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WEAK CONVERGENCE OF EMPIRICAL PROCESSES OF SEQUENCES OF STATIONARY RANDOM VARIABLES

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Abstract

In this paper, the author improves Yoshihara's result (J. Multivariate Anal. 8(1978), 584—588) and proves the weak convergence of empirical processes for sequences of ρ-mixing strictly stationary random variable with $\rho(n)=0(n^{-\frac{1}{2}-\theta})$, $\theta>0$.

Moreover, the author simplifies the complex proof of weak convergence of empirical processes with random index and gets the corresponding result for α-mixing stationary random variables.

Let $\{\xi_n\}$ be a stationary sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ and let ξ_n have a uniform distribution over $[0, 1]$. The sequence $\{\xi_n\}$ is said to satisfy the condition of α-mixing, ρ-mixing or φ-mixing, if the following conditions are satisfied respectively:

1.) If for any $A\in\mathscr{F}^0_{-\infty}=\mathscr{F}\{\xi_k: k\leqslant 0\}$, $B\in\mathscr{F}^\infty_n=\mathscr{F}\{\xi_k: k\geqslant n\}$,

$$|P(AB)-P(A)P(B)|\leqslant\alpha(n)\downarrow 0 \quad (n\to\infty);$$

2.) if for any $\mathscr{F}^0_{-\infty}$-measurable random variable ξ, any $\mathscr{F}^\infty_n$-measurable random variable η, $E|\xi|^2<\infty$, $E|\eta|^2<\infty$,

$$|E\xi\eta-E\xi E\eta|/\sqrt{\operatorname{Var}\xi\operatorname{Var}\eta}\leqslant\rho(n)\downarrow 0 \quad (n\to\infty);$$

3.) if for any $A\in\mathscr{F}^0_{-\infty}$, $B\in\mathscr{F}^\infty_n$,

$$|P(AB)-P(A)P(B)|\leqslant\varphi(n)P(A),\ \varphi(n)\downarrow 0 \quad (n\to\infty).$$

The empirical processes $\{Y_n\}$ are defined as follows:

$$Y_n(t,\ \omega)=\sqrt{n}\,(F_n(t,\ \omega)-t) \quad (0\leqslant t\leqslant 1), \tag{1}$$

where we denote by $F_n(t, \omega)$ the empirical distribution function of $\xi_1(\omega), \cdots, \xi_n(\omega)$. Suppose that $Y=\{Y(t, \omega): 0\leqslant t\leqslant 1\}$ are real Gaussian processes, with $EY(t)=0$ and covariance function

$$EY(s)Y(t)=Eg_s(\xi_0)g_t(\xi_0)+\sum_{k=1}^{\infty}Eg_s(\xi_0)g_t(\xi_k)+\sum_{k=1}^{\infty}Eg_s(\xi_k)g_t(\xi_0), \tag{2}$$

where $g_t(\alpha)=I_{[0,t)}(\alpha)-t$.

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In [1—4], the weak convergence of empirical processes for sequences of stationary random variables has been discussed. Yoshihara proved that Y_n weakly converges to Y in the case of α-mixing and φ-mixing, in [3, 4], when $\alpha(n)=O(n^{-5/2-\theta})$ and $\varphi(n)=O(n^{-1-\theta})$, $\theta>0$. In this paper we, first of all, weaken the mixing condition and prove the same result in Theorem 1, when $\{\xi_n\}$ is ρ-mixing, $\rho(n)=O(n^{-1/2-\theta})$, $\theta>0$. Since $\rho(n)\leqslant 2\sqrt{\varphi(n)}$, Theorem 1 improves the result in [4].

In [6, 7], the weak convergence of empirical processes for random number of independent random variables has been considered, but the proof is very complex. In Theorem 2, we give a simple proof in the case of dependent variables, that improves the result in [6, 7]

Theorem 1. *Let $\{\xi_n\}$ be a stationary ρ-mixing sequence, $\rho(n)=O(n^{-1/2-\theta})$, $\theta>0$. If the series on the right hand side of (2) are convergent absolutely, then Y_n weakly converges to Y.*

Proof It follows from [4, 5] that the finite-dimensional distributions of Y_n converges weakly to those of Y. In order to prove that the $\{Y_n\}$ is tight, from [4], it is sufficient to prove that the following lemma is valid.

