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THE AUTOMORPHISMS OF NON-DEFECTIVE ORTHOGONAL GROUPS $\Omega_8(V)$ AND $O'_8(V)$ IN CHARACTERISTIC 2*

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Abstract

Let V be a non-defective 8-dimensional quadratic space over a field F of characteristic 2, $F\neq\mathbf{F}_2$. We prove that if there is an exceptional automorphism of either $\Omega_8(V)$ or $O'_8(V)$, then V^α has a Cayley algebra structure for some α in $\dot{F}$. Moreover, every exceptional automorphism of $O'_8(V)$ has exactly one of the following forms:

$$\varphi_1\circ\Phi_g \text{ or } \varphi_2\circ\Phi_g,$$

where Φ_g is an automorphism of $O'_8(V)$ given by conjugation by a semilinear automorphism of V which preserves the quadratic structure, and φ_1 and φ_2 are the automorphisms induced by triality principle. Every exceptional automorphism of $\Omega_8(V)$ is the restriction of a unique exceptional automorphism of $O'_8(V)$.

A. Hahn[8] has completely determined all the automorphisms of $P\Omega_8(V)$ and $PO'_8(V)$ in characteristic not 2. We now treat the case char $F=2$. Throughout this paper V is a non-defective n-dimensional quadratic space over a field F of characteristic 2 with $F\neq\mathbf{F}_2$. The quadratic form and the associated symplectic form are respectively denoted by Q and $(\ ,\ )$, with $(x, y)=Q(x+y)+Q(x)+Q(y)$. $O_n(V)$, $O_n^+(V)$, $O'_n(V)$ and $\Omega_n(V)$ are, respectively, the orthogonal group, the rotation group, the spinor subgroup and the commutator subgroup of the orthogonal group on V.

§ 1. Preliminary Results

We assume familiarity with the theory of quadratic forms, orthogonal groups, and the residual space method. Refer to [3, 5, 7, 11, 12].

Definition 1. *Let v be a non-zero vector in V, and let U be a non-zero subspace of V. We call v a singular vector (non-singular vector, resp.) if $Q(v)=0$ ($Q(v)\neq 0$, resp.). We call U non-defective (defective, totally defective, resp.) if $U\cap U^*=0$ ($U\cap$*

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$U^* \neq 0$, $U \subseteq U^*$, *resp.*), *where* $U^* = \{x \in V \mid (x, U) = 0\}$. U *is degenerate if* U *is defective and there is a singular vector in* $U \cap U^*$.

Convention. *Let* $\sigma, \sigma_i \in O_n(V)$. *The residual spaces of* σ *and* σ_i *will always be denoted by* R *and* R_i *respectively. We use* Δ *to denote either* $\Omega_n(V)$ *or* $O'_n(V)$. *If* Λ *is an automorphism of* Δ, *and* $\sigma, \sigma_i \in \Delta$, *then* Σ *and* Σ_i *will be used for* $\Lambda(\sigma)$ *and* $\Lambda(\sigma_i)$ *respectively, and the residual spaces of* Σ *and* Σ_i *will be denoted by* R' *and* R'_i, *respectively.*

When R has some geometric property, we say that σ has the same property (e. g., non-defective, defective, totally defective, degenerate, etc.).

Lemma 1.1. *Let* $\sigma \in O_n(V)$ *and* $\sigma \neq 1$. *Then* $\sigma^2 = 1$ *if and only if* σ *is totally defective. In particular, a plane rotation* σ *is non-defective if and only if* $\sigma^2 \neq 1$.

Proof See 1.1 of [6].

Lemma 1.2. *Every plane rotation in* $O'_n(V)$ *is either non-defective or degenerate.*

Proof See 1.4a of [6].

Proposition 1.3. *Suppose* $n \geq 6$. *Let* Λ *be an automorphism of* Δ, *and let* $\sigma \in \Delta$ *be a non-defective plane rotation. Then* $\Sigma = \Lambda(\sigma)$ *is a non-defective rotation with residual index* 2 *or* n.

Proof See Proposition 2.3 of [2].

Proposition 1.4. *Suppose* $n \geq 6$. *Let* Λ *be an automorphism of* Δ. *If there is a non-defective plane rotation* $\sigma \in \Delta$ *such that* res $\Sigma \neq n$, *then* Λ *has the standard type* Φ_g.

Proof See Propositions 2.3, 2.4 and Section 3 of [2].

Proposition 1.5. *Let* Λ *be an automorphism of* Δ. *If there is a degenerate plane rotation* $\sigma \in \Delta$ *such that* res $\Sigma < \frac{n}{2}$, *then* Λ *has the standard type* Φ_g.

Proof See the proof of Proposition 2.5 and Section 3 of [2].

Definition 2. *Suppose* Λ *is an automorphism of* Δ. *We say that* Λ *is exceptional if* Λ *does not have the type* Φ_g.

Clearly, Λ is exceptional if and only if Λ^{-1} is exceptional.

§ 2. Cayley Rotations

In order to study exceptional automorphisms of Δ, we need the concept of Cayley rotations. Refer to Section 1B of [8].

Definition 3. $\sigma \in O_n(V)$ *is called a Cayley rotation on* V *if its minimal polynomial on* V *has the form* $\lambda^2 + \beta\lambda + 1$ *with* $\beta \neq 0$. *We call* β *the residual trace of* σ, *and denote* $\beta = \text{res tr}(\sigma)$.

Lemma 2.1 *Suppose* $n = 2$. *The set of Cayley rotations on* V *is* $O_2^+(V)$ *excluding* 1. *If* σ *is a Cayley rotation on* V *and* x *is a nonsingular vector, then* $\{x, \sigma x\}$ *is a basis* *of V. Given any β in $\dot{F}$, there are at most two Cayley rotations on V with residual trace β. If σ is one of them, then σ^{-1} is the other.*

Lemma 2.2. *Let $\sigma \in O_n(V)$ be a Cayley rotation on V. Suppose U is a σ-invariant non-defective subspace of V. Then $\sigma|_U$ is a Cayley rotation on U, and* $\operatorname{res}\operatorname{tr}(\sigma|_U)=\operatorname{restr}(\sigma)$.

The above two lemmas are easy to check. We omit the proof.

