Chin. Ann. of Math.
7B (1) 1986

NUMBERS OF CONJUGATE CLASSES OF SYMMETRIC AND ALTERNATING GROUPS

WANG EFANG (王萼芳)*

Abstract

Let $d(n)$ be the excess of the number of even conjugate classes of S_n over that of odd conjugate classes of S_n, and $q(n)$ the number of splitting classes of S_n. In this paper a recurrence formula for $d(n)$ and one for $q(n)$ are given. As a recurrence formula for the number $p(n)$ of conjugate classes of S_n is known[1], one can make use of $p(n)$, $d(n)$ and $q(n)$ to calculate the numbers of even (odd) conjugate classes of S_n and that of conjugate classes of A_n. By means of a graphical method the author proves the identity $d(n)=q(n)$ when $n\geqslant 2$, which seems to have been obtained first by Sylvester[3] by use of generating functions.

Let σ be a permutation on n letters. The ordered lengths $a_1, a_2, \cdots, a_s (a_j \geqslant a_{j+1})$ of the disjoint cyclic factors of σ form a partition $(a_1, a_2, \cdots, a_s)$ of n. We know that in the symmetric group on n letters S_n, two permutations are conjugate if and only if the corresponding partitions are equal. Hence, there is a one-to-one correspondence between conjugate classes of S_n and partitions of n, and the number of conjugate classes of S_n is equal to the number of partitions of n. An n-partition α is said to be odd if it corresponds to an odd permutation class, otherwise even. The number of odd (even) conjugate classes is equal to the number of odd (even) n-partitions.

A recursion formula for the number of n-partitions $p(n)$ can be found in [1]. In the first part of the present paper we give a recursion formula for $d(n)$, the excess of the number of even n-partitions over the number of odd n-partitions. From $p(n)$ and $d(n)$, we can find the number of the even and odd coniugate classes of S_n immediately.

In order to discuss the number of conjugate classes of the alternating group A_n, we have to calculate the number of splitting classes of S_n. In the second part of the present paper, we give a recursion formula for the number of splitting classes $q(n)$.

In the last part, we use a graphical method to prove the identity $d(n)=q(n)$ when $n\geqslant 2$, which seems to have been mentioned first by Sylvester[3]. From this

Manuscript received July 26, 1983.
* Department of Mathematics, Beijing University, Beijing, China.

identity, we deduce that the number of even conjugate classes of $S_n (n>2)$ is always greater than the number of odd conjugate classes. Finally, as a consequence of this identity, we prove again another Sylvester's theorem.

§ 1. Numbers of Odd and Even Conjugate Classes of S_n

Let α be a partition of n. By $s(\alpha)$ we denote the number of parts of α. By the weight $w(\alpha)$ of α, we mean the smallest part of α. It is easy to see that α is an even partition iff n and $s(\alpha)$ have the same odevity.

We call an n-partition of weight k an (n, k)-partition.

Let $(a_1, a_2, \cdots, a_s)$ be an n-partition, $s\geqslant 2$. Then

$$\rho=(a_1, a_2, \cdots, a_{s-1})$$

is an $(n-a_s, a_{s-1})$-partition. β is called the shortening of α, and α is called the extending of β. Every n-partition α with $s(\alpha)\geqslant 2$ is an extending of an $(n-w(\alpha))$-partition, and different partitions give different extendings. There are some important properties of extending and shortening:

1. If β is the shortening of α, then $w(\beta)\geqslant w(\alpha)$.
2. An $(n-k)$-partition β can be extented to an n-partition iff $w(\beta)\geqslant k$.
3. Let α be a partition with more than one part, then α and its shortening have the same odevity iff $w(\alpha)$ is odd.

By $d(n, k)$ we denote the excess of the number of even n-partitions with weight greater than or equal to k over the number of odd n-partitions with weight greater than or equal to k. Then $d(n, 1)$ is the excess of the number of even n-partitions over the number of odd n-partitions, and we write $d(n)=d(n, 1)$. Because there is only one n-partition with one part: (n), and the odevities of (n) and n are opposite, we have the following recurrence formula:

$$\begin{cases} d(n, 1)=d(n-1, 1)-d(n-2, 2)+d(n-3, 3)-\cdots \\ \qquad +(-1)^{k+1}d(n-k, k)+\cdots+(-1)^n d(1, n-1)+(-1)^{n+1}, \\ d(n, 2)=-d(n-2, 2)+d(n-3, 3)-\cdots \\ \qquad +(-1)^{k+1}d(n-k, k)+\cdots+(-1)^n d(1, n-1)+(-1)^{n+1}, \\ \cdots\cdots\cdots \\ d(n, n)=(-1)^{n+1}. \end{cases} \tag{1}$$

Here we take $d(l, m)=0$ when $l<m$.