Lemma *Let $\{\xi_n\}$ be as in Theorem 1, $\rho(n)=O(n^{-1/2-\theta})$, $\theta>0$. Put*

$$z_i=I_{[s,t]}(\xi_i)-(t-s),\ Ez_i=0,\ Ez_i^2=\tau>0,$$

$$S_n=\sum_{k=1}^{n} z_k.$$

Then for any $\varepsilon_0>0$, there exist $\theta_1>0$, $\theta_2>0$ such that

$$P\{|S_n/\sqrt{n}|\geqslant\varepsilon_0\}\leqslant K(n^{-\theta_1}\tau+\tau^{1+\theta_2}), \tag{3}$$

when n is so large and $\tau<\varepsilon_0/\sqrt{n}$. Here (and below) K is a positive constant (and can assume different values on each of its appearance, even within the same formula).

Proof Let

$$r=[\log_2 n-1],\ p=2^{[(N-1)r/(2N)]},\ m=2^{r-[r(N-1)/(2N)]}.$$

We may assume that $0<\theta<1/2$, and take the nature number $N>2/(5\theta)$, where $[\alpha]$ denotes the integral part of real number α. Put

$$\eta_j=\sum_{i=1}^{p} z_{jp+i},\quad T_k=\sum_{j=1}^{k}\eta_{2j-2}\quad(k=1,\cdots,m),$$

$$T'_m=\sum_{j=1}^{m}\eta_{2j-1},\quad T''_m=S_n-T_m-T'_m.$$

From the conditions of the lemma, it follows that

$$E\eta_0^2=E\Big(\sum_{i=1}^{p}z_i\Big)^2\leqslant p\tau+2\tau((p-1)\rho(1)+\cdots+\rho(p-1))\leqslant Kp^{1+\lambda}\tau,$$

where $\lambda=1/2-\theta$. Futhermore

$$E|\eta_0\eta_j|\leqslant E\eta_0^2\rho(jp)\leqslant Kp^{1+\lambda}\tau(jp)^{-(1-\lambda)}\leqslant Kp^{2\lambda}j^{-(1-\lambda)}\tau,$$

$$ET_k^2\leqslant Kkp^{1+\lambda}\tau+k\sum_{j=1}^{k-1}E|\eta_0\eta_{2j-2}|\leqslant K(kp^{1+\lambda}+k^{1+\lambda}p^{2\lambda})\tau\leqslant Kkp^{2\lambda}(p^{1-\lambda}\vee k^{\lambda})\tau,$$

where $p^{1-\lambda}\vee k^{\lambda}$ is just the same as $\max(\rho^{1-\lambda}, k^{\lambda})$. Since $N>2/(5\theta)>(1+2\theta)/(5\theta)$, we can take δ such that

$$(2(N-1)(1-2\theta)/(N+1-2\theta))\vee(\{2(N+1)-4(3N-1)\theta\}/\{N-1+2(2N-1)\theta\})<\delta<2. \tag{4}$$

Checking the proof of Lemma 2.1 in [5], we see that Lemma 2.1 is also true for the δ chosen above. It follows from (2.7) and Lemma 2.1 in [5] that we have

$$E|T_{2m}|^{2+\delta}\leqslant(2+\varepsilon)E|T_m|^{2+\delta}+K(E|T_m|^2)^{1+\delta/2}.$$

There exist positive constants ε and β such that $(2+\varepsilon)\,2^{-(1+\delta/(2N))}<\beta<1$. By a discussion as in [4], it follows from Lemma 2.1 of [5] that

$$\begin{aligned}E|T_m|^{2+\delta}&\leqslant K(2+\varepsilon)^{r-\left[\frac{N-1}{2N}r\right]-\left[\frac{r\delta}{M}\right]}\left(E|T(2^{\left[\frac{r\delta}{M}\right]})|^2\right)^{1+\delta/2}\\&+K\sum_{i=1}^{r-\left[\frac{N-1}{2N}r\right]-\left[\frac{r\delta}{M}\right]}(2+\varepsilon)^{i-1}\left(E|T(2^{r-\left[\frac{N-1}{2N}r\right]-i}|^2\right)^{1+\delta/2}.\end{aligned}\tag{5}$$