Proposition 2.3. *$\sigma\in O_n(V)$ is a Cayley rotation on V if and only if $Fx+F\sigma x$ is a σ-invariant non-defective plane for any non-singular vector x in V.*

Proof Assume that σ is a Cayley rotation on V with residual trace β, and x is a non-singular vector. By simple calculation, we have $(x, \sigma x)=\beta Q(x)\neq 0$, so $Fx+F\sigma x$ is a non-defective plane. Clearly, $Fx+F\sigma x$ is σ-invariant.

Conversely, suppose $Fx+F\sigma x$ is a σ-invariant non-defective plane for any non-singular vector x in V. Then we can take a splitting $V=\pi_1\perp\cdots\perp\pi_{\frac{n}{2}}$ with $\pi_i=Fx_i+F\sigma x_i$ where $Q(x_i)\neq 0$ for $i=1, \cdots, \frac{n}{2}$. We see that $1\neq\sigma|_{\pi_i}\in O_2^+(\pi_i)$, so $\sigma|_{\pi_i}$ is a Cayley rotation on π_i. Let $\beta_i=\operatorname{res}\operatorname{tr}(\sigma|_{\pi_i})$. Claim $\beta_i=\beta_j$ for all $1\leqslant i, j\leqslant\frac{n}{2}$. Otherwise suppose $\beta_1\neq\beta_2$. Choose a non-singular vector y in π_2 with $Q(y)\neq Q(x_1)$. Then x_1+y is a non-singular vector, but $F(x_1+y)+F\sigma(x_1+y)$ is not σ-invariant. This is a contradiction. Hence we have $\beta_1=\cdots=\beta_{\frac{n}{2}}=\beta$. Thus $\sigma^2+\beta\sigma+1=0$, and σ is a Cayley rotation on V.

Proposition 2.4. *$\sigma\in O_n(V)$ is a Cayley rotation on V if and only if*

(a) $\operatorname{res}\sigma=n$;

(b) *there is a splitting $V=\pi_1\perp\cdots\perp\pi_{\frac{n}{2}}$ into non-defective planes π_i, each invariant under σ;*

(c) *$\sigma|_{\pi_i}$ is a Cayley rotation on π_i for each i;*

(d) $\operatorname{res}\operatorname{tr}(\sigma|_{\pi_i})=\operatorname{res}\operatorname{tr}(\sigma|_{\pi_j})$ *for all i, j.*

Proof By the definition of Cayley rotations and Proposition 2.3.

Proposition 2.5. *Suppose $n=8$. Any Cayley rotation σ is in $O_8'(V)$.*

Proof By Proposition 2.4, there is a splitting $V=\pi_1\perp\pi_2\perp\pi_3\perp\pi_4$ such that $\sigma\pi_i=\pi_i$ for each i. Take a non-singular vector x_i in π_i. By simple calculation we have $\sigma|_{\pi_i}=\tau_{x_i+\sigma x_i}\tau_{x_i}$, $\theta(\sigma|_{\pi_i})=Q(x_i+\sigma x_i)Q(x_i)\dot{F}^2=\beta Q(x_i)^2\dot{F}^2=\beta\dot{F}^2$, and so $\theta(\sigma)=\beta^4\dot{F}^2=\dot{F}^2$. Thus, $\sigma\in O_8'(V)$.

Proposition 2.6. *Suppose $n\geqslant 6$. Let Λ be an exceptional automorphism of Δ. If $\sigma\in\Delta$ is a non-defective plane rotation, then $\Sigma=\Lambda(\sigma)$ is a Cayley rotation.*

Proof By Proposition 1.4, $\operatorname{res}\Sigma=n$ since Λ is exceptional. Then $(\Sigma+1)V=V$.

Suppose, if possible, Σ is not a Cayley rotation. Claim there is a non-defective plane $\pi=Fx+F\Sigma x$ for some x in V such that $\Sigma\pi\neq\pi$. In fact, take non-singular $y_1\in V=(\Sigma+1)V$ with $y_1=\Sigma x_1+x_1$. Since $(\Sigma x_1, x_1)=Q(y_1)\neq 0$, $\pi_1=Fx_1+F\Sigma x_1$ is a non-defective plane. If π_1 is not Σ-invariant, π_1 will do. If π_1 is Σ-invariant, then $\pi_1 \perp \pi_1^* = V = (\Sigma+1)V = (\Sigma+1)(\pi_1 \perp \pi_1^*) = (\Sigma+1)\pi_1 \perp (\Sigma+1)\pi_1^* \subseteq \pi_1 \perp \pi_1^*$, which implies $\pi_1^* = (\Sigma+1)\pi_1^*$. Take non-singular $y_2 \in \pi_1^* = (\Sigma+1)\pi_1^*$ with $y_2 = \Sigma x_2 + x_2$. Then $\pi_2 = Fx_2 + F\Sigma x_2$ is a non-defective plane, and so on. If $V = \pi_1 \perp \cdots \perp \pi_{\frac{n}{2}}$, and each π_i is Σ-invariant, then $\Sigma|_{\pi_i} \in O_2^+(\pi_i)$. Since Σ is not a Cayley rotation on V by hypothesis, we can assume $\beta_1 = \text{res tr}(\Sigma|_{\pi_1}) \neq \text{res tr}(\Sigma|_{\pi_2}) = \beta_2$ by Proposition 2.4. Take non-singular z_1 and z_2 in π_1 and π_2 respectively such that $\beta_1 Q(z_1) + \beta_2 Q(z_2) \neq 0$, and put $x = z_1 + z_2$. Then $\pi = Fx + F\Sigma x$ is a non-defective plane, and $\Sigma\pi \neq \pi$.

Now take a plane rotation $\rho' \in \Omega_2(\pi) \perp 1$. Clearly $\rho'\Sigma \neq \Sigma\rho'$. The residual space of $\rho'\Sigma\rho'^{-1}\Sigma^{-1}$ lies in the ternary subspace $Fx + F\Sigma x + F\Sigma^2 x$, and so $\rho'\Sigma\rho'^{-1}\Sigma^{-1}$ is a plane rotation. Put $\rho = \Lambda^{-1}(\rho')$. Then $\Lambda^{-1}(\rho'\Sigma\rho'^{-1}\Sigma^{-1}) = \rho\sigma\rho^{-1}\sigma^{-1}$, being the product of two plane rotations, has residual index at most 4.

