Chin. Ann. of Math.
7B (1) 1986

ON THE FIRST KIND OF RELATIVE k-JET COHOMOLOGY OF SINGULARITIES OF MAPGERMS

XIAO ERJIAN (肖尔健)*

Abstract

In this paper the author generalizes the computations about the first kind of k-jet cohomology in [5] to mapgerms. The main results are as follows:

$$H^{\circ}(\Omega^{\cdot}_{\varphi,k-\cdot,x})=\underbrace{\mathcal{O}_{M,x}\oplus\cdots\oplus\mathcal{O}_{M,x}}_{\binom{\cdot+k}{k}},$$

$$H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})=0,\ 0<p<m-\dim\mathcal{O}_{M,x}/I(\varphi)_x-1 \text{ or } p=m.$$

There exists an integer s, such that

$$(I(\varphi)_x)^s H^p(\Omega_{\varphi,k-\cdot,x}),\ m-\dim\mathcal{O}_{M,x}/I(\varphi)_x-1\leqslant p\leqslant m-1.$$

Hence, $H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})$ are finitely generated $\mathcal{O}_{M,x}/(I(\varphi)_x)^s$-modules. If $\dim_{\mathbf{C}}\mathcal{O}_{M,x}/I(\varphi)_x<\infty$, then

$$H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})=0,\ 0<p<m-1 \text{ or } p=m,$$

$$\dim_{\mathbf{C}} H^{m-1}(\Omega^{\cdot}_{\varphi,k-\cdot,x})=\sum_{r=1}^{k-m}\dim_{\mathbf{C}}\Omega^{m,r}_{\varphi,k-r-m,x}<\infty.$$

In [5] we define two kinds of relative k-jet cohomology and compute them for C^∞ hypersurfaces with finite multiplicities at singular points. Professor Greuel posed the following problem: which results in [5] can be generalized to complete intersections with isolated singularities. In this paper we generalize the computations about the first kind of k-jet cohomology in [5] to holomorphic mapgerms, real analytic mapgerms and finitely determined C^∞ mapgerms on respective manifolds. All results except Theorem 4 are true not only for isolated singularities but also for nonisolated singularities.

Let $K=\mathbf{C}$ (or $\mathbf{R}$). M is an m-dimensional complex manifold (or real analytic manifold). $\mathcal{O}_M$ is the sheaf of germs of holomorphic (or real analytic) functions on M. $\mathcal{O}_{M,x}$ is the stalk of $\mathcal{O}_M$ at x. $J_{M,k}$ is the sheaf of germs of holomorphic (or real analytic) cross sections of the k-jet, $k<\infty$, bundle on M. $J_{M,k,x}$ is the stalk of $J_{M,k}$ at x. Let $U\subset M$ be an open set with local system of coordinates $(x_1, \cdots, x_m)$, then $J_{M,k}(U)=\sum_{|\alpha|\leqslant k}\mathcal{O}_M(U)\xi^\alpha$, where $\alpha=(\alpha_1, \cdots, \alpha_m)$'s are multi-indexes and $\xi_1, \cdots, \xi_m$

Manuscript received February 26, 1983.

* Institute of Mathematics, Fudan University, Shanghai, China.

are formal coordinates of $J_{M,k}$. The differential form sheaf with respect to $\xi_1, \cdots, \xi_m$ is $\Omega_{M,k-1}$. $\Omega_{M,k-1}(U)=\sum_{i=1}^{m} J_{M,k-1}(U)D\xi_1$, where $D\xi_1, \cdots, D\xi_m$ are formal differentials (see [5] for details). $\Omega_{M,k-1}$ is a locally free $J_{M,k-1}$ module with locally free basis $D\xi_1, \cdots, D\xi_m$. Let $\Omega^p_{M,k-p}=\Lambda^p\Omega_{M,k-p}$. We get a sheaf complex

$$\Omega^{\cdot}_{M,k-\cdot}: J_{M,k}\xrightarrow{D}\Omega^1_{M,k-1}\to\cdots\to\Omega^{p-1}_{M,k-p+1}\xrightarrow{D}\Omega^p_{M,k-p}\to\cdots\to\Omega^m_{M,k-m}.$$

If $F\in J_{M,k}(U)$, $F=\sum_{|\alpha|\le k}F^\alpha(x)\xi^\alpha$, $F^\alpha(x)\in\mathscr{O}_M(U)$,

$$DF=\sum_{i=1}^{m}\frac{\partial F}{\partial\xi_1}D\xi_i=\sum_{i=1}^{m}\Big(\sum_{|\alpha|\le k-1}\frac{1}{\alpha_i+1}F^{\alpha+1_i}(x)\xi^\alpha\Big)D\xi_i,$$

where $\alpha+1_i=(\alpha_1, \cdots, \alpha_i+1, \cdots, \alpha_m)$.

Let $j_k:\mathscr{O}_M\to J_{M,k}$ be $j_k f=\sum_{|\alpha|\le k}\frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}\xi^\alpha$, $f\in\mathscr{O}_M$.