From (1) we can obtain

$$\begin{cases} d(n, n)=(-1)^{n+1}, \\ d(n, k)=d(n, k+1)+(-1)^{k+1}d(n-k, k), \end{cases} \tag{2}$$

$$n=1, 2, 3, \cdots; \quad k=n-1, n-2, \cdots, 2, 1.$$

Since there is just one n-partition with weight greater than $\left[\frac{n}{2}\right]$: (n); and just one n-partition with weight equal to $\left[\frac{n}{2}\right]$ also: $\left(\left[\frac{n+1}{2}\right], \left[\frac{n}{2}\right]\right)$, and the odevities of (n), $\left(\left[\frac{n+1}{2}\right], \left[\frac{n}{2}\right]\right)$ are opposite, we have the following improvements of the recurrence formulas (1) and (2):

$$\begin{cases} d(n, 1) = d(n-1, 1) - d(n-2, 2) + \cdots + (-1)^{\left[\frac{n}{2}\right]+1} d\left(\left[\frac{n+1}{2}\right], \left[\frac{n}{2}\right]\right) + (-1)^{n+1}, \\ d(n, 2) = -d(n-2, 2) + d(n-3, 3) - \cdots + (-1)^{\left[\frac{n}{2}\right]+1} d\left(\left[\frac{n+1}{2}\right], \left[\frac{n}{2}\right]\right) + (-1)^{n+1}, \\ \cdots\cdots\cdots \\ d\left(n, \left[\frac{n}{2}\right]-1\right) = (-1)^{\left[\frac{n}{2}\right]} d\left(\left[\frac{n+1}{2}\right]+1, \left[\frac{n}{2}\right]-1\right) \\ \qquad + (-1)^{\left[\frac{n}{2}\right]+1} d\left(\left[\frac{n+1}{2}\right], \left[\frac{n}{2}\right]\right) + (-1)^{n+1}, \\ d\left(n, \left[\frac{n}{2}\right]\right) = 0, \\ d\left(n, \left[\frac{n}{2}\right]+1\right) = \cdots = d(n, n) = (-1)^{n+1}. \end{cases} \tag{3}$$

$$\begin{cases} d(n, n) = d(n, n-1) = \cdots = d\left(n, \left[\frac{n}{2}\right]+1\right) = (-1)^{n+1}, \\ d(n, k) = d(n, k+1) + (-1)^{k+1} d(n-k, k), \end{cases} \tag{4}$$

$$n = 1, 2, 3, \cdots, k = \left[\frac{n}{2}\right], \left[\frac{n}{2}\right]-1, \cdots, 2, 1.$$

By the recurrence formula (4), $d(n)$ can be calculated quickly.

We denote the numbers of odd and even n-partitions by $p_0(n)$ and $p_e(n)$ respectively. Then $p_0(n)$ and $p_e(n)$ can be expressed by $p(n)$ and $d(n)$:

$$p_0(n) = \frac{1}{2}(p(n) - d(n)),$$

$$p_e(n) = \frac{1}{2}(p(n) + d(n)).$$

§ 2. Numbers of A-classes of S_n and A_n

Let σ, τ be two permutations in S_n. If there is a permutation $\rho \in A_n$ such that $\sigma = \rho^{-1}\tau\rho$, we say that σ and τ are A-conjugate. Obviously, A-conjugate permutations are always conjugate, but conjugate permutations are not always A-conjugate. Hence some S-classes split into A-classes, which are the so-called splitting classes. We know[2] that the odd conjugate class is always not splitting, and an even conjugate class is splitting iff the lengths of its disjoint cyclic factors are different odd numbers. Moreover, whenever a class is splitting, it splits into two A-classes. Thus, the number of A-conjugate classes of S_n equals the sum of the number of conjugate classes of S_n and the number of splitting classes of S_n; the number of conjugate classes of A_n, which is just the number of A-conjugate classes of A_n, is equal to the sum of the number of even conjugate classes of S_n and the number of splitting classes of S_n.