Applying Proposition 1.4 to the automorphism Λ^{-1}, we see that $\rho'\Sigma\rho'^{-1}\Sigma^{-1}$ is defective since Λ^{-1} is exceptional. Then $\rho'\Sigma\rho'^{-1}\Sigma^{-1}$ is an involution by Lemma 1.1, and so $\rho\sigma\rho^{-1}\sigma^{-1}$ is an involution. Thus $\rho\sigma\rho^{-1}\sigma^{-1}$ is totally defective, and the residual index of $\rho\sigma\rho^{-1}\sigma^{-1}$ can not be 4 since R is non-defective. So res $\rho\sigma\rho^{-1}\sigma^{-1} = 2$. By Proposition 1.5 Λ is a standard automorphism. This is a contradiction.

Therefore, Σ must be a Cayley rotation.

Remark. If we replace Δ by $O_n^+(V)$ in Proposition 2.6, then, clearly, the proof can pass through, and the result is also true.

§ 3. Composition Algebras

Definition 4. *Let V be a finite-dimensional vector space over a field F of characteristic 2 with a non-defective quadratic form Q. We say that Q permits composition (or say that V is a composition algebra) if it is possible to define a bilinear product $x \cdot y$ (denoted by xy in brief) such that $Q(xy) = Q(x)Q(y)$ for all x, y in V.*

If V is a composition algebra, we can always assume that V has the identity element e by modifying the product. In fact, choose $v \in V$ with $Q(v) \neq 0$ and put $u = Q(v)^{-1}v^2$. Then $Q(u) = 1$ and hence $Q(xu) = Q(x) = Q(ux)$ for all x in V. Thus the multiplications R_u and L_u are in $O_n(V)$, and so they are invertible, and their inverses are also in $O_n(V)$. We now define a new product $x \odot y$ by $x \odot y = (R_u^{-1}x) \cdot (L_u^{-1}y)$. It is easy to verify that $Q(x \odot y) = Q(x)Q(y)$ and $u^2 \odot x = x \odot u^2 = x$. We now revert to the original notation xy for $x \odot y$. Thus $e = u^2$ is the identity. Clearly, $Q(e) = 1$.

It follows by linearization from the equality $Q(xy) = Q(x)Q(y)$ that

$$(xy, x'y) = (x, x')Q(y), \tag{1}$$

$$(xy, xy') = Q(x)(y, y'), \tag{2}$$

$$(xy', x'y) + (xy, x'y') = (x, x')(y, y') \tag{3}$$

for all x, y, x', y' in V.

Define $\bar{x} = \tau_e x = x + (e, x)e$, where τ_e is the symmetry determined by e. Then we have $(e, \bar{x}) = (e, x)$ and $Q(\bar{x}) = Q(x)$. From (3) we immediately get

$$(xy', y) + (y', \bar{x}y) = 0, \tag{4}$$

$$(x'y, x) + (x', x\bar{y}) = 0. \tag{5}$$

Now $(xy', y) = (y', \bar{x}y) = (y'\bar{y}, \bar{x}) = (\bar{y}, \bar{y}'\bar{x})$. Putting $y' = e$, we have $(x, y) = (\bar{y}, \bar{x}) = (\bar{x}, \bar{y})$. Then $(\overline{xy'}, \bar{y}) = (xy', y) = (\bar{y}, \bar{y}'\bar{x})$ for all x, y, y' in V. Since V is non-defective, $\bar{y}'\bar{x} = \overline{xy'}$ for all x, y' in V. So $x \mapsto \bar{x}$ is an involution (anti-automorphism of period two) in V.

For any y in $(Fe)^*$, we have $(x\bar{x}, y) = (x, yx) = (e, y)Q(x) = 0$ by equalities (5) and (1). Hence $x\bar{x} \in Fe$, and so $x\bar{x} = Q(x)e$ for all x in V. Now

$$(x, \bar{a}(ay)) = (ax, ay) = Q(a)(x, y) = (x, Q(a)y) = (x, (\bar{a}a)y).$$

Since V is non-defective, we obtain

$$\bar{a}(ay) = (\bar{a}a)y, \tag{6}$$

and so

$$a(ay) = (aa)y. \tag{7}$$

Similarly

$$(ya)a = y(aa) \tag{8}$$

for all a, y in V.

Equalities (7) and (8) are the conditions for an alternative algebra. So a composition algebra must be alternative. We replace a by $a+y$ in (7) and derive

$$y(ay) = (ya)y, \tag{9}$$

and so we can denote $yay = (ya)y = y(ay)$.

Proposition 3.1. *In alternative algebras the Moufang identities*

$$(xax)y = x[a(xy)],$$

$$y(xax) = [(yx)a]x,$$

$$(xy)(ax) = x(ya)x$$

hold for all x, y, a.

Proof See Pages 28—29 *of* [13].

Example 1 Let $V = Fx + Fy$ be a non-defective 2-dimensional quadratic space over F with

$$V \cong \alpha \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}.$$

Denote $V^\alpha = V$ as a vector space. Put $Q^\alpha(v) = \alpha^{-1}Q(v)$. Then Q^α is a quadratic form of V^α and

$$V^\alpha \cong \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}.$$

Define $xx=x$, $xy=yx=y$, $yy=x+\beta y$. Then V^α is a composition algebra called a quadratic algebra. The identity is x.

Example 2 Let V be a non-defective 4-dimensional quadratic space over F. If

$$V\cong\alpha\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp ab\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}$$

with respect to some certain basis $\{x_1, x_2, x_3, x_4\}$, then Q^α permits composition, and V^α is a composition algebra called a quaternion algebra. Indeed, we can define $x_1x_i=x_ix_1=x_i$ for $i=1, 2, 3, 4$, $x_2x_2=x_1+\beta x_2$, $x_2x_3=x_4$, $x_3x_2=\beta x_3+x_4$, $x_2x_4=x_3+\beta x_4$, $x_4x_2=x_3$, $x_3x_3=bx_1$, $x_3x_4=b(\beta x_1+x_2)$, $x_4x_3=bx_2$, and $x_4x_4=bx_1$.

Example 3 Let V be a non-defective 8-dimensional quadratic space over F. If

$$V\cong\alpha\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp ab\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp ac\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp abc\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}$$

with respect to some certain basis $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, then V^α is a composition algebra called a Cayley algebra or octonion algebra. Tne multiplication table is as follows.

