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ON THE JOINT SPECTRUM FOR N-TUPLE OF HYPONORMAL OPERATORS

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Abstract

Let $A=(A_1, \cdots, A_n)$ be an n-tuple of double commuting hyponormal operators. It is proved that: 1. The joint spectrum of A has a Cartesian decomposition: $\mathrm{Re}[Sp(A)]=Sp(\mathrm{Re}A)$, $\mathrm{Im}[Sp(A)]=Sp(\mathrm{Im}A)$; 2. The joint resolvent of A satisfies the growth condition: $\|\widehat{(A-z)}\|=\frac{1}{\mathrm{dist}(z, Sp(A))}$; 3. If $0\notin\sigma(A_i)$, $i=1, 2, \cdots, n$, then

$$\|A\|=r_{sp}(A).$$

If $A_1, A_2, \cdots A_n$ are mutually commuting linear bounded operators on Hilbert space H, tnen the joint spectrum of n-tuples $A=(A_1, \cdots, A_n)$ can be defined in terms of the Koszul complex by J. L. Taylor. Several analysts have investigated the joint spectral properly of an n-tuple of hyponormal operators. In this paper, we shall give some new results about it, for example, the property of the Cartesian decompositon of joint spectrum, of the growth of joint resolvent, of the joint normaloid, etc.

§ 1. Definitions and Preliminaries

We denote the Taylor joint spectrum of commuting n-tuple $A=(A_1, \cdots, A_n)$ by $Sp(A, H)$. We shall say that a point $z=(z_1, \cdots, z_n)$ of $\mathscr{C}^n$ is in the joint approximate point spectrum $\sigma_\pi(A)$ if there exists a sequence $\{x_k\}_{k=1}^\infty\subset H$, $\|x_k\|=1$ such that

$$\|(A_i-z_i)x_k\|\to 0 \quad (k\to\infty),\ i=1, 2, \cdots, n.$$

We say that $z\in\mathscr{C}^n$ is in the joint compressive spectrum of A, if $z\in\sigma_\pi(A^*)$, where $A^*=(A_1^*, \cdots, A_n^*)$. We denote the joint norm of A:

$$\|A\|=\sup\left\{\left(\sum_{i=1}^n\|A_iX\|^2\right)^{\frac{1}{2}}: x\in H, \|x\|=1\right\},$$

the joint spectral radius:

$$r_{sp}(A)=\sup\left\{\left(\sum_{i=1}^\infty|\lambda_i|^2\right)^{\frac{1}{2}}: \lambda=(\lambda_1, \cdots, \lambda_n)\in Sp(A)\right\},$$

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the joint numerical range:

$$W(A)=\{((A_1x, x), (A_2x, x), \cdots, (A_nx, x)): x\in H, \|x\|=1\}$$

and the joint numerical radius: $\omega(A)=\sup\{|\lambda|: \lambda\in W(A)\}$.

If $\omega(A)=\|A\|$, we say A is joint normaloid.

Muneo chō has proved $r_{sp}(A)\leqslant\omega(A)\leqslant\|A\|$ and $\omega(A)=\|A\|$ iff $r_{sp}(A)=\|A\|$[6,7].

Now, we quote some theorems which will be used in our discussion.

Theorem A (Taylor). *If A is a commuting n-tuple of operators, U is a neighbourhood of $Sp(A)$, $f_1, \cdots, f_m$ are analytic functions on U. Let f: $U\to C^m$ be defined by $f(z)=(f_1(z), \cdots, f_m(z))$ and let $f(A)=(f_1(A), \cdots, f_m(A))$. Then we have*

$$Sp(f(A), H)=f(Sp(A, H)).$$

Theorem B (Curto)[5]. *Let H be a complex Hilbert space, $A=(A_1, \cdots, A_n)$ be an n-tuple of mutually commuting linear bounded operators, $E(H, A)=\{E_p^n(H), d_p^{(n)}\}$ be a chain complex induced by A, where $d_p^{(n)}$: $E_p^n(H)\to E_{p-1}^n(H)$ are the boundary operators. Let d_i^* denote the conjugate operator of $d_i=d_i^n$ and construct an operator $\hat{A}$ on $H\otimes\mathscr{C}^{2n-1}$ as follows*

$$\hat{A}=\begin{bmatrix} d_1 & & \\ d_2^* & d_3 & \\ & d_4^* & \ddots \\ & & \ddots \end{bmatrix}.$$

Then $A=(A_1, \cdots, A_n)$ is regular in the sense of Taylor's if and only if $\hat{A}$ has an inverse.

Theorem C (Curto[5], Corollary 3.14). *Let $A=(A_1, \cdots, A_n)$ be a commuting n-tuple, ϕ: $\{1, \cdots, n\}\to\{1, *\}$ be a function and $\phi(A_i)=A_i^{\phi(i)}$.* Assume that *$\phi(A_i)\phi(A_j)=\phi(A_j)\phi(A_i)$ for all i, j. Then $Sp(\phi(A))=\{\phi(\lambda): \lambda\in Sp(A)\}$.*

§ 2. The Joint Spectrum of an n-tuple of Seminormal Operators

If $A\in B(H)$, $A^*A-AA^*\geqslant 0$, we say A is hyponormal. If $A^*A-AA^*\leqslant 0$, we say A is cohyponormal. Operator A will be said to be seminormal, if A is either hyponormal or cohyponormal.