Here M is so large that $(N-1)(1/2+\theta)/(2N)>\delta(1/2-\theta)/M$, which implies $p^{1-\lambda}\geqslant 2^{[r\delta/M]\lambda}$. Multiplying two hands of the above inequality by $2^{-r(1+\delta/2)}$, we get

$$\begin{aligned}&2^{-r(1+\delta/2)}E|T_m|^{2+\delta}\\&\leqslant K2^{-r\left(1+\frac{\delta}{2}\right)+\left(1+\frac{\delta}{2N}\right)\left(r-\left[\frac{N-1}{2N}r\right]-\left[\frac{r\delta}{M}\right]\right)+\left[\frac{N-1}{2N}r\right](1+\lambda)\left(1+\frac{\delta}{2}\right)+\left[\frac{r\delta}{M}\right]\left(1+\frac{\delta}{2}\right)}\beta^{r-\left[\frac{N-1}{2N}r\right]-\left[\frac{r\delta}{M}\right]}\tau^{1+\frac{\delta}{2}}\\&\quad+K\sum_{i=1}^{r-\left[\frac{N-1}{2N}r\right]-\left[\frac{r\delta}{M}\right]}2^{-r\left(1+\frac{\delta}{2}\right)}(2+\varepsilon)^{i-1}\tau^{1+\frac{\delta}{2}}\times\\&\quad\times\left\{p^{1+\lambda}\left(2^{r-\left[\frac{N-1}{2N}r\right]-i}\right)\vee p^{2\lambda}\left(2^{r-\left[\frac{N-1}{2N}r\right]-i}\right)^{1+\lambda}\right\}^{1+\frac{\delta}{2}}.\end{aligned}\tag{6}$$

Because of $\tau<\varepsilon_0/\sqrt{n}$, $\tau^{\delta/2}\leqslant Kn^{-\delta/4}\leqslant K2^{-r\delta/4}$. From (4), if M is so large, the powers of 2 in the first term on the right hand side of (6) is

$$\begin{aligned}&-r\left(1+\frac{\delta}{2}\right)+\left(1+\frac{\delta}{2N}\right)\left(r-\left[\frac{N-1}{2N}r\right]-\left[\frac{r\delta}{M}\right]\right)+\left[\frac{N-1}{2N}r\right](1+\lambda)\left(1+\frac{\delta}{2}\right)\\&\quad+\left[\frac{r\delta}{M}\right]\left(1+\frac{\delta}{2}\right)-\frac{r\delta}{4}\leqslant r\left(\frac{N-1}{2N}\lambda\left(1+\frac{\delta}{2}\right)-\frac{\delta}{4}\right)\\&=\frac{r}{8N}\{2(N-1)(1-2\theta)-(N+1+2(N-1)\theta)\delta\}<0.\end{aligned}$$

So the first term on the right hand side of (6) does not exceed $Kn^{-\theta_1^{(0)}}\tau$, $\theta_1^{(0)}>0$.

Let us consider the summation of the second term on the right hand side of (6). If $p^{1-\lambda}\geqslant(2^{r-[(N-1)r/(2N)]-i})^{\lambda}$, then the corresponding term in the summation is

$$\begin{aligned}&2^{-r(1+\delta/2)}\{p^{1+\lambda}2^{r-[(N-1)r/(2N)]-i}\tau\}^{1+\delta/2}(2+\varepsilon)^{i-1}\\&\leqslant K((2+\varepsilon))/2^{1+\delta/2})^{i-1}2^{((N-1)r/(2N))\lambda(1+\delta/2)-r(1-\theta)\delta/4}\tau^{1+\theta\delta/2}.\end{aligned}\tag{7}$$

It follows from (4) that the powers of 2 on the right hand side of (7) is

$$\begin{aligned}&r\left(\frac{N-1}{2N}\left(\frac{1}{2}-\theta\right)\left(1+\frac{\delta}{2}\right)-\frac{1-\theta}{4}\delta\right)\\&=\frac{r}{8N}(2(N-1)(1-2\theta)-(N+1-2\theta)\delta)<0.\end{aligned}$$