Let N be an n-dimensional complex (or real analytic) manifold, there are $\mathscr{O}_N$, $J_{N,k}$ and $\Omega^p_{N,k-1}$. If V is an open set on N with local system of coordinates $y_1, \cdots, y_n$. $G\in J_{N,k}(V)$,

$$G=\sum_{|\beta|\le k}G^\beta(y)\xi^\beta,\ \Omega_{N,k-1}(V)=\sum_{i=1}^{n}J_{N,k-1}(V)D\eta_i.$$

$\varphi: M\to N$ is a holomorphic (or real analytic) mapping. $\varphi^*\Omega_{N,k-p}$ is the pullback of $\Omega_{N,k-p}$ by φ. Let

$$\Omega^1_{\varphi,k-1}=\Omega^1_{M,k-1}/J_{M,k-1}\varphi^*\Omega^1_{N,k-1},$$
$$\Omega^p_{\varphi,k-p}=\Lambda^p\Omega^1_{\varphi,k-p}=\Omega^p_{M,k-p}/J_{M,k-p}\varphi^*\Omega_{N,k-p}\Lambda\Omega^{p-1}_{M,k-p}.$$

We have a sheaf complex

$$\Omega^{\cdot}_{\varphi,k-\cdot}: J_{M,k}\xrightarrow{D}\cdots\to\Omega^{p-1}_{\varphi,k-p+1}\xrightarrow{D}\Omega^p_{\varphi,k-p}\to\cdots\to\Omega^m_{\varphi,k-m}.$$

We will compute cohomology groups $H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})$ in the following, where

$$H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})=\mathrm{Ker}(\Omega^p_{\varphi,k-p,x}\xrightarrow{D}\Omega^{p+1}_{\varphi,k-p-1,x})/\mathrm{Im}(\Omega^{p-1}_{\varphi,k-p+1,x}\xrightarrow{D}\Omega^p_{\varphi,k-p,x}).$$

Suppose $\varphi(V)\subset U$, $\varphi|_V=(f_1, \cdots, f_n)$, $f_1, \cdots, f_n\in\mathscr{O}_M(U)$. Let $I(\varphi)$ be the ideal sheaf of $\mathscr{O}_M$ generated by the coefficients of $df_1\wedge\cdots\wedge df_n$ where d's are ordinary differentials and $C(\varphi)=\mathscr{O}_M/I(\varphi)$. Let $F_i=j_pf_i$, $i=1, \cdots, n$. $I_k(\varphi)$ is the ideal sheaf or $J_{M,k}$ generated by the coefficients of $DF_1\wedge\cdots\wedge DF_n$ and $C_k(\varphi)=J_{M,k}/I_k(\varphi)$.

Notations: For $G\in J_{M,k,x}$ write $G=\sum_{l=0}^{k}G^l$ the homogeneous expansion with respect to ξ such that $G^0\in\mathscr{O}_{M,x}$ denotes the initial term,

$\mathscr{O}_{M,x}$ is an m-dimensional Cohen-Macaulay ring. $\mathrm{depth}_{I(\varphi)_x}\mathscr{O}_{M,x}=m-\dim C(\varphi)_x$.

Lemma 1. a) *$J_{M,k,x}$ is an m-dimensional Cohen-Macaulay ring* b) *If $G_1, \cdots, G_1\in J_{M,k,x}$ is a $J_{M,k,x}$-regular sequence, $G^0_1, \cdots, G^0_l$ is an $\mathscr{O}_{M,x}$-regular sequence.*

Proof a) By Lemma 1 in [5], if $G_1, \cdots, G_m\in J_{M,k,x}$ such that $G^0_1,\cdots, G^0_m$ is an $\mathscr{O}_{M,x}$-regular sequence, $\dim_K J_{M,k,x}/(G_1, \cdots, G_m)<\infty$. Hence $G_1, \cdots, G_m$ is a system of parameters and dim $J_{M,k,x} \leqslant m \leqslant$ depth $J_{M,k,x}$. But depth $J_{M,k,x} \leqslant$ dim $J_{M,k,x}$. a) follows.

If $G_1, \cdots, G_l$ is a $J_{M,k,x}$-regular sequence, there exist $G_{l+1}, \cdots, G_m \in J_{M,k,x}$ such that $G_1, \cdots, G_m$ is a system of parameters of $J_{M,k,x}$ (see [3]). $\dim_K J_{M,k,x}/(G_1, \cdots, G_m) < \infty$. Hence $\dim_K \mathcal{O}_{M,x}/(G_1^0, \cdots, G_m^0) < \infty$ and $G_1^0, \cdots, G_m^0$ is an $\mathcal{O}_{M,x}$-regular sequence. b) follows.

Remark. By Lemma 1 b) and Lemma 1 a) in [5], $G_1, \cdots, G_m \in J_{M,k,x}$ is a $J_{M,k,x}$-regular sequence if and only if $G_1^0, \cdots, G_m^0$ is an $\mathcal{O}_{M,x}$ -regular sequence.

Corollary. m-dim $C(\varphi)_x = \text{depth}_{I_k(\varphi)_x} J_{M,k,x} = m$-dim$C_k(\varphi)_x$.

G.-M. Greuel[1] and K. Saito[4] proved the following generalized de-Rham lemma. We write it into the form we need here.

Generalized de-Rham lemma: R is a noetherian commutative ring. A is a free R-module with finite rank. $A^p = \wedge^p A$ the p-th exterior product of A. $\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_k \in A$. I is an ideal of A generated by the coefficients of $\lambda_1 \wedge \cdots \wedge \lambda_r \wedge \mu_1 \wedge \cdots \wedge \mu_k$. Let

$$A^p \xrightarrow{\mu_1 \wedge \cdots \wedge \mu_k \wedge} A^{p+k},$$

$$Z^p = \left\{ \omega \in A^p \,|\, \mu_1 \wedge \cdots \wedge \mu_k \wedge \omega \in \sum_{i=1}^{r} \lambda_i \wedge A^{p+k-1} \right\},$$

$$H^p = Z^p \Big/ \sum_{i=1}^{r} \lambda_i \wedge A^{p-1} + \sum_{j=1}^{k} \mu_j \wedge A^{p-1}.$$

Then a) There exist s an integer $s \geqslant 0$, such that

$$I^s H^p = 0.$$

b) If $0 \leqslant p < \text{depth}_I R$,

$$H^p = 0.$$

Notations: B is an R-module. Let $T_N(B) = \{x \in B \,|\,$ there is a nonzero divisor $a \in R$, $ax = 0\}$.