For convenience, we call a partition corresponding to a splitting class a splitting partition. That is to say, a splitting partition is a partition with distinct odd parts.

By $q(n, k)$ we denote the number of splitting n-partitions with weight greater than or equal to k. Then $q(n, 1)$ is just the number of splitting n-partitions, denoted by $q(n)$.

It is easy to see the following facts:

1°. Any partition with even weight is not splitting.

2°. If $a_s=a_{s-1}$, then $(a_1, a_2, \cdots, a_{s-1}, a_s)$ is not splitting.

Hence we have

Lemma 1. *An n-partition $(a_1, a_2, \cdots, a_s)$ $(n\geqslant 2)$ is splitting iff*

(1) *when $s=1$: n is odd;*

(2) *when $s\geqslant 2$: a_s is odd, $a_s<a_{s-1}$ and $(a_1, \cdots, a_{s-1})$ is a splitting partition of $n-a_s$.*

Corollary 1. $q(n, 2l)=q(n, 2l+1)$.

We know that the only n-partition with weight $\left[\frac{n}{2}\right]$ is $\left(\left[\frac{n+1}{2}\right], \left[\frac{n}{2}\right]\right)$. This partition is not splitting because $\left[\frac{n+1}{2}\right]=\left[\frac{n}{2}\right]$ when n is even, and one of the consecutive integers $\left[\frac{n}{2}\right]$ and $\left[\frac{n+1}{2}\right]$ is even when n is odd. The only n-partition with weight greater than $\left[\frac{n}{2}\right]$: (n), is splitting iff n is odd. Hence we get a recurrence formula for $q(n, k)$. By $\langle k\rangle$ we denote the greatest odd number not exceeding k. Let

$$\delta_n=\begin{cases}1, & \text{when } n \text{ is odd,}\\ 0, & \text{when } n \text{ is even.}\end{cases}$$

Then

$$\left\{\begin{aligned}
&q(n, 1)=q(n-1, 3)+q(n-3, 5)+\cdots+q\left(n-\left\langle\frac{n}{2}\right\rangle, \left\langle\frac{n}{2}\right\rangle+2\right)+\delta_n,\\
&q(n, 3)=q(n, 2)\\
&\qquad=q(n-3, 5)+q(n-5, 7)+\cdots+q\left(n-\left\langle\frac{n}{2}\right\rangle, \left\langle\frac{n}{2}\right\rangle+2\right)+\delta_n,\\
&\cdots\cdots\cdots\\
&q\left(n, \left\langle\frac{n}{2}\right\rangle\right)=q\left(n, \left\langle\frac{n}{2}\right\rangle-1\right)=q\left(n-\left\langle\frac{n}{2}\right\rangle, \left\langle\frac{n}{2}\right\rangle+2\right)+\delta_n,\\
&q(n, k)=\delta_n \qquad \left(k\geqslant\left\langle\frac{n}{2}\right\rangle\right).
\end{aligned}\right. \tag{5}$$

Here we also take $q(l, m)=0$ when $l<m$.

We can calculate $q(n, k)$ by the following recurrence formulas:

$$\begin{cases} q(n, n)=q(n, n-1)=\cdots=q\left(n, \left[\frac{n}{2}\right]\right)=\delta_n, \\ q(n, 2l+1)=q(n, 2l)=q(n-2l-1, 2l+3)+q(n, 2l+3), \end{cases}$$

$$n=1, 2, 3, \cdots; \ l=\left[\frac{n}{4}\right], \left[\frac{n}{4}\right]-1, \cdots, 2, 1.$$

§ 3. The Proof of $q(n)=d(n)$

In this section we prove on identity about the number of splitting classes of S_n and the difference between the numbers of even and odd classes of S_n.

Theorem 1. $q(n)=d(n)\ (n\geqslant 2)$.