An n-tuple of operators $A=(A_1, \cdots, A_n)$, $A_i\in B(H)$, will be said to be double commuting, if $A_iA_j=A_jA_i$, $A_iA_j^*=A_j^*A_i$, $i\neq j$, $i, j=1, 2, \cdots, n$.

Let $A_k=B_k+iC_k$, $k=1, 2, \cdots, n$, be the Cartesian decomposition of $A_k\in B(H)$. We denote

$$\mathrm{Re}A=(\mathrm{Re}A_1, \cdots, \mathrm{Re}A_n)=(B_1, \cdots, B_n),$$
$$\mathrm{Im}\,A=(\mathrm{Im}\,A_1, \cdots, \mathrm{Im}\,A_n)=(C_1, \cdots, C_n),$$

where A is double commuting, and so Re A and Im(A) are commuting n-tuples. Thus, we can define their joint spectrum.

Lemma 2.1. *Let $A=(A_1, \cdots, A_n)$ be an n-tuple of normal operators, Then we have Cartesian decomposition of the joint spectrum:*

$$\mathrm{Re}[Sp(A)]=Sp(\mathrm{Re}A), \mathrm{Im}[Sp(A)]=Sp(\mathrm{Im}A).$$

Proof For an n-tuple of normal operators, it is well known that the joint spectral mapping theorem holds. Since the mappings $(z_1, \cdots, z_n)\to(\mathrm{Re}z_1, \cdots, \mathrm{Re}z_n)$, $(z_1, \cdots, z_n)\to(\mathrm{Im}z_1, \cdots, \mathrm{Im}z_n)$ are continued, we can prove this lemma by operator calculus.

Q.E.D.

Now, we recall the definition of symbol of an operator (cf. [1]). Let $T\in B(H)$, $\{A(t)\,|\,0\leqslant t<\infty\}$ be a contractive semigroup of operators with one parameter. Its generator is iA, i. e. $A(t)=\exp(iAt)$. For $t<0$, we set $A(t)=A(-t)^*$. If $S_A^{\pm}(T)=\operatorname*{s-lim}_{t\to\pm\infty} A(t)TA(-t)$ exists, we shall call $S_A^{\pm}(T)$ the symbol of T for A. We denote

$$S_A^{\pm}=\{T\in B(H);\ S_A^{\pm} \text{ exists}\}.$$

The following theorem is a generalization of Xia's theorem (cf. [1] II. Theorem 1.6).

Theorem 2.2. *Let $A=(A_1, \cdots, A_n)$ be a double commuting n-tuple of operators. $A_k=B_k+iC_k$ is the Cartesian decomposition of A_k, $k=1, 2, \cdots, n$. We have*

(i) *If $C_j\in S_{B_j}^{\pm}$, $j=1, 2, \cdots, n$, then* $\mathrm{Re}\,\sigma_\pi(A)\supset\sigma_\pi(\mathrm{Re}A)$;

(ii) *If $B_j\in S_{C_j}^{\pm}$, $j=1, 2, \cdots, n$, then* $\mathrm{Im}\,\sigma_\pi(A)\supset\sigma_\pi(\mathrm{Im}A)$.

Proof We confine the proof to (i), and that of (ii) is similar.

Let $B=B_1+B_2+\cdots+B_n$, $B(t)=\exp(iBt)\,(t\geqslant0)$, $B(t)=B(-t)^*(t<0)$, $B_j(t)=\exp(iB_jt)\,(t\geqslant0)$, $B_j(t)=B_j(-t)^*(t<0)$. Since $A=(A_1, \cdots, A_n)$ is double commuting, it is easy to see that $\{B_i(t), B_j(t): i, j=1, 2, \cdots, n\}$ is a commuting tuple for any t. Morever, $B_i(t)$ and $B_i(-t)$ commute with C_j, $i\neq j$. By our present hypothesis, for each j,

$$S_B^{\pm}(C_j)=\operatorname*{s-lim}_{t\to\pm\infty} B(t)C_jB(-t)=\operatorname*{s-lim}_{t\to\pm\infty} B_1(t)\cdots B_n(t)C_jB_n(-t)\cdots B_1(t)$$
$$=\operatorname*{s-lim}_{t\to\pm\infty} B_j(t)C_jB_j(-t)=S_{B_j}^{\pm}(C_j).$$

For simplicity, we denote this limit by $C_j^{\pm}$, $j=1, 2, \cdots, n$.