$\mathcal{O}_{M,x}$ is an integer domain. $G \in J_{M,k,x}$. If $G^0 \neq 0$, G is not a zero divisor. If $G^0 = 0$, let $L \in J_{M,k,x}$, $L^1 = 0$, $1 < k$. Then $GL = 0$. Hence G is not a zero divisor if and only if $G^0 = 0$.

The mapping $DF: \Omega^p_{\varphi,k,x} \to \Omega^{p+n}_{M,k,x}$ is defined as follows: $\omega \in \Omega^p_{\varphi,k,x}$, $DF(\omega) = DF_1 \wedge \cdots \wedge DF_n \wedge \omega \in \Omega^{p+n}_{M,k,x}$.

Proposition. $T_N(\Omega^p_{\varphi,k,x}) = \text{Ker}(DF)$.

Proof $\omega \in T_N(\Omega^p_{\varphi,k,x})$. There exists a nonzero divisor $a \in J_{M,k,x}$ such that $a\omega = 0$. $aDF(\omega) = 0$. $\Omega^{p+n}_{M,k,x}$ is a free $J_{M,k,x}$-module. Hence $DF(\omega) = 0$.

By generalized de-Rham lamma, $I_k^s(\varphi)$ Ker$(DF) = 0$. There are nonzero divisors in $I_k^s(\varphi)$. Hence Ker$(DF) \subset T_N(\Omega^p_{\varphi,k,x})$.

Corollary. *For* $m-n+1 \leqslant p \leqslant m$, $\Omega^p_{\varphi,k,x} = T_N(\Omega^p_{\varphi,k,x})$.

Lemma 2. *R is a noetherian commutative ring. A is a free R-module of finite* *rank.* $A^p=\wedge^p A$. $\theta_1, \cdots, \theta_n \in A$. *I is an ideal of R generated by the coefficients of* $\theta_1\wedge\cdots\wedge\theta_n$. *Suppose* $\omega_{i_1\cdots i_r}\in A^p$, $i_1, \cdots, i_r=1, \cdots n$, *are symmetric with respect to the lower indexes* $i_1, \cdots, i_r$ *and*

$$\sum_{a=1}^{n}\theta_a\wedge\omega_{i_1\cdots i_{r-1}a}=0.$$

Then 1) *there exists a positive integer s such that for any* $h\in I$, *there are* $\omega_{i_1\cdots i_r i_{r+1}}\in A^{p-1}, i_1, \cdots, i_{r+1}=1, \cdots, n$, *symmetric with respect to* $i_1, \cdots, i_{r+1}$ *and*

$$h^s\omega_{i_1\cdots i_r}=\sum_{a=1}^{n}\theta_a\wedge\omega_{i_1\cdots i_r a}.$$

2) *If* $p<\mathrm{depth}_I R$,

$$\omega_{i_1\cdots i_r}=\sum_{a=1}^{n}\theta_a\wedge\omega_{i_1\cdots i_r a},$$

where $\omega_{i_1\cdots i_r i_{r+1}}\in A^{p-1}$, $i_1, \cdots, i_{r+1}=1, \cdots, n$, *are symmetric with respect to* $i_1, \cdots, i_{r+1}$.

Proof 1) We prove it by induction on n. Let $h\in I$.

The case $n=1$ follows immediately from generalized de-Rham lemma.

Suppose it is true for n-1.

Case n. We use induction on r.

$r=1$. $\sum_{a=1}^{n}\theta_a\wedge\omega_a=0$ implies $\theta_n\wedge\omega_n=-\sum_{a=1}^{n-1}\theta_a\wedge\omega_a$. By generalized de-Rhan lemma there exists an integer u and $\bar{\omega}_{na}\in A^{p-1}$, $a=1, \cdots, n-1$, such that

$$h^u\cdot\omega_n=\sum_{a=1}^{n}\theta_a\wedge\bar{\omega}_{na}.$$

Substitute it into

$$\sum_{a=1}^{n-1}\theta_a\wedge h^u\omega_a=0.$$

We get

$$\sum_{a=1}^{n-1}\theta_a\wedge(h^u\omega_a-\theta_n\wedge\bar{\omega}_{na})=0.$$

By the induction hypothesis on $n-1$ and $I\subset$the ideal generated by the coefficients of $\theta_1\wedge\cdots\wedge\theta_{n-1}$, there exists an integer u' and $\omega_{i_1i_2}\in A^{p-1}$, $i_1, i_2=1, \cdots, n-1$, symmetric with respect to i_1 and i_2 such that

$$h^{u'}(h^u\omega_{i_1}-\theta_n\wedge\bar{\omega}_{ni_1})=\sum_{a=1}^{n-1}\theta_a\wedge\omega_{i_1a}.$$

Let $\omega_{ni_1}=h^{u'}\bar{\omega}_{ni_1}$, $u+u'=t$. We have

$$h^t\omega_{i_1}=\sum_{a=1}^{n}\theta_a\wedge\omega_{i_1a}.$$

Suppose it is true for $r-1$.