If $p^{1-\lambda}<(2^{r-[(N-1)r/(2N)]-i})$, then the corresponding term in the summation is

$$2^{-r(1+\delta/2)}\{p^{2\lambda}(2^{r-[(N-1)r/(2N)]-i})^{1+\lambda}\tau\}^{1+\delta/2}(2+\varepsilon)^{i-1}$$
$$\leqslant K((2+\varepsilon)/2^{1+\alpha})^{i-1}2^{r\{\lambda-(N-1)(1-\lambda)/(2N)-(1-\theta)\delta/(4+2\delta)\}(1+\delta/2)}\tau^{1+\theta\delta/2}, \qquad (8)$$

where $1+\alpha=(1+\lambda)(1+\delta/2)$. It follows from (4) that the powers of 2 on the right hand side of (8) is

$$r(1+\delta/2)(1/2-\theta-(N-1)(1/2+\theta)/(2N)-(1-\theta)\delta/(4+2\delta))$$
$$=(r/8N)\{2(N+1)-4(3N-1)\theta-((N-1)+2(2N-1)\theta)\delta\}<0.$$

Denote $i_0=\max\{i: p^{1-\lambda}<(2^{r-[(n-1)r/(2N)]-i})^{\lambda}\}$. Therefore the second term on the right hand side of (6) does not exceed

$$K\Big\{\sum_{i=1}^{i_0}\Big(\frac{2+\varepsilon}{2^{1+\alpha}}\Big)^{i-1}+\sum_{i=i_0}^{r-[(N-1)r/(2N)]-[r\delta/M]}\Big(\frac{2+\varepsilon}{2^{1+\delta/2}}\Big)^{i-1}\Big\}\tau^{1+\theta\delta/2}$$
$$\leqslant K\Big\{\sum_{i=1}^{\infty}\Big(\Big(\frac{2+\varepsilon}{2^{1+\alpha}}\Big)^{i-1}+\Big(\frac{2+\varepsilon}{2^{1+\delta/2}}\Big)^{i-1}\Big)\Big\}\tau^{1+\theta\delta/2}\leqslant K\tau^{1+2\theta_2},$$

where $4\theta_2=\theta\delta$, the constant K does not depend on r and p. It follows that

$$2^{-r(1+\delta/2)}E|T_m|^{2+\delta}\leqslant K(n^{-\theta_1^{(0)}}\tau+\tau^{1+2\theta_2}).$$

For $T''_m=S_n-T_m-T'_m$, put $0\leqslant n_1=n-2mp<2^{r+1}$. If $n_1\leqslant 2p$, since $|z_i|\leqslant 1$, $|T''_m|\leqslant n_1<2p$. If $n_1>2p$, let us write $r_1=[\log_2 n_1-1]$, $p_1=2^{[(N-1)r_1/(2N)]}$, $m_1=2^{r_1}/p_1$, and define $T_k^{(1)}$, $T_{m_1}^{(1)\prime}$, $T_{m_1}^{(1)\prime\prime}=T''_m-T_{m_1}^{(1)}-T_{m_1}^{(1)\prime}$ as above. By the same argumentation, we have

$$2^{-r_1(1+\delta/2)}E|T_{m_1}^{(1)}|^{2+\delta}\leqslant K(n^{-\theta_1^{(1)}}\tau+\tau^{1+2\theta_2}).$$

Here the number of terms in $T_{m_1}^{(1)\prime\prime}$ is $n_2=n_1-2m_1p_1<2^{r_1+1}<2^{r+1}$, therefore $r_1<r$. For any given n, take this step s times ($s\leqslant r=[\log_2 n-1]$) so that $n_i\geqslant 2p$ $(i=2, \cdots, s-1)$, $n_s>2p$, then $|T_{m_s}^{(s)\prime\prime}|<2p<\varepsilon\sqrt{n}/r$ and for $T_{m_i}^{(i)}$ $(i=2, \cdots, s-1)$ we have

$$2^{-r_i(1+\delta/2)}E|T_{m_i}^{(i)}|^{2+\delta}\leqslant K(n^{-\theta_1^{(i)}}\tau+\tau^{1+2\theta_2}).$$

