*Chin. Ann. of Math*
**5B** (4) 1984

# AN NONEXISTENCE THEOREM FOR HARMONIC MAPS WITH SLOWLY DIVERGENT ENERGY

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## Abstract

It is proved that a harmonic map or a relative harmonic map from Euclidean space $R^n(n\neq 2)$ into an $m$-dimensional Riemannian manifold $M_m$ with finite energy or slowly divergent energy must be a constant map. Some physical applications are also presented.

It is known that the harmonic map from Euclidean space $R^n(n>2)$ to any $m$ dimensional Riemannian space $M_m$ with finite energy must be a constant map, i. e. the image of $R^n$ is a fixed point [1-3]. In this paper, we will use a certain technique, similar to that in the author's previous work on massive Yang-Mills fields[4], to show that the "finite energy" hypothesis can be weakened. The method is quite different from that in the papers[1—3].

We also show that this conclusion for the harmonic maps holds for relative harmonic maps, too.

As is known, the energy of a harmonic map $\phi$ from $R^n\to M_m$ is

$$E(\phi)=\int_{R^n} e(\phi)d^n x. \tag{1}$$

In local coordinates, the energy density $e(\phi)$ can be expressed in the form

$$e(\phi)=\sum_i g_{\alpha\beta}(\phi)\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^i}=\sum_i g_{\alpha\beta}\phi_i^\alpha\phi_i^\beta \tag{2}$$

$$(\alpha,\ \beta=1,\ \cdots,\ m;\ i,\ j=1,\ \cdots,\ n),$$

where $g_{\alpha\beta}$ is the metric tensor of $M_m$. The equations for harmonic maps are

$$\sum_i \phi^\alpha_{i|i}=0. \tag{3}$$

where "$|$" denotes the covariant derivatives, i. e.

$$\phi^\alpha_{i|i}=\frac{\partial\phi_i^\alpha}{\partial x^i}+\Gamma^\alpha_{\beta\gamma}\phi_i^\beta\phi_i^\gamma. \tag{4}$$

Here $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of $M_m$.

For each harmonic map $\phi$, we have a symmetric tensor

$$T_{ij}=\frac{1}{2}\left(\sum_l g_{\alpha\beta}\phi_l^\alpha\phi_l^\beta\right)\delta_{ij}-g_{\alpha\beta}\phi_i^\alpha\phi_j^\beta, \tag{5}$$

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Manuscript received February 7, 1983.

which satisfies the conservative law[5]

$$\frac{\partial T_{ij}}{\partial x^j}=0. \tag{6}$$

It is noted that (6) also follows from the Noether's theorem in the field theory.

In the previous works, it was assumed that the energy $E(\phi)$ of the harmonic map $\phi$ from $R^n\to M_m$ is finite, now the condition is weakened to

$$\int\frac{e(\phi)}{\psi(r)}\,d^nx<\infty, \tag{7}$$

where $\psi(r)$ is a positive, continuous function of $r$ satisfying

$$\int_a^\infty \frac{dr}{r\psi(r)}=\infty \quad \text{(for a certain constant } a>0). \tag{8}$$

If $\psi(r)=1$, the energy is finite. But the energy may be infinite, for example, in the case $\psi(r)=O(\log r)$ (as $r\to\infty$). Hence when (7) holds true, the energy $E(\phi)$ of a harmonic map $\phi$ may be either finite or infnite.

**Difinition.** *The energy is called "slowly divergent" if* $\int e(\phi)]d^nx=\infty$ *and* (7) *holds.*

**Theorem 1.** *Let* $\phi$: $R^n\to M_m$ *be a harmonic map from* $n(n>2)$ *dimensional Euclidean space* $R^n$ *into an m-dimensional Riemannian manifold* $M_m$. *Suppose that the energy* $E(\phi)$ *of* $\phi$ *is finite or slowly divergent, then* $\phi$ *is a constant map.*

*Proof* For any piecewise continuous function $\sigma(r)$, we have

$$\begin{aligned}0&=\int_0^R\sigma(r)dr\int_{|x|\leq r}x^j\frac{\partial T_{ij}}{\partial x^i}\,d^nx\\&=\int_0^R\sigma(r)dr\int_{|x|\leq r}\left\{\frac{\partial}{\partial x^i}(x^jT_{ij})-\sum_i T_{ii}\right\}d^nx\\&=\int_0^R\sigma(r)dr\int_{|x|=r}(x^jT_{ij})\frac{x^i}{r}\,dS-\int_0^R\sigma(r)dr\int_{|x|\leq r}\sum_i T_{ii}d^nx.\end{aligned} \tag{9}$$