	1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_2	x_2	$x_1+\beta x_2$	x_4	$x_3+\beta x_4$	x_6	$x_5+\beta x_6$	βx_7+x_3	x_7
x_3	x_3	βx_3+x_4	bx_1	$b(\beta x_1+x_2)$	x_7	x_8	bx_5	bx_6
x_4	x_4	x_3	bx_2	bx_1	x_8	$x_7+\beta x_8$	$b(\beta x_5+x_6)$	bx_5
x_5	x_5	βx_5+x_6	x_7	x_8	cx_1	$c(\beta x_1+x_2)$	cx_8	cx_4
x_6	x_6	x_5	x_3	$x_7+\beta x_3$	cx_2	cx_1	$c(\beta x_3+x_4)$	cx_3
x_7	x_7	x_8	bx_5	$b(\beta x_5+x_6)$	cx_3	$c(\beta x_3+x_4)$	bcx_1	bcx_2
x_8	x_8	$x_7+\beta x_3$	bx_6	bx_5	cx_4	cx_3	$bc(\beta x_1+x_2)$	bcx_1

Theorem 3.2. *Let V be a finite-dimensional vector space over F with a non-defective quadratic form Q. If V is a composition algebra, then V is one of the above three examples.*

Proof Proceed as in Theorem 1 of [9] or Pages 422—426 of [10].

Remark. N. Jacobson pointed out that there are five types of composition algebras in characteristic 2 (See Page 428 of [10]). His non-degeneracy, however, is different from our non-defectiveness.

Proposition 3.3. *Suppose $(V, \cdot)$ is a Cayley algebra, and $(V, \odot)$ is also a Cayley algebra. Then there exists a g in $O_8(V)$ such that g: $(V, \cdot)\to(V, \odot)$ is an isomorphism of Cayley algebras.*

Proof Let e be the identity of $(V, \cdot)$. Choose x in V with $Q(x)=1$ and $(e, x)=\beta\neq 0$. Then $\pi=Fe+Fx$ is a non-defective plane, and is a subalgebre of $(V, \cdot)$. Take

y in π^* with $Q(y)\neq 0$. Then πy is orthogonal to π. Put $U=\pi\perp\pi y$. U is a non-defective subalgebra of $(V, \cdot)$. Select non-singular $z\in U^*$. Then Uz is orthogonal to U, and we have $V=\pi\perp\pi y\perp\pi z\perp\pi(yz)$. (Note that $x(yz)\neq(xy)z$, but $\pi(yz)=(\pi y)z$.)

Suppose now e' is the identity of $(V, \odot)$. Since $Q(e)=Q(e')=1$, by the transitivity theorem, there is a $g_1\in O_8(V)$ such that $g_1e=e'$. Put $x'=g_1x$. Then $\pi'=Fe'+Fx'$ is a non-defective subalgbra of $(V, \odot)$. Clearly, g_1: $(\pi, \cdot)\to(\pi', \odot)$ is an isomorphism of quadratic algebras.

Put $y'=g_1y$. It is easily seen that $\pi'\odot y'$ is orthogonal to π'. Denote $U'=\pi'\perp\pi'\odot y'$. This is a non-defective subalgebra of $(V, \odot)$. With respect to the basis $\{e', x', y', x'\odot y'\}$

$$U'\cong\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp b\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix},$$

where $b=Q(y')$, and U has the same expression with respect to $\{e, x, y, xy\}$. So by the transitivity theorem, there is a $g_2\in O_8(V)$ such that $g_2e=e'$, $g_2x=x'$, $g_2y=y'$, $g_2(xy)=x'\odot y'$. Hence g_2: $(U, \cdot)\to(U', \odot)$ is an isomorphism of quaternion algebras.

Now let $z'=g_2z$. As above, $U'\odot z'$ is orthogonal to U'. With respect to the two bases $\{e, x, y, xy, z, xz, yz, x(yz)\}$ and $\{e', x', y', x'\odot y', z', x'\odot z', y'\odot z', x'\odot(y'\odot z')\}$, V has the same expression

$$V\cong\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp b\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp c\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp bc\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}.$$

Therefore, by the transitivity theorem again, there exists a g in $O_8(V)$ such that $ge=e'$, $gx=x'$, $gy=y'$, $g(xy)=x'\odot y'$, $gz=z'$, $g(xz)=x'\odot z'$, $g(yz)=y'\odot z'$, $g(x(yz))=x'\odot(y'\odot z')$. Such a g must be an isomorphism of the two Cayley algebras.

Proposition 3.4. *Suppose V is a Cayley algebra, and $\Sigma\in O_8(V)$ is a Cayley rotation. Then there is a Cayley multiplication $\odot$ for V such that $\Sigma v=(\Sigma e)\odot v$ for all v in V, where e is the identity of both $(V, \cdot)$ and $(V, \odot)$.*

Proof As in the proof of Proposition 3.3, there is a basis $\{e, x, y, xy, z, xz, yz, x(yz)\}$ such that

$$V\cong\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp b\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp c\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix}\perp bc\begin{bmatrix}1 & \beta\\ 0 & 1\end{bmatrix},$$

where x can be taken as Σe, Put $\pi=Fe+Fx$.

Let $\pi_1=Fy+F\Sigma y$. Clearly $\Sigma\pi_1=\pi_1$, and $\pi_1\cong\pi y$. Put $U=\pi\perp\pi_1$. Then $U\cong\pi\perp\pi y$ and $U^*\cong\pi z\perp\pi(yz)$. Choose $h\in U^*$ with $Q(h)=Q(z)=c$. Put $\pi_2=Fh+F\Sigma h\subset U^*$. Then $\pi_2\cong\pi z$. Write $U^*=\pi_2\perp\pi_3$. By Witt theorem, $\pi_3\cong\pi(yz)$. Choose $k\in\pi_3$ with $Q(k)=Q(yz)=bc$. Now $\{e, x, y, \Sigma y, h, \Sigma h, k, \Sigma k\}$ is a basis of V. With respect to this basis, the expression of V is the same as above. By the transitivity theorem, there is a g in $O_8(V)$ such that $ge=e$, $gx=x$, $gy=y$, $g(\Sigma y)=xy$, $gh=z$, $g(\Sigma h)=xz$, $gk=yz$, and $g(\Sigma k)=x(yz)$.

Define $\odot: V\times V\to V$ by $u\odot v=g^{-1}(gu\cdot gv)$. It is easy to verify that $(V, \odot)$ is a Cayley algebra, and $\Sigma v=(\Sigma e)\odot v$ for all v in V.