Since $n_1>n_2>\cdots>n_{s-1}\geqslant 2p\geqslant Kn^b$ ($b(>0)$ doesn't depend on n), there exists a $\theta_1(>0)$, which doesn't depend on n, such that $2\theta_1<\theta_1^{(i)}$ $(i=0, 1, \cdots, s-1)$. Put $T_m=T_{m_0}^{(0)}$. It follows that

$$P\{|S_n|\geqslant\varepsilon\sqrt{n}\}\leqslant 2\sum_{i=0}^{s-1}P\{|T_{m_i}^{(i)}|\geqslant\varepsilon\sqrt{n}/r\}+P\{|T_{m_s}^{(s)\prime\prime}|\geqslant\varepsilon\sqrt{n}/r\}$$
$$\leqslant 2(r/\varepsilon)^{2+\delta}\sum_{i=0}^{s-1}n^{-(1+\delta/2)}E|T_{m_i}^{(i)}|^{2+\delta}\leqslant Kr^{3+\delta}(n^{-2\theta_1}\tau+\tau^{1+2\theta_2}).$$

Note that $r^{3+\delta}/n^{\theta_1}$ and $r^{3+\delta}\tau^{\theta_2}$ are bounded when n is so large, which implies inequlity (3). The proof of the lemma is completed.

Rmark I wish to thank the referee, who has pointed out that the inequality $|T''_m|\leqslant 2p$ is not always true for any case in [4]. In fact, if $n=2^{r+1}+2^r$, the number of terms in T''_m is $n_1=n-2mp=2^r$. Then the inequality does not necessarily hold in this case.

Theorem 2. *Let $\{\xi_n\}$ be a stationary α-mixing sequence, $\{N_n\}$ be a sequence of random variables with $N_n/n\to\mu(P.)$, where μ is a positive random variables. If $Y_n\Rightarrow Y$, then $Y_{N_n}\Rightarrow Y$.*

Proof Write

$$Y_n(t,\ \omega)=\sqrt{n}\,(F_n(t,\ \omega)-t)=n^{-1/2}\sum_{k=1}^{n}(I(0\leqslant\xi_k(\omega)<t)-t).$$

If we can prove that Y_n weakly converges to Y in Renyi-mixing, denoted by $Y_n\Rightarrow Y$ (R-mixing), then it follows from Theorem 7 in [8], that $Y_{N_n}\Rightarrow Y$.

Put

$$Y'_n(t,\ \omega)=n^{-1/2}\sum_{k=p_n}^{n}(I(0\leqslant\xi_k(\omega)<t)-t),$$

where $p_n\to\infty$, $p_n/\sqrt{n}\to 0$. We have

$$\sup_{0\leqslant t\leqslant 1}|Y_n(t,\ \omega)-Y'_n(t,\ \omega)|\leqslant 2p_n/\sqrt{n}\to 0. \tag{9}$$

Hence $Y'_n\Rightarrow Y$. Let us prove $Y'_n\Rightarrow Y$ (R-mixing), i. e. for any Y-continuity set A and any $E\in\mathscr{F}$,

$$P\{Y'_n\in A,\ E\}\to P\{Y\in A\}P\{E\}\quad (n\to\infty). \tag{10}$$

Let $\mathscr{B}_0=\bigcup_{n=1}^{\infty}\mathscr{F}(\xi_1,\ \cdots,\ \xi_n)$ be a class of finite-dimensional sets of the stationary sequence $\{\xi_n\}$. Then for any $E\in\mathscr{B}_0$, there exists a k such that $E\in\mathscr{F}(\xi_1,\ \cdots,\ \xi_k)$. It follows from α-mixing that we have

$$|P\{Y'_n\in A,\ E\}-P\{Y'_n\in A\}P\{E\}|\leqslant\alpha(p_n-k)\to 0\quad (n\to\infty).$$

Because of this, we can prove, as in Theorem 4.5 of [1], that for any $\mathscr{F}$-measurable integrable function g and any Y-continuity set A, we have

$$\int_{(Y'_n\in A)}g\,dP\to P\{Y\in A\}\int g\,dP\quad (n\to\infty).$$

Particularily, if take $g=I_E$, $E\in\mathscr{F}$, then (10) is obtained, i. e. $Y'_n\Rightarrow Y$ (R-mixing). By (9), $\rho(Y'_n,\ Y_n)\to 0$. It follows from the lemma of [9] that $Y_n\Rightarrow Y$ (R-mixing). The proof is complete.

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