For the proof we require the graphical representation of partitions. Every n-partition can be represented by a two-dimensional array of dots. One dimension indicates the size of each part and the other the number of parts. The array is called the Ferrars graph of this partition. If $\alpha=(a_1, a_2, \cdots, a_s)$ is a partition of n, then its Ferrars graph has a rows, and the i th row contains a_i dots. We denote the Ferrars graph of α by $\Gamma(\alpha)$.

Let Γ be a Ferrars graph. By reflecting Γ in the main diagonal, we obtain another Ferrars graph Γ', which is called the conjugate of Γ. The n-partition corresponding to $(\Gamma(\alpha))'$ is called the conjugate of α, denoted by α^ϕ. It is clear that conjugation is an involution of the partitions of any integer, as the conjugate of the conjugate of α is again α.

For convenience, we denote the greatest part of the partition $=(a_1, a_2, \cdots, a_s)$ by $l(\alpha)$. The number of occurrences of a_i in α is called the multiplicity of a_i in α. The multiplicity of a_1 in α is simply called the multiplicity of α and denoted by $r(\alpha)$.

It is easy to see that

$$s(\alpha^\phi)=l(\alpha),\ l(\alpha^\phi)=s(\alpha),\ w(\alpha^\phi)=r(\alpha),\ r(\alpha^\phi)=w(\alpha).$$

Lemma 2. *If n-partition α is splitting, then $s(\alpha)\leqslant n$.*

Proof Because α has $s(\alpha)$ distinct odd parts, and the sum of $s(\alpha)$ smallest distinct odd numbers is $(s(\alpha))^2$, we have

$$(s(\alpha))^2\leqslant n,$$

and so

$$s(\alpha)\leqslant n.$$

Lemma 3. *Let E_0 denote the set of all even n-partitions β which satisfy:*

1. $l(\beta)\leqslant r(\beta)$,

2. $l(\beta)$, $r(\beta)$ *and n have the same odevity,*

3. *The multiplicities of the parts different from $l(\beta)$ of β are all even.*
Then

$$q(n)=|E_0|.$$

Proof Let Q be the set of all splitting n-partitions.

If $\alpha=(a_1, a_2, \cdots, a_s)\in Q$, then

$$a_i\geqslant 2(s-i)+1, \quad i=1, 2, \cdots, s.$$

Put

$$a_i'=a-2(s-i)-1.$$

Then $a_1', a_2', \cdots, a_s'$ are all non-negative even numbers. And from

$$a_i\geqslant a_{i+1}+2, \quad i=1, 2, \cdots, s-1,$$

we know that

$$a_1'\geqslant a_2'\geqslant\cdots\geqslant a_s'\geqslant 0,$$
$$s^2+a_1'+a_2'+\cdots+a_s'=n.$$

Let

$$\beta=(s+a_1', s+a_2', \cdots, s+a_s')^\phi.$$

Then

$$l(\beta)=s, r(\beta)=s+a_s', s(\beta)=s+a_1'.$$

Hence

$$l(\beta)\leqslant r(\beta),$$

$l(\beta)$ and $s(\beta)$ have the same odevity. Furthemore, $l(\beta)=s$ and n have the same odevity, because α is an even partition. Finally, since $a_1', a_2', \cdots, a_s'$ are all even, the differences of any two of them are also even. Thus, the multiplicities of all parts (except $l(\beta)$) of β are all even. Therefore, $\beta\in E_0$. Obviously, distinct splitting partitions give distinct β's. So we have $q(n)\leqslant|E_0|$.

On the other hand, if $\beta=(b, b_2, \cdots, b_t)\in E_0$, then

$$(l(\beta))^2\leqslant l(\beta)\cdot r(\beta)\leqslant n.$$

Thus

$$l(\beta)\leqslant\sqrt{n}.$$

Let $s=l(\beta)$. The conjugate partition β^ϕ of β has s parts. Suppose

$$\beta^\phi=(c_1, c_2, \cdots, c_s).$$

From the properties of E_0, we know that

$$c_i\geqslant c_s=r(\beta)\geqslant l(\beta)=s, \quad i=1, 2, \cdots, s-1.$$

Set

$$d_i=c_i-s, \quad i=1, 2, \cdots, s.$$

Then $d_1, d_2, \cdots, d_s$ are all even, and

$$d_1\geqslant d_2\geqslant\cdots\geqslant, d_s\geqslant 0,$$
$$d_1+d_2+\cdots+d_s=n-s^2.$$

Let

$$a_i = d_i + 2(s-i) + 1, \quad i = 1, 2, \cdots, s,$$
$$\alpha = (a_1, a_2, \cdots, a_s).$$

Since

$$a_1 + a_2 + \cdots + a_s = n,$$
$$a_1 > a_2 > \cdots > a_s,$$

and $a_1, a_2, \cdots, a_s$ are distinct odd numbers, α is a splitting n-partition. Obviously, distinct β's give distinct α's. Thus $|E_0| \leqslant q(n)$.