Similarly, we can show that $(C_1^{\pm}, \cdots, C_n^{\pm})$, $(B_1+iC_1^{\pm}, \cdots, B_n+iC_n^{\pm})$ are also commuting tuples of normal operators (A is double commuting). Put

$$C^{\pm}=(C_1^{\pm}, \cdots, C_n^{\pm}),\ B+iC^{\pm}=(B_1+iC_1^{\pm}, \cdots, B_n+iC_n^{\pm}).$$

Now, let $b=(b_1, \cdots, b_n)\in\sigma_\pi(\mathrm{Re}A)=\sigma_\pi(B)=\sigma_\pi[\mathrm{Re}(B+iC^{\pm})]$. We have $Sp(A)=\sigma_\pi(A)$[8], if A is a commuting tuple of normal operators. Thus, by Lemma 2.1, there exists $c=(c_1, \cdots, c_n)\in Sp(C^{\pm})$ and a sequence $\{g_m\}$, $g_m\in H$, $\|g_m\|=1$, $m=1, 2, \cdots$(or $m=-1, -2, \cdots$) such that

$$\|(B_i-b_jI)g_m\|\to0,\ \|C_j^{\pm}-c_jI)g_m\|\to0,\ (m\to\infty),\ j=1, 2, \cdots n. \qquad (*)$$

By the definition of symbol of operators, and $C_j^{\pm}=S_B^{\pm}(C_j)$ we can find a real number

$t_m^{\pm}$ for each g_m such that

$$\|(\exp(it_m^{\pm}B)C_j\exp(-it_m^{\pm}B)-C_j^{\pm})g_m\|<\frac{1}{|m|}.$$

Denote the class of operators which commute with B by $[B]'$. Since $B=B_1+\cdots+B_n$ is selfadjoint, we have $C_j^{\pm}=S_B^{\pm}(C_j)\in[B]'$ (cf. [1], II, Lemma 1.1). Thus

$$\|[C_j\exp(-it_m^{\pm}B)-C_j^{\pm}\exp(-it_m^{\pm}B)]g_m\|<\frac{1}{m}. \qquad (**)$$

Let $f_m=\exp(-it_mB)g_m$. Then $\|f_m\|=1$. Hence by (*), (**) and B_j, $C_j^{\pm}\in[B]'$, $j=1$, $2, \cdots, n$, it follows that

$$\|(B_j-b_jI)f_m\|=\|(B_j-b_jI)g_m\|\to 0$$

$$\|(C_j-c_jI)f_m\|\leqslant\|(C_j-C_j^{\pm})f_m\|+\|(C_j^{\pm}-c_jI)f_m\|$$

$$=\|(C_j-C_j^{\pm})f_m\|+\|(C_j^{\pm}-c_jI)g_m\|\to 0,\quad m\to\pm\infty,\ j=1,2,\cdots,n.$$

Then $b=(b_1,\cdots,b_n)\in\operatorname{Re}(\sigma_\pi(A))$. Q.E.D.

Corollary 2.3. *If $A=(A_1,\cdots,A_n)$ is a double commuting tuple of hyponormal operators, then*

$$\operatorname{Re}(\sigma_\pi(A))=\sigma_\pi(\operatorname{Re}A),\quad \operatorname{Im}(\sigma_\pi(A))=\sigma_\pi(\operatorname{Im}A).$$

This result was obtained by Wei (cf. [9]) early.

Proof Since $A_j=B_j+iC_j$ are hyponormal, we have $A_j\in S_{B_j}^{\pm}\cap S_{C_j}^{\pm}$, and $C_j\in S_{B_j}^{\pm}$, $B_j\in S_{C_j}^{\pm}$, $j=1,2,\cdots,n$. (cf. [1], II, Theorem 2.6). Thus, by Theorem 2.2, we have $\operatorname{Re}(\sigma_\pi(A))\supset\sigma_\pi(\operatorname{Re}A)$, and $\operatorname{Im}(\sigma_\pi(A))\supset\sigma_\pi(\operatorname{Im}A)$. On the other hand, in general, $\sigma_{j\pi}(T)=\sigma_\pi(T)$, where $T=X+iY$ is hyponormal, $\sigma_{j\pi}(T)=\{\lambda=x+iy:\ \exists f_n\in H$, $\|f_n\|=1$ such that

$$\lim_{n\to\infty}\|X-xI)f_n\|=\lim_{n\to\infty}\|(Y-yI)f_n\|=0\}.$$

Therefore

$$\operatorname{Re}\sigma_\pi(A)\subset\sigma_\pi(\operatorname{Re}A),\ \operatorname{Im}\sigma_\pi(A)\subset\sigma_\pi(\operatorname{Im}A).$$

Q.E.D.

Theorem 2.4. *If $A=(A_1,\cdots,A_n)$ is a double commuting tuple of hyponormal operators, then its joint spectrum has a Cartesian decomposition.*

Proof It is sufficient to prove that $\operatorname{Re}[Sp(A)]=Sp[\operatorname{Re}A]$. From Corollary 2.3 we can see that

$$Sp(\operatorname{Re}A)=\sigma_\pi(\operatorname{Re}A)=\operatorname{Re}[\sigma_\pi(A)]\subset\operatorname{Re}[Sp(A)],$$

where the first equality may be followed by the fact tnat $\operatorname{Re}(A)$ is a commuting tuple of normal operators. We shall prove $Sp(\operatorname{Re}A)\supset\operatorname{Re}(Sp(A))$ under an induction.

