $r=\frac{1}{\sqrt{3}}$.

Substituting (7), (8), (9) and (3) into (6), we get

$$I\geqslant 0.294689y^2-0.029749y-0.0027998=\varphi(y)$$
$$\geqslant\varphi(0.3260029)=0.01882.$$

Here we have used the fact $y\geqslant 0.3260024$ which can be obtained by solving the inequality $y\geqslant\frac{\rho^2-r^4}{2(\rho^2+r^4)}(1+y^2)$ by Lemma2. Thus the case $n\neq 2$ has been proved.

§ 4. The proof of the theorem for $n=2$

Let $z=re^{i\theta}$ and denote $b_3^{(2)}e^{2i\theta}=-x-iy$, $b_5^{(2)}e^{4i\theta}=s+it$. We have

$$\operatorname{Re}\left\{\frac{1}{\sqrt{3}}e^{i\theta}\frac{S_2'\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)}{S_2\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)}\right\}=\frac{F(x,\ y,\ s,\ t)}{81\left|S_2\left(\frac{1}{\sqrt{3}}e^{i\theta}\right)\right|^2},$$

where

$$F(x,\ y,\ s,\ t)=81+27(x^2+y^2)+5(s^2+t^2)-108x+54s-24(xs+ty)$$
$$=27x^2+5s^2-24xs+54s-108x+81+3\left(3y-\frac{4}{3}t\right)^2-\frac{t^2}{3}.$$

It is known that

$$|b_3^{(2)}|^2=x^2+y^2\leqslant 1, \qquad (10)$$
$$|b_5^{(2)}|^2=s^2+t^2\leqslant\left(e^{-\frac{2}{3}}+\frac{1}{2}\right)^2<1.014^2,$$
$$|b_3^{(2)}|^2+3|(b_3^{(2)})^2-b_5^{(2)}|^2=x^2+y^2+3(x^2-y^2-s)^2+3(t-2xy)^2\leqslant 1.$$

From (10) we have

$$2x^2\leqslant s+x^2+y^2+\sqrt{\frac{1-x^2-y^2}{3}}\leqslant s+\frac{13}{12}<s+1.1.$$

In order to prove the theorem in the case of $n=2$, it is sufficient to prove

$$G(x, s)=27x^2+5s^2-24xs+54s-108x+80>0$$

under the conditions of $|x|\leqslant 1$, $|s|\leqslant 1.014$ and $s\geqslant 2x^2-1.1$.

Because $\frac{\partial G}{\partial x}=54x-24s-108<0$ and $\frac{\partial G}{\partial s}=10s+24x+54>0$, the minimun point of $G(x, s)$ lies on the curve of $s=2x^2-1.1$. By calculation, we have

$$\begin{aligned} G(x,\ 2x^2-1.1) &= 20x^4-48x^3+113x^2-81.6x+26.65 \\ &= x^2(20x^2-48x+29)+(84x^2-81.6x+26.65) \\ &= x^2(20(x-1)^2-8x+9)+84\left(x-\frac{1}{2}\right)^2+2.4x+5.65>0. \end{aligned}$$

This ends the proof for $n=2$. Thus the proof of the theorem is complete.

References

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[2] Sun Kun g, *Science Record*, **4**.: 4(1951), 333—341.
[3] 胡克、潘一飞，数学研究与评论，(1983)外文版第一期。