The lemma is proved.

Let O be the set of all odd n-partitions, and E the set of all even n-partitions. We shall define an injection ψ from O into E, and its image set is $E - E_0$. Then

$$|O| = |E - E_0| = |E| - |E_0| = |E| - q(n).$$

Therefore

$$d(n) = |E| - |O| = q(n).$$

This is what we claim.

Proof of Theorem 1 If $\alpha = (a_1, a_2, \cdots, a_s)$ is an odd n-partition, then s and n have opposite odevities.

Case 1. $l(\alpha) = a_1$ and n have the same odevity.

In this case, $s(\alpha^\phi) = l(\alpha)$ has the same odevity as n. Hence α^ϕ is an even partition. Let

$$\alpha^\psi = \alpha^\phi.$$

Then $\alpha^\psi \in E$. Because $l(\alpha^\psi) = l(\alpha^\phi) = s(\alpha)$, $l(\alpha^\psi)$ and n have opposite odevities. Moreover because $\phi^2 = I$, any even n-partition β is an image of ψ, whenever $l(\beta)$ and n have opposite odevities, and the images of distinct α's are distinct.

Case 2. $l(\alpha) = a_1$ and n have opposite odevities and $r(\alpha) \leqslant w(\alpha)$.

Let $\Gamma(\alpha)$ be the Ferrars graph of α. Since $r(\alpha) \leqslant w(\alpha)$, the number of dots on the last column of $\Gamma(\alpha)$ is less than or equal to the number of dots on the last row of $\Gamma(\alpha)$. We delete the last column of $\Gamma(\alpha)$ and add a new smallest row of $r(\alpha)$ dots. Then we get a Ferrars graph Γ_1 of $s(\alpha)+1$ rows. Let α^ψ be the n-partition corresponding to the conjugate graph Γ_1'. Then

$$s(\alpha^\psi) = a_1 - 1 \ (a_1 > 1 \text{ since } \alpha \text{ is odd}),$$
$$l(\alpha^\psi) = s(\alpha) - 1, \quad r(\alpha^\psi) \leqslant w(\alpha^\psi).$$

Hence $s(\alpha^\psi)$, $l(\alpha^\psi)$ and n have the same odevity, α^ψ is an even partition, and $r(\alpha^\psi) \leqslant w(\alpha^\psi)$.

From the definition of α^ψ, we know that every even partition with multiplicity less than or equal to its weight is an image of ψ, if its greatest part has the same odevity as n, except $n = m^2$ and $\alpha_0 = (\underbrace{m, m, \cdots, m}_{m \text{ times}})$. but this α_0 belongs to E_0.

Case 3. $l(\alpha)$ and n have opposite odevities, $r(\alpha) > w(\alpha)$, and the multiplicity r_s

of a_s is odd.

In this case, $s>r_s$, otherwise, $a_1=a_s$, $\alpha=(\underbrace{a_1, a_1, \cdots, a_1}_{r_s \text{ times}})$, $n=a_1r_s$, n and a_1 have the same odevity, contradicting to the hypothesis.

In the Ferrars graph $\Gamma(\alpha)$ of α, the number of dots on the last row is less than the number of dots on the last column. The numbers of the dots of the last r_s rows are all equal to $w(\alpha)$, but the number of dots on the $(s-r_s)$th row is greater than $w(\alpha)$.

We delete the last r_s rows from $\Gamma(\alpha)$, and add the conjugate graph of these r_s rows to the right of the last column. Thus we get a Ferrars graph Γ_2, let α^ψ be the n-partition of Γ_2'.