Case r: Take $i_1=n$. We get

$$\sum_{a=1}^{n}\theta_a\wedge\omega_{ni_2\cdots i_{r-1}a}=0.$$

By induction hypothesis on $r-1$,

$$h^u\omega_{ni_2\cdots i_{r-1}i_r}=\sum_{a=1}^{n}\theta_a\wedge\bar{\omega}_{ni_2\cdots i_r a}.$$

Then for $i_1,\cdots,i_{r-1}\leqslant n-1$,

$$\sum_{a=1}^{n-1}\theta_a\wedge h^u\omega_{i_1\cdots i_{r-1}a}+\theta_n\Big(\sum_{a=1}^{n}\theta_a\wedge\bar{\omega}_{i_1\cdots i_{r-1}na}\Big)=0.$$

$$\sum_{a=1}^{n-1}\theta_a\wedge(h^u\omega_{i_1\cdots i_{r-1}a}-\theta_n\wedge\bar{\omega}_{i_1\cdots i_{r-1}na})=0.$$

By the induction hypothesis on $n-1$, we get

$$h^{u'}(h^u\omega_{i_1\cdots i_{r-1}i_r}-\theta_n\wedge\bar{\omega}_{i_1\cdots i_{r-1}ni_r})=\sum_{a=1}^{n-1}\theta_a\wedge\omega_{i_1\cdots i_r a},$$

where $i_1,\cdots,i_r\leqslant n-1$. Let $h^{u'}\bar{\omega}_{ni_1\cdots i_{r-1}i_r}=\omega_{ni_1\cdots i_r}$ and $t=u+u'$. We get

$$h^t\omega_{i_1\cdots i_r}=\sum_{a=1}^{n}\theta_a\wedge\omega_{i_1\cdots i_r a},$$

where $i_1,\cdots,i_r=1,\cdots,n$ and $\omega_{i_1\cdots i_{r+1}}\in A^{p-1}$ are symmetric with respect to $i_1,\ \cdots,\ i_{r+1}$. Because I is finitely generated, we can get the desired s.

2) A similar but simpler induction can prove 2).

We define

$$A^{p,r}=\underbrace{A^p\oplus\cdots\oplus A^p}_{\binom{n+r-1}{r}\text{ copies}}$$

and

$$\psi^{p,r}:A^{p,r}\to A^{p+1,r-1},$$

$$\psi^{p,r}(\{\omega_{i_1\cdots i_r}\})=\Big\{\sum_{a=1}^{n}\theta_a\wedge\omega_{i_1\cdots i_{r-1}a}\Big\},$$

where $\{\omega_{i_1\cdots i_r}\}\in A^{p,r}$, $i_1,\cdots,i_r=1,\cdots,n$, are symmetric with respect to $i_1\cdots,i_r$.

It is clear that $\psi^{p-1,r+1}\psi^{p,r}=0$. Hence we get a complex $A^{\cdot,p-}$:

$$A^{0,p}\xrightarrow{\psi^{0,p}}\cdots\to A^{q\ p-q}\xrightarrow{\psi^{q,p-q}}A^{q+1,p-q-1}\to\cdots\to A^{p,0}=A^p.$$

We define

$$R^q(A^{\cdot,p-\cdot})=\operatorname{Ker}\psi^{q,p-q}/\operatorname{Im}\psi^{q-1,p-q+1}.$$

Corollary. *$I^sR^q(A^{\cdot,p-\cdot})=0$, for some positive integer s, and $R^q(A^{\cdot,p-\cdot})=0$, for $q<\operatorname{depth}_I R$.*

Let $A=\Omega^1_{M,k-p,x}$. We have complexes

$$\Omega^{\cdot,p-\cdot}_{M,k-p,x}:\Omega^{0,p}_{M,k-p,x}\to\cdots\to\Omega^{q,p-q}_{M,k-p,x}\xrightarrow{\psi^{q,p-q}}\Omega^{q+1,p-q-1}_{M,k-p,x}\to\cdots\to\Omega^{p,0}_{M,k-p,x}.$$

Let

$$D^{q,p-q}:\Omega^{q,p-p}_{M,k-p,x}\to\Omega^{q+1,p-q}_{M,k-p-1,x},$$

$$D^{q,p-q}(\{\omega_{i_1\cdots i_{p-q}}\})=\{D\omega_{i_1\cdots i_{p-q}}\}.$$

Clearly $D^{p,0}=D$.

Lemma 3. *$D^{q+1,p-q}D^{q,p-q}=0$ and $D^{q+1,p-q-1}\psi^{q,p-q}+\psi^{q+1,p-q}D^{q,p-q}=0$.*

We have complexes

$$\Omega^{\cdot,r}_{M,k-r-\cdot,x}:\Omega^{0,r}_{M,k-r,x}\xrightarrow{D^{0,r}}\cdots\to\Omega^{q,r}_{M,k-r-q,x}\xrightarrow{D^{q,r}}\Omega^{q+1,r}_{M,k-r-q-1,x}\to\cdots\to\Omega^{m,r}_{M-r-m,x}.$$

Clearly

$$\Omega^{\cdot,0}_{M,k-\cdot,x}=\Omega^{\cdot}_{M,k-\cdot,x}.$$

Lemma 4. $H^q(\Omega^{\cdot,r}_{M,k-r-\cdot,x})=0,\ q\geqslant 1,$

$$H^0(\Omega^{\cdot,r}_{M,k-r-\cdot,x})=\underbrace{\mathcal{O}_{M,x}\oplus\cdots\oplus\mathcal{O}_{M,x}}_{\binom{n+r-1}{r}\text{ copies}}.$$

Proof It follows immediately from Poincare lemma.