For $n=1$, the theorem holds (cf. [1], II, Theorem 3.2).

For $n\geqslant 2$, assume that it holds for a double commuting $(n-1)$-tuple of hyponormal operators. Then we shall prove that the theorem also holds for n. Let $\lambda=(\lambda_1,\cdots\lambda_n)\in Sp(A)$. It is well known that $\sum_{i=1}^{n}(A_i-\lambda_i)(A_i-\lambda_i)^*$ is not invertible[5].

The Berberian extension of $A=(A_1, \cdots, A_n)$ is denoted by $A^0=(A_1^0, \cdots, A_n^0)$[11]. It is easy to see that $A^0=(A_1^0, \cdots, A_n^0)$ is also a double commuting n-tuple of hyponormal operators, and we have

$$\mathrm{Ker}\Big(\sum_{i=1}^{n}(A_i^0-\lambda_i)(A_i^0-\lambda_i)^*\Big)=\bigcap_{i=1}^{n}\mathrm{Ker}(A_i^0-\lambda_i)(A_i^0-\lambda_i)^*\neq 0. \qquad (***)$$

Let

$$\mathscr{M}=\mathrm{Ker}(A_n^0-\lambda_n)(A_n^0-\lambda_n)^*=\mathrm{Ker}(A_n^0-\lambda_n)^*\neq 0.$$

Since A^0 is double commuting, $\mathscr{M}$ reduces $(A_1^0-\lambda_1), \cdots, (A_{n-1}^0-\lambda_{n-1})$, and $\mathrm{Re}(A_1^0-\lambda_1), \cdots, \mathrm{Re}(A_{n-1}^0-\lambda_{n-1})$. By (***) we have

$$0\neq\mathrm{Ker}\Big(\sum_{i=1}^{n-1}[(A_i^0-\lambda_i)|_{\mathscr{M}}][(A_i^0-\lambda_i)|_{\mathscr{M}}]^*=\mathscr{M}\cap\Big(\bigcap_{i=1}^{n-1}\mathrm{Ker}(A_i^0-\lambda_i)(A_i^0-\lambda_i)^*\Big).$$

Since $(A_1^0|_{\mathscr{M}}, \cdots, A^0|_{\mathscr{M}})$ is double commuting $(n-1)$-tuple of hyponormal operators, and $\mathrm{Re}(A_i^0|_{\mathscr{M}})=(\mathrm{Re}A_i^0)|_{\mathscr{M}}$, we see that $(\mathrm{Re}(A_1^0-\lambda_1)|_{\mathscr{M}}, \cdots, \mathrm{Re}(A_{n-1}^0-\lambda_{n-1})|_{\mathscr{M}})$ is not regular in the sense of Taylor's (by the assumption of the induction).

Howerer, if we restrict the n-tuple of operators

$$T=(\mathrm{Re}\ A_1^0-\mathrm{Re}\ \lambda_1, \cdots, \mathrm{Re}\ A_{n-1}^0-\mathrm{Re}\ \lambda_{n-1}, (A_n^0-\lambda_n)(A_n^0-\lambda_n)^*)$$

in the subspace $\mathscr{M}$, we shall see that $((\mathrm{Re}\ A_1^0-\mathrm{Re}\ \lambda_1|_{\mathscr{M}}, \cdots, (\mathrm{Re}\ A_{n-1}^0-\mathrm{Re}\ \lambda_{n-1})|_{\mathscr{M}}, 0)$ is singular, and so is T. It is well known that the n-tuple T is regular if and only if $\sum_{i=1}^{n}T_i^*T_i$ is regular, where $T=(T_1, \cdots, T_n)$ is a normal commuting n-tuple[8]. Hence we have

$$\Big(\bigcap_{i=1}^{n-1}\mathrm{Ker}(\mathrm{Re}\ A_i^0-\mathrm{Re}\ \lambda_i)\Big)\cap\mathrm{Ker}(A_n^0-\lambda_n)^*\neq 0.$$

Let $\mathscr{N}=\bigcap_{i=1}^{n-1}\mathrm{Ker}(\mathrm{Re}\ A_i^0-\mathrm{Re}\ \lambda_i)$. Then $\mathscr{N}$ reduces $(A_n^0-\lambda_n)^*$ and $\mathrm{Re}(A_n^0-\lambda_n)$. Since $\mathscr{N}\cap\mathscr{M}\neq 0$, $(A_n^0-\lambda_n)|_{\mathscr{N}}$ is hyponormal, it follows that $(A_n^0-\lambda_n)|_{\mathscr{N}}$ is singular, whence $\mathrm{Re}\ A_n^0|_{\mathscr{N}}-\mathrm{Re}\ \lambda_n=\mathrm{Re}(A_n^0-\lambda_n)|_{\mathscr{N}}$ is also singular. As before, we can say that $(\mathrm{Re}\ A_1^0-\mathrm{Re}\ \lambda_1, \cdots, \mathrm{Re}\ A_{n-1}^0-\mathrm{Re}\ \lambda_{n-1}, \mathrm{Re}\ A_n^0-\mathrm{Re}\ \lambda_n)$ is singular, because this tuple is singular on reduced subspace $\mathscr{N}$. Then, owing to the relations $\sigma_\pi(A_i)=\sigma_\pi(A_i^0)=\sigma_0(A_i^0)$, $\sigma_\pi(A)=\sigma_\pi(A^0)=\sigma_0(A^0)$, where $\sigma_0(A)$ denotes the point spectrum of A, we have