From (5), we obtain

$$\sum_i T_{ii}=\left(\frac{n}{2}-1\right)\left(\sum_i g_{\alpha\beta}\phi_i^\alpha\phi_i^\beta\right)\geq 0. \tag{10}$$

Suppose $\phi$ is not a constant harmonic map, then there exist positive constants $R_1$ and $\epsilon$ such that

$$\int_{|x|\leq R}\sum_i T_{ii}d^nx>\epsilon>0 \tag{11}$$

for $R\geq R_1$.

Choose $\sigma(r)\geq 0$ such that $\sigma(r)=0$ iff $r\leq R_1$. Then, from (9) we have

$$0<\int_0^R\sigma(r)dr\int_{|x|=r}x^jT_{ij}\frac{x^i}{r}\,dS-\epsilon\int_{R_1}^R\sigma(r)dr. \tag{12}$$

Since $|T_{ij}|\leq\frac{3}{2}e(\phi)$, we have

$$0<\frac{3}{2}n^2\int_{|x|\leq R}r\sigma(r)e(\phi)d^nx-\epsilon\int_{R_1}^R\sigma(r)dr. \tag{13}$$

Furthermore, we choose $\sigma(r)$ such that for $r>R_1$

$$\sigma(r)=\frac{1}{r\psi(r)}, \tag{14}$$

then from (7) and (8), it is easily seen that if $R$ is sufficiently large, the right side of (13) should be negative. This is a contradiction. Hence $\phi$ must be a constant map. Theorem 1 is proved.

A relative harmonic map $\phi$ is a generalization of the harmonic map. It is defined in [6] by the formula

$$\sum_i g_{\alpha\beta}\phi^{\alpha}_{i|i}\phi^{\beta}_{k}=0 \tag{15}$$

in local coordinates. We can verify that for relative harmonic maps, equation (6) holds true also. Repeating the previous argument, we have

**Theorem 2.** *There exists no relative harmonic map from $R^n\to M_m(n>2)$ with finite energy or slowly divergent energy other than a constant map.*

In theoretical physics, the Chiral field (or the nonlinear $\sigma$-model) in 4-dimensional Minkowski spacetime $R^{3,1}$ is just a harmonic map $\phi$ from $R^{3,1}$ to a homogeneous Riemannian manifold $M_m$. If the field is static, then $\phi$ is a harmonic map from $R^3$ to $M_m$. Hence the physical significance of Theorem 1 can be expressed as

**Theorem 3.** *In $n+1(n>2)$ dimensional Minkowski spacetime $R^{n,1}$, there does not exist any static nontrivial Chiral field with finite energy or slowly divergent energy.*

It is known that Ernst equations are the basic equations for obtaining axial symmetric stationary solutions to the pure Einstein gravitational equation and are the Euler equations of the energy integral

$$\int\frac{\sum_i\left[\left(\frac{\partial\Phi}{\partial x^i}\right)^2+\left(\frac{\partial\Psi}{\partial x^i}\right)^2\right]}{\Phi^2}\,d^nx.$$

Hence a solution to the Ernst equations is a harmonic map from $R^n$ to the hyperbolic plane with metric

$$dS^2=\frac{1}{\Phi^2}(d\Phi^2+d\Psi^2),\qquad \Phi>0$$

in Poincare representation.

Thus we have

**Theorem 4.** *A nontrivial solution to the Ernst equations with $n>2$ must have infinite energy. Furthermore, the energy cannot be slowly divergent. Here the energy is in the sense of harmonic maps.*

**Remark 1.** There are many nontrivial harmonic maps $\phi$ from $S^2$ to certain manifolds with finite energy $E(\phi)$. By using sterographic projection from $S^2\to R^2$, we will obtain nontrivial harmonic maps from $R^2\to M$ with finite energy. So $n=2$ is actually an exceptional case.

**Remark 2.** We shall specialize that the condition for the energy in our theorem cannot be omitted, because we can find many regular harmonic maps from $R^n$, whose energy does not diverge so slowly. For example, the linear function

$$\phi = a_i x^i + a \quad (a_i \text{ are constants not all zero})$$

is a harmonic map from $R^n$ to $R$ with a constant energy density.

Further results on harmonic maps from noncompact Riemannian manifolds will appear in [7].

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