§ 4. Triality Principle

In this section we assume that V is a Cayley algebra with the identity e. Denote $\tau_e x$ by $\bar{x}$. $\Gamma O_8(V)$ is the group of semilinear automorphisms of V which preserve Q. We have the following triality principle.

Theorem 4.1. *Let $\sigma\in\Gamma O_8(V)$. Then there exist σ_1 and σ_2 in $\Gamma O_8(V)$ such that either*

$$\sigma(xy)=\sigma_1(x)\sigma_2(y) \text{ for all } x,\ y \text{ in } V$$

or

$$\sigma(xy)=\sigma_1(y)\sigma_2(x) \text{ for all } x,\ y \text{ in } V.$$

Moreover, σ_1 and σ_2 are unique up to a scalar, and only one of the equalities holds.

Proof. Since $\Gamma O_8(V)$ is generated by the symmetries and the left multiplications, it suffices to prove the theorem in the two cases.

1) Assume $\sigma=\tau_a$, where $Q(a)\neq 0$. Take σ_1 and σ_2 by $\sigma_1(x)=a\bar{x}$ and $\sigma_2(x)=Q(a)^{-1}\,xa$. Then σ_1 and σ_2 are in $\Gamma O_8(V)$. Using Moufang identities, we have

$$\begin{aligned}\sigma_1(y)\sigma_2(x)&=(a\bar{y})Q(a)^{-1}(\bar{x}a)=Q(a)^{-1}(a\bar{y})(\bar{x}a)=Q(a)^{-1}a(\overline{yx})a\\&=Q(a)^{-1}[a(\overline{xy})]a=Q(a)^{-1}[(xy)\bar{a}+(a,\ xy)e]a\\&=xy+Q(a)^{-1}(a,\ xy)a=\tau_a(xy).\end{aligned}$$

2) Assume $\sigma=L_a$, where $Q(a)\neq 0$. Take σ_1 and σ_2 by $\sigma_1(x)=\mathrm{axa}$ and $\sigma_2(x)=Q(a)^{-1}\bar{a}x$. Then $\sigma_1, \sigma_2\in\Gamma O_8(V)$. Using Moufang identities again, we have

$$\begin{aligned}\sigma_1(x)\sigma_2(y)&=(\mathrm{axa})Q(a)^{-1}(\bar{a}y)=Q(a)^{-1}a\{x[a(\bar{a}y)]\}\\&=Q(a)^{-1}a[x(Q(a)y)]=a(xy)=L_a(xy).\end{aligned}$$

The last result is easy since V is not a commutative algebra.

We now define $\Gamma O_8^+(V)$ to be the set of all σ in $\Gamma O_8(V)$ such that the first equality in Theorem 4.1 holds, and $\Gamma O_8^-(V)$ the set of all σ in $\Gamma O_8(V)$ such that the second equality holds. Clearly, $\Gamma O_8^+(V)\cap O_8(V)=O_8^+(V)$ and $\Gamma O_8^-(V)\cap O_8(V)=O_8^-(V)$.

If $\sigma\in\Gamma O_8(V)$, then there is an α_σ in $\dot{F}$ such that $Q(\sigma x)=\alpha_\sigma Q(x)$ for all x in V. We define $\hat{\sigma}=\alpha_\sigma^{-1}\tau_e\sigma\tau_e$, i. e., $\hat{\sigma}(x)=\alpha_\sigma^{-1}\overline{\sigma(\bar{x})}$.

The following results (Corollaries 4.2—4.4) are very easy to check.

Corollary 4.2. *If $\sigma\in\Gamma O_8^+(V)$, then $\sigma_1, \sigma_2, \hat{\sigma}\in\Gamma O_8^+(V)$, and*

$$\sigma_1(xy)=\sigma(x)\hat{\sigma}_2(y),$$
$$\sigma_2(xy)=\hat{\sigma}_1(x)\sigma(y),$$
$$\hat{\sigma}(xy)=\hat{\sigma}_2(x)\hat{\sigma}_1(y)$$

for all x, y in V.

Corollary 4.3. *If $\sigma\in\Gamma O_8^-(V)$, then σ_1, σ_2, $\hat{\sigma}\in\Gamma O_8^-(V)$, and*

$$\sigma_1(xy)=\sigma(y)\hat{\sigma}_2(x),$$
$$\sigma_2(xy)=\hat{\sigma}_1(y)\sigma(x),$$
$$\hat{\sigma}(xy)=\hat{\sigma}_2(y)\hat{\sigma}_1(x)$$

for all x, y in V.

Corollary 4.4. *Let φ_1, φ_2, ε be mappings: $P\Gamma O_8^+(V)\to P\Gamma O_8^+(V)$ by $\varphi_1(\bar{\sigma})=\bar{\sigma}_1$, $\varphi_2(\bar{\sigma})=\bar{\sigma}_2$ and $\varepsilon(\bar{\sigma})=\bar{\hat{\sigma}}$. Then φ_1, φ_2, ε are automorphisms of $P\Gamma O_8^+(V)$. Moreover, $\varphi_1^2=\varphi_2^2=\varepsilon^2=1$, $\varphi_1\varphi_2=\varepsilon\varphi_1=\varphi_2\varepsilon$, $\varphi_2\varphi_1=\varepsilon\varphi_2=\varphi_1\varepsilon$. Therefore φ_1, φ_2 and ε generate a group isomorphic to the symmetric group S_3.*

Corollary 4.5. *If $\sigma\in O_8'(V)$, then σ_1 and σ_2 may be chosen in $O_8'(V)$.*

Proof See Corollary 4 of [4].

Corollary 4.6. *φ_1, φ_2 and ε induce three automorphisms of $O_8'(V)$. They generate a group isomorphic to S_3.*

Proof Use the fact that the center of $O_8'(V)$ is $\{1\}$.

φ_1 and φ_2 are exceptional automorphisms of $O_8'(V)$. To see this, we take a vector x with $(e, x)\neq 0$ and $Q(x)=1$. Consider $\sigma=\tau_x\tau_e\in O_8'(V)$. By straightforward calculation we have $\varphi_1(\sigma)=\Sigma$, where $\Sigma(v)=xv$ for all v in V. We see that Σ is a Cayley rotation since $x^2+\beta x+1=0$, where $\beta=(e, x)\neq 0$. Thus res $\Sigma=8$, and so φ_1 is a exceptional automorphism of $O_8'(V)$. Proceed similarly with φ_2.