$$\mathrm{Re}\ \lambda=(\mathrm{Re}\ \lambda_1, \cdots, \mathrm{Re}\ \lambda_n)\in Sp(\mathrm{Re}A^0)=\sigma_\pi(\mathrm{Re}A^0)=\sigma_\pi(\mathrm{Re}\ A)=Sp(\mathrm{Re}\ A),$$

which proves this theorem. Q.E.D.

Corollary 2.5. *If $A=(A_1, \cdots, A_n)$ is a double commuting n-tuple of seminormal operators, then its joint spectrum has a Cartesian decomposition.*

Proof There exists a function ϕ: $\{1, 2, \cdots, n\}\to\{1, *\}$ such that $\phi(A)=(\phi(A_1), \phi(A_2), \cdots, \phi(A_n))$ is a double commution n-tuple of hyponormal operators, where $\phi(A_i)=A_i^{\phi(i)}$. By § 1, Theorem C and Theorem 2.4, we can come to the conclusion.

Definition 2.6. *If $A=(A_1, \cdots, A_n)$ is a commuting n-tuple of operators, we call*

$(A-\lambda)$ *for* $\lambda \notin Sp(A)$ *the joint resolvent of* A.

Lemma 2.7. (Muneo Chō)[7] *If* $A=(A_1, \cdots, A_n)$ *is a double commuting n-tuple of hyponormal operators, then for any* $\lambda=(\lambda_1, \cdots, \lambda_n)\in \mathscr{C}^n$ we have

$$\inf\left\{\left(\sum_{i=1}^{n}\|(A_i-\lambda_i)^* x\|^2\right)^{\frac{1}{2}}: \|x\|=1\right\}=\text{dist}(\lambda, Sp(A)).$$

Theorem 2.8. *If* $A=(A_1, \cdots, A_n)$ *is a double commuting n-tuple of hyponormal operators, then for any* $z=(z_1, \cdots, z_n)\notin Sp(A)$, *we have*

$$\|(A-z)^{-1}\|=[\text{dist}(z, Sp(A))]^{-1}.$$

Proof It is well known that (cf. [5], p. 135)

$$(\widehat{A-z})(\widehat{A-z})^*=\begin{bmatrix} \sum_{i=1}^{n}(A_i-z_i)(A_i-z_i)^* & 0 & 0 \\ 0 & \sum_{i=1}^{n}{}^{f_2}(A_i-z_i) & \\ 0 & & \ddots \quad \sum_{i=1}^{n}{}^{f_l}(A_i-z_i) \end{bmatrix},$$

where f_l: $(1, 2, \cdots, n)\to(0, 1)$.

$${}^{f_l}(A_i-z_i)=(A_i-z_i)^*(A_i-z_i) \text{ or } (A_i-z_i)(A_i-z_i)^*, \text{ if } f_l(i)=0 \text{ or } 1.$$

Since $A=(A_1, \cdots, A_n)$ is hyponormal, for any f: $(1, 2, \cdots, n)\to(0.1)$, we have

$$\sum_{i=1}^{u}{}^{f}(A_i-z_i)\geqslant\sum_{i=1}^{n}(A_i-z_i)(A_i-z_i)^*.$$

Thus

$$\begin{aligned}\|(A-z)^{-1}\|&=\|[(A-z)^{-1}]^*(A-z)^{-1}\|=\sup_l\left\|\left(\sum_{i=1}^{n}{}^{f_l}(A_i-z_i)\right)^{-1}\right\|\\ &=\left\|\left(\sum_{i=1}^{n}(A_i-z_i)(A_i-z_i)^*\right)^{-1}\right\|\\ &=\left(\inf\left\{\left(\sum_{i=1}^{n}(A_i-z_i)(A_i-z_i)^*x, x\right): \|x\|=1\right\}\right)^{-1}.\end{aligned}$$

By Lemma 2.7, we have

$$\|(\widehat{A-z})^{-1}\|^2=\left[\inf_{\|x\|=1}\sum_{i=1}^{n}\|(A_i-z_i)^*x\|^2\right]^{-1}=[\text{dist}(z, Sp(A)]^{-2}.$$

This completes the proof.

Q.E.D.

Now, we consider the seminormal operators. Let $A=(A_1, \cdots, A_n)$ be a double commuting n-tuple of seminormal operators, ϕ: $(1, 2, \cdots, n)\to(1, *)$, and $\phi(A_i)=A_i^{\phi(i)}$. We set $\phi(A)=(\phi(A_1), \cdots, \phi(A_n))$. Curto[5] showed that $Sp(\phi(A), H)=\{\phi(\lambda): \lambda\in Sp(A)\}$. If p is a permutation of $(1, 2, \cdots, n)$, $p(A)=(A_{p(1)}, \cdots, A_{p(n)})$, then $\hat{A}=U\widehat{p(A)}V$, U, V are unitary operators (cf. [5], p. 137). Thus, $\|\hat{A}\|=\|\widehat{p(A)}\|$. If A exsists, then $p(A)$ also has an inverse (cf. [3]).