We define a double complex

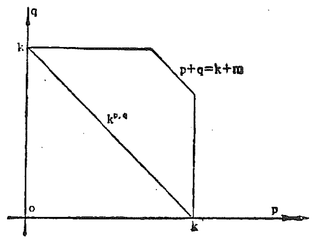
$$K^{p,q}=\begin{cases}\Omega^{p+q-k,k-q}_{M,k-p,x}, & k\leqslant p+q\leqslant k+m,\ 0\leqslant p\leqslant k, 0\leqslant q\leqslant k,\\ 0, & \text{otherwise},\end{cases}$$

and $'d^{p,q}=D^{p+q-k,k-q}$, $''d^{p,q}=\psi^{p+q-k,k-q}$.

$$''E_1^{q,p}={}'H^p(K^{*,q})=\begin{cases}\overbrace{\mathcal{O}_{M,x}\oplus\cdots\oplus\mathcal{O}_{M,x}}^{\binom{n+p-1}{p}\text{copies}}, & p=k-q,\\ 0, & \text{otherwise}.\end{cases}$$

Hence

$$H^r(K^{\cdot\cdot})=\begin{cases}\overbrace{\mathcal{O}_{M,x}+\cdots+\mathcal{O}_{M,x}}^{\text{L copies}}, & r=k,\\ 0, & \text{otherwise},\end{cases}$$



where $$L=\sum_{p=0}^{k}\binom{n+p-1}{p}=\binom{n+k}{r}.$$

We denote $t=m\text{-}\dim C(\varphi)_x$ from now on.

$$'E_1^{p,q}={}''H^q(K^{p,\cdot})$$

$$=\begin{cases}\Omega^p_{\varphi,k-p,x}, & q=k,\\ R^{p-k+q}(\Omega^{p-k+\cdot,k-\cdot}_{M,k-p}), & k+t\leqslant p+q<k+m,\ 0\leqslant q<k,\ 0\leqslant p<k\\ \Omega^{m,p-m}_{M,k-p,x}/\psi^{m-1,p-m+1}\Omega^{m-1,p-m+1}_{M,k-p,x}, & p+q=k+m,\ p\geqslant m,\\ 0, & \text{otherwise}.\end{cases}$$

For $q\neq k$, $p+q<k+t$ or $p+q>k+m$, we have

$$'E_1^{p,q}={}'E_2^{p,q}=\cdots={}'E_r^{p,q}=\cdots={}'E_\infty^{p,q}=0.$$

By $H^r(K^{\cdot\cdot})=0$, $r\neq k$, we have $'E^{p,q}=0$, $p+q\neq k$. Because $'E_2^{p,q}={}'H^p(''H^q(K^{\cdot\cdot}))$,

$$'E_2^{p,k}=H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x}).$$

For $0\leqslant p<t-1$,

$$'E_2^{p,k}=\cdots={}'E_r^{p,k}=\cdots={}'E_\infty^{p,k}=H^{p+k}(K^{\cdot\cdot}).$$

For $p=m$,

$$'E_2^{m,k}=\cdots={}'E_r^{m,k}=\cdots={}'E_\infty^{m,k}=H^{m+k}(K^{\cdot\cdot}).$$

Theorem 1. $$H^0(\Omega^{\cdot}_{\varphi,k-\cdot,x})=\underbrace{\mathcal{O}_{M,x}\oplus\cdots\oplus\mathcal{O}_{M,x}}_{\binom{n+k}{k}\text{ copies}},$$

$$H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})=0,\ 0<p<m\text{-}\dim C(\varphi)_x-1 \text{ or } p=m.$$

Proposition 2. *If $a\in I(\varphi)_x$, let $S_a=\{1,a,a^2,\cdots\}$ be a multiplicative set. Then for $t-1\leqslant p\leqslant m-1$, the localizations*

$$S_a^{-1}H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})=0.$$

Proof We consider the localized double complex $S_a^{-1}K^{p,q}$.

Because $S_a^{-1}\mathcal{O}_{M,x}$ is a flat $\mathcal{O}_{M,x}$-module, two spectral sequence of $S_a^{-1}K^{p,q}$ are as follows

$$''E_1^{q,p}(S_a^{-1}K^{\cdot\cdot})=\begin{cases}\underbrace{S_a^{-1}\mathcal{O}_{M,x}\oplus\cdots\oplus S_a^{-1}\mathcal{O}_{M,x}}_{\binom{n+p-1}{p}\text{ coipes}}, & p+q=k,\\ 0, & \text{otherwise.}\end{cases}$$

$$H^r(S_a^{-1}K^{\cdot\cdot})=\begin{cases}\underbrace{S_a^{-1}\mathcal{O}_{M,x}\oplus\cdots\oplus S_a^{-1}\mathcal{O}_{M,x}}_{\binom{n+k}{k}\text{ copies}}, & r=k,\\ 0, & \text{otherwise.}\end{cases}$$

On the other hand, if we denote $\Omega_{\varphi,k-p,x}^{m,p-m}=\Omega_{M,k-p,x}^{m,p-m}/\psi^{m-1,p-m+1}\Omega_{M,k-p,x}^{m-1,p-m+1}$,

$$'E^{p,q}(S_a^{-1}K)=\begin{cases}S_a^{-1}\Omega_{\varphi,k-p,x}^{p}, & q=k,\\ S_a^{-1}R^{p+q-k}(\Omega_{M,k-p}^{p-k+\cdot,k-\cdot}), & k+t\leqslant p+q<k+m,\ 0\leqslant q<k,\ 0\leqslant p\leqslant m,\\ S_a^{-1}\Omega_{\varphi,k-p,x}^{m,p-m}, & p+q=k+m, p\geqslant m,\\ 0, & \text{otherwise.}\end{cases}$$