§ 5. The Automorphisms

In this section $\dim V=8$. We shall determine all the automorphisms of $O_8'(V)$ and $\Omega_8(V)$.

Lemma 5.1. *Suppose Σ is a Cayley rotation. U is a non-defective 4-dimensional subspace of V, and $\Sigma U=U$. Then there is a non-defective plane π in U such that $U=\pi+\Sigma\pi$.*

Proof Since $\Sigma|_U$ is a Cayley rotation on U, there is a splitting

$$U=(Fx_1+F\Sigma x_1)\perp(Fx_2+F\Sigma x_2)$$

with x_1 and x_2 non-singular. We take $\pi=Fx_1+F(\Sigma x_1+\Sigma x_2)$.

Lemma 5.2. *Suppose Λ is an exceptional automorphism of Δ. Let σ and $\rho'\in\Delta$ be non-defective plane rotations. put $\Sigma=\Lambda(\sigma)$, $\rho=\Lambda^{-1}(\rho')$, and $\theta=\sigma\rho\sigma^{-1}\rho^{-1}$, $\theta'=\Lambda(\theta)=\Sigma\rho'\Sigma^{-1}\rho'^{-1}$. Then* res $\theta=4$ *if and only if* res $\theta'=4$. *When* res $\theta=4$, *θ is non-defective if and only if θ' is non-defective.*

Proof Assume res $\theta=4$. Then the residual space of θ is $R+\rho R$. If res $\theta'\neq 4$, then res $\theta'=2$ since $0\neq$ res $\theta'\leqslant 4$. By Proposition 1.4 and Lemma 1.1 θ' is an involution since Λ is exceptional, and so θ is also an involution. It implies that θ is totally defective. But the residual space of θ contains a non-defective plane R. This is a contradiction. Similarly, if res $\theta'=4$, then res $\theta=4$.

Suppose now res $\theta=4$ and θ is non-defective. If θ' is defective, then res $\theta'^2\leqslant 2$. Since the residual space of θ' contains a non-defective plane, i. e., the residual space of ρ', we have $\theta'^2\neq 1$, and so res $\theta'^2=2$. Consider now $\Lambda^{-1}(\theta'^2)=\theta^2$. Since Λ^{-1} is exceptional, θ'^2 can not be non-defective, and so θ'^2 is a degenerate plane rotation. Thus, $\theta'^4=1$, and $\theta^4=1$. Therefore θ^2 is totally defective, and the residual space of θ^2 is a proper subspace of $R+\rho R$. Then res $\theta^2=2$. Applying Proposition 1.5, we get a contradiction. Similarly, if res $\theta=4$ and θ' is non-defective, then θ is non-defective.

Theorem 5.3. *Suppose Δ has an exceptional automorphism Λ. Then there exists an α in $\dot{F}$ such that V^α is a Cayley algebra.*

Proof 1) Take a non-defective plane rotation $\sigma\in\Delta$. Then $\Sigma=\Lambda(\sigma)$ is a Cayley rotation. By Proposition 2.4, there is a splitting $V=W_1\perp W_2\perp W_3\perp W_4$ where $W_i=Fx_i+F\Sigma x_i$, $Q(x_i)\neq 0$ for each i. Put $U=W_1\perp W_2$. By Lemma 5.1, there are non-defective planes $R_1'\subseteq U$ and $R_2'\subset U^*$ with $R_1'=Fx_1+F(\Sigma x_1+\Sigma x_2)$ such that $U=R_1'+\Sigma R_1'$ and $U^*=R_2'+\Sigma R_2'$.

Take non-defective planes rotation $\Sigma_i\in\Omega_2(R_i')\perp 1$ for $i=1, 2$. Put $\Sigma_3=\Sigma\Sigma_1\cdot\Sigma^{-1}\Sigma_1^{-1}$ and $\Sigma_4=\Sigma\Sigma_2\Sigma^{-1}\Sigma_2^{-1}$. Clearly $R_3'=U$ and $R_4'=U^*$. Let $\sigma_i=\Lambda^{-1}(\Sigma_i)$ for $i=1$, 2, 3, 4. Then σ_1 and σ_2 are Cayley rotations by Proposition 2.6, and res $\sigma_3=$res$\sigma_4=4$ by Lemma 5.2. Since $\Sigma_1\Sigma_4=\Sigma_4\Sigma_1$, $\sigma_1\sigma_4=\sigma_4\sigma_1$, we have $\sigma_1R_4=R_4$. So $R_3=R+\sigma_1R\subseteq R+\sigma_1R_4=R_4$. Therefore we can put $R_3=R_4=W$, which is non-defective by Lemma 5.2.

2) Take non-singular y in R. Put $R_5=Fy+F\sigma_1y\subset W$. Then R_5 is a non-defective plane. We have $R_5\neq R$ since $\Sigma_1\Sigma\neq\Sigma\Sigma_1$. Choose a plane rotation $\sigma_5\in\Omega_2(R_5)\perp 1$ and put $\Sigma_5=\Lambda(\sigma_5)$. Then $\sigma_5\sigma_1=\sigma_1\sigma_5$, and so $\Sigma_5\Sigma_1=\Sigma_1\Sigma_5$. Thus, $\Sigma_5R_1'=R_1'$.

Moreover, claim $\Sigma_5U=U$. In fact, select a non-defective plane rotation $\sigma_6\in\Delta$ with residual space $R_6\subset W^*$, and put $\Sigma_6=\Lambda(\sigma_6)$. Then Σ_6 is a Cayley rotation. Since $\sigma_6\sigma_3=\sigma_3\sigma_6$, $\Sigma_6\Sigma_3=\Sigma_3\Sigma_6$, we have $\Sigma_6U=U$. By Lemma 5.1, there is a non-defective plane $R_7'\subset U$ such that $U=R_7'+\Sigma_6R_7'$. Choose a plane rotation $\Sigma_7\in\Omega_2(R_7')\perp 1$. Then $\Sigma_7\Sigma_4=\Sigma_4\Sigma_7$, $\sigma_7\sigma_4=\sigma_4\sigma_7$, where $\sigma_7=\Lambda^{-1}(\Sigma_7)$. Hence $\sigma_7W=W$ and $\sigma_7W^*=W^*$. Put $\Sigma_8=\Sigma_6\Sigma_7\Sigma_6^{-1}\Sigma_7^{-1}$, and $\sigma_8=\Lambda^{-1}(\Sigma_8)$. Since $R_8'=R_7'+\Sigma_6R_7'=U$, we have res $\sigma_8=4$ by Lemma 5.2. Now $R_8=R_6+\sigma_7R_6\subseteq R_6+\sigma_7W^*=W^*$, so $R_8=W^*$. But $\sigma_5\sigma_8=\sigma_8\sigma_5$, so $\Sigma_5\Sigma_8=\Sigma_8\Sigma_5$. It implies $\Sigma_5U=U$.