Theorem 2.9. *If* $A=(A_1, \cdots, A_n)$ *is a double commuting n-tuple of seminormal operators, then for any* $z=(z_1, \cdots, z_n)\notin Sp(A)$, *we have*

$$\|(\widehat{A-z})^{-1}\|=\frac{1}{\operatorname{dist}(z, Sp(A))}.$$

Proof There exists a mapping ϕ: $\{1, 2, \cdots, n\}\to\{1, *\}$ such that $\phi(A)$ is a double commuting of hyponormal operators. We can prove that $\|\hat{A}^{-1}\|=\|\widehat{\phi(A)}^{-1}\|$. In fact, it is sufficient to show that for ϕ_1: $(A_1, \cdots A_n)\to(A_1^*, A_2, \cdots, A_n)$, $\|\hat{A}\|=\|\widehat{\phi_1(A)}\|$. Now, we set ϕ_2: $(A_1, A_2, \cdots, A_n)\to(A_1^*, -A_2, \cdots, -A_n)$. Then $(\hat{A})^*=\widehat{\phi_2(A)}$, $\widehat{\phi_1(A)}(\widehat{\phi_1(A)})^*=\widehat{\phi_2(A)}(\phi_2(A))^*$. Thus, $\|\widehat{\phi_1(A)}\|=\|\widehat{\phi_2(A)}\|=\|\hat{A}\|$. Similarly, we can also prove that $\|\widehat{\phi_1(A)}^{-1}\|=\|\hat{A}^{-1}\|$ if A has an inverse. By Theorem 2.8, we have $\|\widehat{\phi_1(A-z)}^{-1}\|=[\operatorname{dist}(\phi(z), Sp(\phi(A))]^{-1}$. Since $Sp(\phi(A))=\phi(Sp(A))$, we have

$$\operatorname{dist}(z, Sp(A))=\operatorname{dist}(\phi(z), \phi[Sp(A)])=\operatorname{dist}(\phi(z), Sp(\phi(A)).$$

On the other hand, $\|(\widehat{A-z})^{-1}\|=\|\phi(\widehat{A-z})^{-1}\|$. Thus

$$\|(\widehat{A-z})^{-1}\|=[\operatorname{dist}(z, Sp(A))]^{-1}.$$

Q.E.D

§ 3. The Normaloid Property of an n-Tuple of Semi-hyponormal Operators

Theorem 3.1. *If $A=(A_1, \cdots, A_n)$ is a double commuting n-tuple of semi-hyponormal operators, then $S_p(A)=\sigma_\rho(A)$, where $\sigma_\rho(A)$ is the joint compressive spectrum of A.*

Proof It is sufficient to prove $Sp(A)\subset\sigma_\rho(A)$. If $Z=(Z_1, \cdots, Z_n)\in Sp(A)$, then $(A-Z)$ is not invertible (§ 1. Theorem B). Clearly, $(\widehat{A-Z})^*(\widehat{A-Z})$ and $(\widehat{A-Z})\cdot(\widehat{A-Z})^*$ are diagonal metrices on the space $H\otimes C^{2n-1}$ with diagonal entries $\sum_{i=1}^{n} {}^{f_i}(A_i-Z_i)$. Hence we can find an operator

$$\sum_{i=1}^{n} {}^{f_{i0}}(A_i-Z_i)=\sum_t(A_t-Z_t)^*(A_t-Z_t)+\sum_s(A_s-Z_s)(A_s-Z_s)^*$$

which has no inverse. Thus, there are $\{x_n\}\subset H$, $\|x_n\|=1$, such that

$$\left(\sum_{i=1}^{n} {}^{f_{i0}}(A_i-Z_i)\,x_m, x_m\right)=\sum_t\|(A_t-Z_t)x_m\|^2+\sum_s\|(A_s-Z_s)x_m\|^2\to 0.$$

By [1], Theorem. I 2.5, $\|(A_t-Z_t)x_m\|\to 0$ implies $\|(A_t-Z_t)^*x_m\|\to 0$. Hence $\|(A_i-Z_i)^*x_m\|\to 0$ $(m\to\infty)$, $i=1, 2, \cdots n$. Therefore, $Z\in\sigma_\rho(A)$. This completes the proof.

Q. E. D.

Muneo Chō and Makoto Takaguchi have proved that every double commuting n-tuples of hyponormal operators satisfies $r_{sp}(A)=\|A\|$. In the case of semi-hyponormal operators, we conjecture that it remains true. But we now only prove two particular cases.