By Lemma 1, $I_{k-p}^r(\varphi)_x R^{p+q-k}(\Omega_{M,k-p,x}^{p-k+s,k-\cdot})=0$. But $j_{k-p}a\in I_{k-p}(\varphi)$. We denote $\alpha=j_{k-p}a$. Hence $\alpha^r R^{p+q-k}(\Omega_{M,k-p,x}^{p-k+\cdot,k-\cdot})=0$. $\alpha^r=a^r+b(\xi)$, where deg $b(\xi)\geqslant 1$. Henec $(b(\xi))^{k-p}=0$. $\alpha^r=a^r(1+b(\xi)/a^r)$ in $S_a^{-1}J_{M,k,x}$. So α^r is invertible in $S_a^{-1}J_{M,k,x}$. Hence

$$S_a^{-1}R^{p+q-k}(\Omega_{M,k-p}^{p-k+\cdot,k-\cdot})=0.$$

On the other hand $DF(\Omega_{\varphi,k-p,x}^{m,p-m})=0$. By generalized de-Rham lemma $I_{k-p}^s(\varphi)_x\Omega_{\varphi,k-p,x}^{m,p-m}=0$. Similar arguments can prove $S_a^{-1}\Omega_{\varphi,k-p,x}^{m,p-m}=0$.

We get

$$'E_1^{p,q}(S_a^{-1}K^{\cdot\cdot})=\begin{cases}S_a^{-1}\Omega_{\varphi,k-p,x}^{p}, & q=k,\\ 0, & \text{otherwise.}\end{cases}$$

and

$$'E_2^{p,q}(S_a^{-1}K^{\cdot\cdot})=\begin{cases}S_a^{-1}H^p(\Omega_{\varphi,k-\cdot,x}^{\cdot}), & q=k,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore

$$S_a^{-1}H^p(\Omega_{\varphi,k-\cdot,x}^{\cdot})=0,\ t-1\leqslant p\leqslant m-1.$$

Theorem 2. *There exists a positive integer s such that*

$$(I(\varphi)_x)^s H^p(\Omega_{\varphi,k-\cdot,x}^{\cdot})=0\quad\text{for } t-1\leqslant p\leqslant m-1.$$

Hence $H^p(\Omega_{\varphi,k-\cdot,x}^{\cdot})$ are finitely generated $\mathcal{O}_{M,x}/I(\varphi)_x^s$-module.

Proof It is an immediate consequence of Propsition 2, because $I(\varphi)_x$ is finitely generated.

Let $\Omega_{\varphi,k-p,x}^{q,p-q}=\Omega_{M,k-p,x}^{q,p-q}/\psi^{q-1,p-q+1}\Omega_{M,k-p,x}^{q-1,p-q+1}$. We get complexes

$$\Omega_{\varphi,k-r-\cdot,x}^{\cdot,r}:\ \Omega_{\varphi,k-r,x}^{0,r}\xrightarrow{D^{0,r}}\cdots\to\Omega_{\varphi,k-r-q,x}^{q,r}\xrightarrow{D^{q,r}}\Omega_{\varphi,k-r-q-1,x}^{q+1,r}\to\cdots\to\Omega_{\varphi,k-r-m,x}^{m,r}.$$

Proposition 3. $H^q(\Omega_{\varphi,k-r-\cdot,x}^{\cdot,r})=0,\ 0<q<t-1,\ q=m,$

$$S_a^{-1}H^q(\Omega_{\varphi,k-r-\cdot,x}^{\cdot,r})=0,\ t-1\leqslant q\leqslant m-1,\ a\in I(\varphi)_x.$$

Hence $H^q(\Omega_{\varphi,k-r-\cdot,x}^{\cdot,r})$, $t-1\leqslant q\leqslant m-1$, are finitely generated $\mathcal{O}_{M,x}/I(\varphi)_x^s$ module for some positive integer s.

Proof We define a double complex

$$K^{p,q}(r)=\begin{cases} K^{p,q}, & q=k-r, \\ 0, & \text{otherwise.} \end{cases}$$

Similar to the proofs of Theorem 1, Proposition 2 and Theorem 2, we can prove Proposition 3.

We define a complex

$$\Omega^{q,r}_{\varphi,k-r-q,x}(t)=\begin{cases} \Omega^{q,r}_{\varphi,k-r-q,x}, & q\leqslant t, \\ 0, & q>t. \end{cases}$$

Because $R^q(\Omega^{\cdot,p-\cdot}_{M,k-p,x})=0$, $p\leqslant t-1$, we have short exact sequences of complexes

$$0\to\Omega^{\cdot,r+1}_{\varphi,k-r-\cdot,x}(t)\xrightarrow{\overline{\psi}}\Omega^{\cdot,r}_{M,k-r-\cdot,x}(t)\xrightarrow{\pi}\Omega^{\cdot,r}_{\varphi,k-r-\cdot,x}(t)\to 0,$$

where $\overline{\psi}$'s are induced by ψ's and π's are natural projections. We write the last three terms of the short exact sequences of complexes explicitly

$$\begin{array}{ccccc}
\cdots\to\Omega^{t-2,r+1}_{\varphi,k-r-t+1,x} & \xrightarrow{D} & \Omega^{t-1,r+1}_{\varphi,k-r-t,x} & \to & 0 \\
\downarrow{\scriptstyle\overline{\psi}} & & \downarrow{\scriptstyle\overline{\psi}} & & \downarrow \\
\cdots\to\Omega^{t-1,r}_{M,k-r-t+1,x} & \xrightarrow{D} & \Omega^{t,r}_{M,k-r-t,x} & \to & 0 \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow \\
\cdots\to\Omega^{t-1,r}_{\varphi,k-r-t+1,x} & \xrightarrow{D} & \Omega^{t,r}_{\varphi,k-r-t,x} & \to & 0.
\end{array}$$