3) Consider now $\sigma_5^{-1}\sigma$. The residual space of $\sigma_5^{-1}\sigma$ is contained in $R+R_5$. Since $R\neq R_5$ and $y\in R\cap R_5$, we see that $\sigma_5^{-1}\sigma$ is a plane rotation. By Proposition 2.6, $\Lambda(\sigma_5^{-1}\sigma)=\Sigma_5^{-1}\Sigma$ is a Cayley rotation. Let β, β_1 and β_2 be the residual traces of Cayley rotations Σ, Σ_5 and $\Sigma_5^{-1}\Sigma$ respectively.

4) Take $u=\beta_2 x_3+\beta\ \Sigma_5\, x_3+\beta_1\, \Sigma x_3$. Straightforward calculation shows $u\in(W_1\perp W_2\perp W_3)^*=W_4$, and $Q(u)=(\beta^2+\beta_1^2+\beta_2^2+\beta\beta_1\beta_2)Q(x_3)$. Since $x_1\in R'_1=\Sigma_5 R'_1$, we can assume $\Sigma_5\, x_1=\alpha_1 x_1+\alpha_2(\Sigma\, x_1+\Sigma\, x_2)$. It is easy to see that $\alpha_1=\beta^{-1}\beta_2$ and $\alpha_2=\beta^{-1}\beta_1$. Then we have $(\beta^2+\beta_1^2+\beta_2^2+\beta\beta_1\beta_2)Q(x_1)=\beta_1^2 Q(x_2)$.

Thus $Q(u)=\beta_1^2 Q(x_1)^{-1}Q(x_2)Q(x_3)$.

Take $x'_4=\beta_1^{-1}u$. Then $Q(x'_4)=Q(x_1)^{-1}Q(x_2)Q(x_3)$. With respect to the basis $\{x_1, \Sigma x_1, x_2, \Sigma x_2, x_3, \Sigma x_3, x'_4, \Sigma x'_4\}$ of V,

$$V\cong\alpha\begin{bmatrix}1&\beta\\0&1\end{bmatrix}\perp\alpha b\begin{bmatrix}1&\beta\\0&1\end{bmatrix}\perp\alpha c\begin{bmatrix}1&\beta\\0&1\end{bmatrix}\perp\alpha bc\begin{bmatrix}1&\beta\\0&1\end{bmatrix},$$

where $\alpha=Q(x_1)$, $b=\alpha^{-1}Q(x_2)$, and $c=\alpha^{-1}Q(x_3)$. By Example 3, Q^α permits composition, and V^α is a Cayley algebra.

Theorem 5.4. *The following statements are equivalent:*

1) *$O_8^+(V)$ has exceptional automorphisms.*

2) *V^α is a Cayley algebra for some α in $\dot F$ and $O_8^+(V)=O'_8(V)$,*

3) *V is a Cayley algebra and F is a perfect field.*

Proof 1)⇒2): Let Λ be an exceptional automorphism of $O_8^+(V)$. Since $DO_8^+(V)=\Omega_8(V)$, Λ induces an automorphism of $\Omega_8(V)$. Take a non-defective plane rotation σ in $O_8^+(V)$. By the remark following Proposition 2.6, $\Sigma=\Lambda(\sigma)$ is a Cayley rotation. Then $\Sigma^2\in\Omega_8(V)$ is also a Cayley rotation, res tr$(\Sigma^2)=($res tr$(\Sigma))^2$. So $\Lambda|_{\Omega_8(V)}$ is an exceptional automorphism of $\Omega_8(V)$. By Theorem 5.3, there is an α in $\dot F$ such that V^α is a Cayley algebra.

Since $O_8^+(V)$ can be generated by non-defective plane rotations, if we denote by $\mathscr{P}$ the set of all non-defective plane rotations, we have $O_8^+(V)=\Lambda(O_8^+(V))=\Lambda\langle\sigma\in\mathscr{P}\rangle=\langle\Lambda(\sigma)\,|\,\sigma\in\mathscr{P}\rangle\subseteq O'_8(V)\subseteq O_8^+(V)$ by Proposition 2.5. So $O_8^+(V)=O'_8(V)$.

2)⇒3): Fix a non-singular vector x in V and denote $b=Q(x)\neq0$. For any non-singular y in V, $\tau_x\tau_y\in O_8^+(V)=O'_8(V)$. It implies that $Q(x)Q(y)\in\dot F^2$, i. e., $Q(y)\in b\dot F^2$. If y is singular, then $Q(y)=0\in bF^2$. So $Q(y)\in bF^2$ for any y in V. Take a non-sigular vector z in V with $(x, z)\neq0$. Then $Q(x+z)=b+Q(z)+(x, z)\in bF^2$, and so $(x, z)\in bF^2$. For any c in F, $Q(cx+z)=bc^2+Q(z)+c(x, z)\in bF^2$. It imlpies $c\in F^2$. Hence $F=F^2$, and F is perfect.

Since V^α is a Cayley algebra and $F=F^2$, it is easily seen that V is a Cayley algebra itself.

3)⇒1): Assume V is a Cayley algebra and F is perfect, then we immediately get $O_8^+(V)=O'_8(V)$, and φ_1 and φ_2 induced by triality are exceptional automorphisms of $O_8^+(V)$.

We now assume that V^α is a Cayley algebra with the identity e for some α in $\dot F$. Note that $\Gamma O_8(V^\alpha)=\Gamma O_8(V)$, $O_8(V^\alpha)=O_8(V)$, and so on. We can replace V^α by V.

Denote by C_0 the set of all non-defective plane rotations in $O'_8(V)$. Put $C_1=\varphi_1(C_0)$ and $C_2=\varphi_2(C_0)$. Denote by C the set of all Cayley rotations.