Corollary 3.2. *If $A=(A_1, \cdots, A_n)$ is a double commuting n-tuple of semi-hyponormal operators, $A_i=U_i|A_i|$, where U_i are unitary,* $\dim\mathscr{R}(A)^\perp=\dim\mathscr{R}(A^*)^\perp$,

$\sigma(U_i)\neq C_1$, $i=1, 2, \cdots, n$, *then*

$$r_{sp}(A)=\|A\|.$$

Proof By Theorem 3.1, it is easy to see that $A-Z$ is invertible iff $\sum_{i=1}^{n}(A_i-Z_i)\cdot(A_i-Z_i)^*$ is invertible. For any $r_i\in\sigma(A_i^*A_i)$, we can find $Z_i\in\sigma(A_i)$ such that $|Z_i|=\sqrt{r_i}$ (cf. [1], II, Theorem 3.3). Now, we may apply the proof which was used by M. Chō and M. Takaguchi[7] to the case of semi-hyponormal operatorss. Then the assertion will be proved.

Q. E. D.

If B is an isometric operator, we set

$$B^{[n]}=\begin{cases}B^n, & n\geqslant 0,\\ (B^*)^n, & n<0.\end{cases}$$

If $\mathscr{T}_B^{\pm}(T)=s\text{-}\lim\limits_{n\to\pm\infty}B^{[n]}TB^{[-n]}$ exists, we say that $\mathscr{T}_B^{\pm}(T)$ is a polar symbol of T relative to B. Set

$$\mathscr{T}_B^{\pm}=\{T\,|\,T\in B(H),\mathscr{T}_B^{\pm}(T)\text{ exists}\}^{[1]}.$$

Lemma 3.3. *If U is a unitary operator, $T=(T_1, T_2, \cdots T_n)$, $T_i\in\mathscr{T}_U^{\pm}\cap(\mathscr{T}_U^{\pm})^*$, $i=1, 2, \cdots, n$. Denote $\mathscr{T}_U^{\pm}(T)=(\mathscr{T}_U^{\pm}(T_1), \cdots, \mathscr{T}_U^{\pm}(T_n))$. Then*

$$\sigma_\pi(\mathscr{T}_U^{\pm}(T))\subset\sigma_\pi(T),\quad \sigma_\rho(\mathscr{T}_U^{\pm}(T))\subset\sigma_\rho(T).$$

Proof We know that $A\to\mathscr{T}_B^{\pm}(A)$ is an involution homomorphism from $\mathscr{T}_U^{\pm}\cap(\mathscr{T}_U^{\pm})^*$ to $[B]'$ (cf. [1]). On the other hand, we have

$$\lambda=(\lambda_1,\cdots,\lambda_n)\notin\sigma_\pi(T)$$

iff there is $\varepsilon_1>0$ such that

$$\sum_{i=1}^{n}(T_i-\lambda_i)^*(T-\lambda_i)>\varepsilon_1 I,$$

$$\lambda=(\lambda_1, \cdots, \lambda_n)\notin\sigma_\rho(T)$$

iff there is $\varepsilon_2>0$ such that

$$\sum_{i=1}^{n}(T_i-\lambda_i)(T_i-\lambda_i)^*>\varepsilon_2 I.$$

Then we can establish the lamma.

Theorem 3.4. *If $T=(T_1, T_2, \cdots, T_n)$ is a double commuting n-tuple of semi-hyponormal operators, and if $0\notin\sigma(T_i)$ $i=1, 2, \cdots, n$, then we have*

(i) $r_{sp}(T)=\|T\|$;

(ii) *Let $T^{-1}=(T_1^{-1}, T_2^{-1}, \cdots, T_n^{-1})$, then*

$$\|T^{-1}\|=r_{sp}(T^{-1})=\sup\left\{\left(\sum_{i=1}^{n}\frac{1}{|\lambda_i|^2}\right)^{\frac{1}{2}}: (\lambda_1, \cdots, \lambda_n)\in Sp(T)\right\}.$$

Proof Let $T_i=U_i|T_i|$ be the polar decomposition of T_i, $i=1, 2, \cdots, n$. Since $0\notin\sigma(T_i)$, we see that U_i is unitary, and $|T_i|$ is invertible. It is easy to see that $(U_1, \cdots, U_n)$ and $(|T_1|, \cdots, |T_n|)$ are double commuting and $U_i|T_j|=|T_j|U_i$, $i\neq j$. Let $U=U_1\cdots U_n$, where U is unitary. Since T_i are semi-hyponormal, we have

$$\mathscr{T}_U^{\pm}(T_i) = s\text{-}\lim_{n\to\mp\infty} U^{[n]}T_iU^{[-n]} = s\text{-}\lim_{n\to\mp\infty} U_i^{[n]}T_iU_i^{[-n]} = \mathscr{T}_{U_i}^{\pm}(T_i).$$

Since U_i are unitary, we have $\mathscr{T}_U^{\pm}(T_i^*) = (\mathscr{T}_U^{\pm}(T_i))^*$, $|T_i| \leqslant \mathscr{T}_U^{\pm}(|T_i|)$[1]. Then $\mathscr{T}_U^{\pm}(T) = (\mathscr{T}_U^{\pm}(T_1), \cdots, \mathscr{T}_U^{\pm}(T_n))$ is a commuting n-tuple of normal operators, and $\mathscr{T}_U^{\pm}(T_i) = U_i\mathscr{T}_U^{\pm}(|T_i|)$, $i=1, 2, \cdots, n$. Now the following equality holds:

$$\|T\|^2 = \sup_{\|x\|=1}\left(\sum_{i=1}^n \|T_ix\|^2\right) = \sup_{\|x\|=1}\left(\sum_{i=1}^n (T_i^*T_ix, x)\right) = \sup_{\|x\|=1}\left(\sum_{i=1}^n \||T_i|x\|^2\right) = \||T_i|\|^2.$$

Thus $\|T\| = \||T|\|$.