Theorem 3. *The following sequences are exact*

$$0\to H^{t-1}(\Omega^{t,r}_{\varphi,k-r-\cdot,x})\xrightarrow{\partial}\Omega^{t-1,r+1}_{\varphi,k-r-\ ,x}/D\Omega^{t-2,r+1}_{\varphi,k-r-t+1,x}$$
$$\to\Omega^{t,r}_{M,k-r-t,x}/D\Omega^{t-1,r}_{M,k-r-t+1,x}\to\Omega^{t,r}_{\varphi,k-r-t,x}/D\Omega^{t-1,r}_{\varphi,k-r-t+1,x}\to 0$$

and $\partial H^{t-1}(\Omega^{\cdot,r}_{\varphi,k-r-\cdot,x})\supset H^{t-1}(\Omega^{\cdot,r+1}_{\varphi,k-r-1,x})$.

Proof Let $[\widetilde{\omega}]\in H^{t-1}(\Omega^{\cdot,r}_{\varphi,k-r-1-\cdot,x})$, $\widetilde{\omega}\in Z(\Omega^{t-1,r+1}_{\varphi,k-r-1-t,x})$. Then $D\overline{\psi}(\widetilde{\omega})=0$. There exists $\eta\in\Omega^{t-1,r}_{M,k-r-t+1,x}$, such that $D\eta=\overline{\psi}\widetilde{\omega}$. Let $\widetilde{\eta}=\pi\eta$. Clearly

$$D\widetilde{\eta}=D\pi\eta=\pi D\eta=\pi\overline{\psi}\widetilde{\omega}=0.$$

Let $[\widetilde{\eta}]$ be the residue class of $\widetilde{\eta}$ in $H^{t-1}(\Omega^{\cdot,r}_{\varphi,k-r-\cdot,x})$. Then $\partial[\widetilde{\eta}]=[\widetilde{\omega}]$. Clearly $[\widetilde{\eta}]$ is independent of the representative.

Especially, for $r=0$, we have exact sequence

$$0\to H^{t-1}(\Omega^{\cdot}_{\varphi,k-\cdot,x})\xrightarrow{\partial}\Omega^{t-1,1}_{\varphi,k-t,x}/D\Omega^{t-2,1}_{\varphi,k-\ +1,x}$$
$$\to\Omega^{t}_{M,k-,x}/D\Omega^{t-1}_{M,k-t+1,x}\to\Omega^{t}_{\varphi,k-t,x}/D\Omega^{t-1}_{\varphi,k-t+1,x}\to 0.$$

On the other hand we have a sequence of $\mathscr{O}_{M,x}/I(\varphi)^{s}_{x}$-modules

$$H^{t-1}(\Omega^{\cdot}_{\varphi,k-\cdot,x})\supset\partial^{-1}H^{t-1}(\Omega^{\cdot,1}_{\varphi,k-1-\cdot,x})\supset\cdots$$
$$\supset\underbrace{\partial^{-1}\cdots\varrho^{-1}}_{r}H^{t-1}(\Omega^{\cdot,r}_{\varphi,k-r-\cdot,x})\supset\cdots\supset\underbrace{\partial^{-1}\cdots\partial^{-1}}_{k-t+1}H^{t-1}(\Omega^{\cdot,k-t+1}_{\varphi,t-1-\cdot,x})=0.$$

Theorem 4. *If* $\dim_K\mathcal{O}(\varphi)_x<\infty$,

a) $H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})=0$, $0<p<m-1$ or $p=m$,

b) $\dim_K H^{m-1}(\Omega^{\cdot}_{\varphi,k-\cdot,x})=\sum\limits_{r=1}^{k-m}\dim_K\Omega^{m,r}_{\varphi,k-r-m,x}<\infty$.

Proof Under, the hypothesis, $\dim C(\varphi)_x=0$. a) follows. Because

$$\Omega^{m,r}_{M,k-r-m,x}=D\Omega^{m-1,r}_{M,k-r-m+1,x},\ H^{m-1}(\Omega^{\cdot,r}_{\varphi,k-r-\cdot,x})\overset{\partial}{\approx}\Omega^{m-1,r+1}_{\varphi,k-r-m,x}/D\Omega^{m-2,r+1}_{\varphi,k-r-m+1,x}.$$

By Theorem 3,

$$H^{m-1}(\Omega^{\cdot,r}_{\varphi,k-r-\cdot,x})/H^{m-1}(\Omega^{\cdot,r+1}_{\varphi,k-r-1-\cdot x})=\Omega^{m-1,r+1}_{\varphi,k-r-m,x}/Z(\Omega^{m-1,r+1}_{\varphi,k-r-m,x})\overset{D}{\approx}\Omega^{m,r+1}_{\varphi,k-r-m-1,x}.$$

Because $\dim_K\mathscr{O}_{M,x}/I(\varphi)_x<\infty$, $\dim_K\mathscr{O}_{M,x}/I(\varphi)^s_x<\infty$. Hence $\dim_K H^{m-1}(\Omega^{\cdot,r}_{\varphi,k-r-\cdot,x})<\infty$. b) follows from Proposition 3.