Proposition 5.5. 1) $\varphi_1(C_0)=C_1$, $\varphi_1(C_1)=C_0$, $\varphi_1(C_2)=C_2$.

2) $\varphi_2(C_0)=C_2$, $\varphi_2(C_1)=C_1$, $\varphi_2(C_2)=C_0$.

3) $\varepsilon(C_0)=C_0$, $\varepsilon(C_1)=C_2$, $\varepsilon(C_2)=C_1$.

4) $C_i\cap C_j=\emptyset$ *for* $i\neq j$.

Proof Use Corollary 4.4.

Proposition 5.6. *Let* $g\in\Gamma O_8(V)$. *Then* $\Phi_g(C_0)=C_0$. *Moreover*

$$g\in\Gamma O_8^+(V)\Leftrightarrow\Phi_g(C_1)=C_1\Leftrightarrow\Phi_g(C_2)=C_2,$$

$$g\in\Gamma O_8^-(V)\Leftrightarrow\Phi_g(C_1)=C_2\Leftrightarrow\Phi_g(C_2)=C_1.$$

Proof Use Corollary 4.4 and Proposition 5.5.

Proposition 5.7. $C=C_1\cup C_2$.

Proof Clearly $C_1\cup C_2\subseteq C$.

Conversely, let $\Sigma\in C$. By Proposition 3.4, there is a Cayley multiplication $\odot$ such that $\Sigma v=(\Sigma e)\odot v$ for all v in V, where e is the identity of both $(V,\cdot)$ and $(V,\odot)$. By Proposition 3.3, there exists a g in $O_8(V)$ such that g: $(V,\cdot)\to(V,\odot)$ is an isomorphism of Cayley algebras. Take $\sigma=\tau_{\Sigma e}\tau_e\in C_0$. Then by straightforward calculation we have $\Phi_g\varphi_1\Phi_g^{-1}(\sigma)=\Sigma$. So we see, by Proposition 5.6, that Σ is in C_1 if $g\in O_8^+(V)$; Σ is in C_2 if $g\in O_8^-(V)$.

Theorem 5.8. *Suppose V^α has a Cayley algebra structure for some α in $\dot{F}$. Let φ_1 and φ_2 be the associated exceptional automorphisms of $O'_8(V)$. Let Λ be any exceptional automorphism of $O'_8(V)$. Then there is a $g\in\Gamma O_8(V)$ such that Λ has exactly one of the following forms:*

$$\Lambda=\varphi_1\circ\Phi_g \quad or \quad \Lambda=\varphi_2\circ\Phi_g,$$

where g is unique up to a scalar. Moreover, any such Λ is an exceptional automorphism of $O'_8(V)$.

Proof By Propositions 2.6 and 5.7, $\Lambda(C_0)\subseteq C=C_1\cup C_2$. Assume that $\Lambda(C_0)\cap C_1\neq\emptyset$. Choose $\sigma\in C_0$ such that $\Lambda(\sigma)\in C_1$. Then $(\varphi_1\circ\Lambda)\sigma\in C_0$. It follows from Proposition 1.4 that $\varphi_1\circ\Lambda$ is an automorphism of standard type. Therefore, $\varphi_1\circ\Lambda=\Phi_g$, and $\Lambda=\varphi_1\circ\Phi_g$ for some $g\in\Gamma O_8(V)$. If $\Lambda(C_0)\cap C_2\neq\emptyset$, use φ_2 and obtain $\Lambda=\varphi_2\circ\Phi_g$.

Using Proposition 5.5, we see that $\Lambda=\varphi_1\circ\Phi_{g_1}$ and $\Lambda=\varphi_2\circ\Phi_{g_2}$ can not occur simultaneously. Clearly, g is unique up to a scalar. Finally, it is easy to see that any such Λ is an exceptional automorphism of $O'_8(V)$.

Define λ: Aut $O'_8(V)\to S_3$ by $\lambda(\varphi_1)=(1,2)$, $\lambda(\varphi_2)=(1,3)$, $\lambda(\varepsilon)=(2,3)$, $\lambda(\Phi_g)=1$ for all $g\in\Gamma O_8^+(V)$. Then λ can be extended to a surjective homomorphism of groups, and $\lambda|_{\langle\varphi_1,\varphi_2,\varepsilon\rangle}$ is an isomorphism onto S_3. Denote its inverse by λ_1.

Define μ: $P\Gamma O_8^+(V)\to$ Aut $O'_8(V)$ by $\mu(\bar{g})=\Phi_g$. Then μ is independent of the choice of g, and is an injective homomorphism of groups. From Theorem 5.8, we have

Theorem 5.9. *The subgroup* $\{\Phi_g \mid g \in \Gamma O_8(V)\}$ *of* Aut $O'_8(V)$ *has index* 3 *or* 1 *in* Aut $O'_8(V)$ *according as* V^α *has a Cayley algebra structure for some* α *in* $\dot{F}$ *or not. If* V^α *is a Cayley algebra for some* α *in* $\dot{F}$, *then the sequence of groups*

$$1 \to P\Gamma O_8^+(V) \xrightarrow{\mu} \operatorname{Aut} O'_8(V) \underset{\lambda_1}{\overset{\lambda}{\rightleftarrows}} S_3 \to 1$$

is split exact.

Theorem 5.10. *Every exceptional automorphism of* $\Omega_8(V)$ *is the restriction of a unique exceptional automorphism of* $O'_8(V)$.

Proof Let Λ be an exceptional automorphism of $\Omega_8(V)$. By Proposition 2.6, $\Lambda(C_0) \subseteq C$. Assume $\Lambda(C_0) \cap C_1 \neq \emptyset$. As in the proof of Theorem 5.8, we obtain a $\sigma \in C_0$ with $(\varphi_1 \circ \Lambda)\sigma \in C_0$. Now $\varphi_1 \circ \Lambda$ is an isomorphism of $\Omega_8(V)$ onto $\varphi_1 \Omega_8(V)$. We can prove, by analogue with Proposition 1.4, that as long as $\varphi_1 \circ \Lambda$ sends a non-defective plane rotation to an element whose residual index is less than 8, then $\varphi_1 \circ \Lambda = \Phi_g$ for some g in $\Gamma O_8(V)$. Proceed similarly if $\Lambda(C_0) \cap C_2 \neq \emptyset$. Finally. the uniqueness is clear.

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