Now, since $|T| = (|T_1|, \cdots, |T_n|)$ is a commuting n-tuple of normal operators, we have $\omega(|T|) = \||T|\| = \|T\|$[6]. Put $\mathscr{T}_U^{+}(|T|) = (\mathscr{T}_U^{+}(|T_1|), \cdots, \mathscr{T}_U^{+}(|T_n|))$. Then $\|T\| = \omega(|T|) \leqslant \omega(\mathscr{T}_U^{+}(|T|))$ by the inequlity $|T_i| \leqslant \mathscr{T}_U^{+}(|T_i|)$, $i=1, 2, \cdots, n$. On the other hand

$$\begin{aligned}\omega(\mathscr{T}_U^{+}(|T|)) &= \sup_{\|x\|=1}\left(\sum_{i=1}^n (\mathscr{T}_U^{+}(|T_i|)x, x)^2\right)^{\frac{1}{2}} \\ &\leqslant \sup_{\|x\|=1}\left(\sum_{i=1}^n \|\mathscr{T}_U^{+}(|T_i|)x\|^2\right)^{\frac{1}{2}} \\ &\leqslant \sup_{\|x\|=1}\left(\sum_{i=1}^n (\mathscr{T}_U^{+}(|T_i|^2)x, x)\right)^{\frac{1}{2}} \\ &= \sup_{\|x\|=1}\left(\mathscr{T}_U^{+}\left(\sum_{i=1}^n |T_i|^2\right)x, x\right)^{\frac{1}{2}} \leqslant \left\|\sum_{i=1}^n |T_i|^2\right\|^{\frac{1}{2}} \\ &= \left\|\sum_{i=1}^n T_i^*T_i\right\|^{\frac{1}{2}} = \omega\left(\sum_{i=1}^n T_i^*T_i\right)^{\frac{1}{2}} = \|T\|.\end{aligned}$$

Thus

$$\omega(\mathscr{T}_U^{+}(|T|)) = \|T\|.$$

Since $\mathscr{T}_U^{+}(|T|) = (\mathscr{T}_U^{+}(|T_1|), \cdots, \mathscr{T}_U^{+}(|T_m|))$ is a commuting n-tuple of normal operators, we see that the convex hull of $Sp(\mathscr{T}_U^{+}(|T|))$ is the closure of $W(\mathscr{T}_U^{+}(|T|))$[6]. Thus, we can find $r = (r_1, \cdots, r_n) \in Sp(\mathscr{T}_U^{+}(|T|))$ such that $|r| = \|T\|$. By the continuous spectral mapping theorem of commuting n-tuples of normal operators[2], and by $\mathscr{T}_U^{+}(T_i) = U_i\mathscr{T}_U^{+}(|T_i|)$ (the polar decomposition of $\mathscr{T}_U^{+}(T_i)$), we can see that there exists $Z = (Z_1, \cdots, Z_n) \in Sp(\mathscr{T}_U^{+}(T))$ such that $|Z_i| = r_i$, $i=1, 2, \cdots, n$. Lemma 3.3 shows that

$$Z \in Sp(\mathscr{T}_U^{+}(T)) = \sigma_\pi(\mathscr{T}_U^{+}(T)) \subset \sigma_\pi(T).$$

Thus $r_{sp}(T) = \|T\|$.

(ii) $T^{-1} = (T_1^{-1}, \cdots, T_n^{-1})$ is also a double commuting n-tuple of semi-hyponormal operators. By (i), we have $\|T^{-1}\| = r_{sp}(T^{-1})$. Since $0 \notin \sigma(T_i)$, $i=1, 2, \cdots, n$. we see that $\left(\frac{1}{Z_1}, \cdots, \frac{1}{Z_n}\right)$ is an analytic mapping on the neighbourhood of $Sp(T)$. By § 1, Theorem A, we have

$$Sp(T^{-1}) = \{(\lambda_1^{-1}, \cdots, \lambda_n^{-1}) \mid \lambda = (\lambda_1, \cdots, \lambda_n) \in Sp(T)\}.$$

Thus

$$r_{sp}(T^{-1}) = \sup_{\lambda \in S_p(T)} \left(\sum_{i=1}^{n} \frac{1}{|\lambda_i|^2} \right)^{\frac{1}{2}}.$$

Q.E.D.

Finally, if $T=(T_1, T_2, \cdots, T_n)$ is a double commuting n-tuple of operators, where T_i are semi-hyponormal or semi-cohyponormal, then Corollary 3.2 and Theorem 3.4 are also true. The proof is omitted.

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