Corollary (i.e. Theorem 2 in 5). *If φ: $M\to K$ is a complex holomorphic (or real analytic) function and* $\dim_K\mathscr{O}_{M,x}/(\partial\varphi/\partial x_i)<\infty$,

$$H^0((\Omega^{\cdot}_{\varphi,k-\cdot,x})=\underbrace{\mathscr{O}_{M,x}\oplus\cdots\oplus\mathscr{O}_{M,x}}_{k+1\text{ copies}},$$

$$H^p(\Omega^{\cdot}_{\varphi,k-\cdot,x})=0,\ p=0,\ m-1,$$

$$\dim{}_kH^{m-1}(\Omega^{\cdot}_{\varphi,k-\cdot,x})=\binom{x}{k-m-1}\dim_K\mathscr{O}_{M,k}/(\partial\varphi/\partial x_i).$$

Proof In this case $n=1$.

$$\Omega^{m,r}_{\varphi,k-r-m,x}=\Omega^{m}_{\varphi,k-r-m,x}\approx J_{M,k-r-m,x}\Big/\left(\frac{\partial\eta}{\partial\xi_1},\ \cdots,\ \frac{\partial\eta}{\partial\xi_m}\right).$$

By Theorem 4,

$$\dim_K H^{m-1}(\Omega^{\cdot}_{\varphi,k-\cdot,x})=\sum_{r=1}^{k-m}\dim_K\Omega^{m}_{\varphi,k-r-m,x}=\binom{x}{k-m-1}\dim_K\mathscr{O}_{M,x}/(\partial\varphi/\partial x_i).$$

If M and N are C^∞ manifolds and φ is a finitely determined C^∞ mapping. There are suitable local systems of coordinates near x and $\varphi(x)$ respectively, such that $f_1,\cdots,f_n$ are polynomials. Let $U\subset M$ be such a coordinate open set on M $x\in U$. We denoteby $\mathscr{O}_{U,x}$ the ring of germs of real analytic functions at x (defined with respect to U), and $\varepsilon_{U,x}$ the ring of germs of C^∞ functions at x. $\varepsilon_{U,x}$ is flat over $\mathscr{O}_{U,x}$ (see (2)).

If $I\subset\mathscr{O}_{U,x}$ is an ideal of $\mathscr{O}_{U,x}$ and

$$\cdots\to X_n\to X_{n-1}\to\cdots\to X_0\to\mathscr{O}_{U,x}/I\to 0$$

is a finitely generated free resolution of $\mathscr{O}_{U,x}/I$ over $\mathscr{O}_{U,x}$. We know

$$\mathrm{Hom}_{\mathscr{O}_{U,x}}(X_n,\mathscr{O}_{U,x})\otimes_{\mathscr{O}_{U,x}}\mathscr{E}_{U,x}\approx\mathrm{Hom}_{\mathscr{E}_{U,x}}(X_n\otimes_{\mathscr{O}_{U,x}}\mathscr{E}_{U,x},\mathscr{E}_{U,x}).$$

Hence

$$\cdots\to X_n\otimes_{\mathscr{O}_{U,x}}\mathscr{E}_{U,x}\to\cdots\to X_0\otimes_{\mathscr{O}_{U,x}}\mathscr{E}_{U,x}\to\mathscr{E}_{U,x}/I\mathscr{E}_{U,x}\to 0$$

is a finitely generated free resolution of $\mathscr{E}_{U,x}/I\mathscr{E}_{U,x}$ over $\mathscr{E}_{U,x}$ and

$$\mathrm{Ext}^n_{\mathscr{O}_{U,x}}(\mathscr{O}_{U,x}/I,\mathscr{O}_{U,x})\otimes_{\mathscr{O}_{U,x}}\mathscr{E}_{U,x}=\mathrm{Ext}^n_{\mathscr{E}_{U,x}}(\mathscr{E}_{U,x}/I\mathscr{E}_{U,x},\mathscr{E}_{U,x}).$$

Therefore the homological $I\mathscr{E}_{U,x}$-codimension of $\mathscr{E}_{U,x}=\inf\{n\,|\,\mathrm{Ext}^n_{\mathscr{E}_{U,x}}(\mathscr{E}_{U,x}/I\mathscr{E}_{U,x},\mathscr{E}_{U,x})\neq 0\}=\inf\{n\,|\,\mathrm{Ext}^n_{\mathscr{O}_{U,x}}(\mathscr{O}_{U,x}/I,\mathscr{O}_{U,x})\neq 0\}=$ homological I-codimension of $\mathscr{O}_{U,x}=m-\dim\mathscr{O}_{U,x}/I=t$.

Because all morphisms,such as $D,D^{p,r}$ and $\psi^{p,r}$, are $\mathscr{O}_{U,x}$-linear, by tensoring all modules in this paper with $\mathscr{E}_{U,x}$ over $\mathscr{O}_{U,x}$ Theorems 1,2,3,4 and the Corollary of

Theorem 4 are all true for finitely determined C^∞ mapgerms, if $\mathscr{O}_{M,x}$'s are substituted by $\mathscr{E}_{U,x}$'s and m-dim $C(\varphi)_x$ by homological $I(\varphi)_x$-codimension$=t$, where $I(\varphi)_x$ is the corrssponding ideal of φ in $\mathscr{E}_{U,x}$.

References

[1] Greuel, G. M., Der Gauss-Manin Zusammenhang isolierter Singularitaten von vollstandigen Durchschnitten, *Math. Ann.*, **214** (1975), 235—266.

[2] Malgrange, B., Ideals of differentiable functions, Oxford, Oxford University Press 1968.

[3] Matsumura, H., Commutative Algebra, 2nd Edition, Benjamin, 1980.

[4] Saito, K., On a generalization of de Rham lemma, *Ann. Inst. Fourier* (Grenoble), **26** (1976), 165-170.

[5] Xiao, E. J., On Cohomology of Singularities of a C^∞ Mapping, *Math. Ann*, 262 (1983), 255—272.