From the definition of α^ψ, we have

$$s(\alpha^\psi)=a_1+r_s, \quad l(\alpha^\psi)=s(\alpha)-r_s,$$
$$w(\alpha^\psi)=w(\alpha), \quad r(\alpha^\psi)=a_{s-r}.$$

Hence $s(\alpha^\psi)$ and n have the same odevity, and α^ψ is an even partition. The greatest part $l(\alpha^\psi)$ of α^ψ has the same odevity as n, $r(\alpha^\psi)>w(\alpha^\psi)$. Moreover, $l(\alpha^\psi)$ is distinct from $w(\alpha^\psi)$. The multiplicity of $w(\alpha^\psi)$ is r_s, an odd number.

We also see that all even n-partitions satisfyng these conditions are images of ψ, and distinct α's give distinct images.

Hence, if β is an even partition with multiplicity greater than weight, and the multiplicity of its last part is odd, then β is not an image of ψ only when its greatest part is equal to the last part. In this case,

$$\beta=(\underbrace{m, m, \cdots, m}_{s \text{ times}}),$$

here s is an odd number, and $s>m$. Since β is even, m must be odd. The partitions of such form belong to E_0.

Case 4. a_1 and n have opposite odevities, $r(\alpha)>w(\alpha)$, but the multiplicity of a_s is even.

After deleting the last r_s rows from $\Gamma(\alpha)$, we get a new graph Γ_1. We consider the $(n-r_sa_s)$-partition γ_1 corresponding to Γ_1. Because $s(\gamma_1)=s(\alpha_1)-r_s$ and r_s is even, γ_1 is also an odd partition. Moreover, $l(\gamma_1)=a_1$ and $n-r_sa_s$ have opposite odevities.

If γ_1 is also in the Case 4, we delete an even number of rows again and consider the partition corresponding to the subgraph, and so on.

Because we delete an even number of columns for each step, and α is an odd partition, we can divide $\Gamma(\alpha)$ into two subgraphs. The upper subgraph Γ_2 is conresponding to a partition γ_2 in the Case 2 or 3, while in the lower subgraph Γ_3, the numbers of rows with the same length are all even. And the number of dots on the first row of Γ_3 is less than the numbers of dots on the last row and last column of Γ_2.

Using the same method as that in case 2 or 3, we get an even partition γ_2' from γ_2. Let γ_3 be the partition with Ferrars graph $(\Gamma_3)'$. Then γ_3 is an even partition, and

$$w(\gamma_2')>l(\gamma_3).$$

Hence, we get an even n-partition (γ_2', γ_3). Set $\alpha^\psi=(\gamma_2', \gamma_3)$.

The greatest part of α^ψ is equal to $l(\gamma_2')$, thus its odevity is the same as n. $r(\alpha^\psi)=r(\gamma_2')>l(\gamma_3)\geqslant w(\alpha^\psi)$. And the multiplicity of the last part of α^ψ is even.

According to the construction of α^ψ, we know that distinct α's have distinct images.

Now we investigate the conditions for which an even partition, with multiplicity greater than weight, with its greatest part having the same odevity as n, and with the multiplicity of the last part even, cannot be taken as an image of ψ. If β is not an image, then, after deleting some parts with even multiplicity from β, we get a partition which is not an image in the case 2 or 3. Hence, it must be $(\underbrace{m, m, \cdots, m}_{r \text{ times}})$; $r\geqslant m$; r, m and n have the same odevity. These partitions all belong to E_0.

As any odd partition must be one of the four cases, and the partitions in E_0 are just all partitions which are not images of ψ, we conclude that ψ is a one-to-one correspondence from O onto $E-E_0$. The theorem is proved.

Because an n-partition is even iff the number of its parts has the same odevity as n. Theorem 1 is an equivalent statement of the Sylvester's identity.

Theorem 2. *The excess of the number of partitions with an even number of parts over that of partitions with an odd number of parts of the same number n is equal to $(-1)^n q(n)$.*

There are corollaries of Theorem 1.

Corollary 1. *In S_n $(n>2)$, the number of even conjugate classes is always greater than that of odd conjugate classes.*

Corollary 2. *The number of A-conjugate classes of S_n is equal to $p(n)+d(n)$. The number of conjugate classes of A_n is equal to $\frac{1}{2}\{p(n)+3d(n)\}$.*

Corollary 3 (Sylvester). $p(n)-q(n)$ *is even.*

Proof From Theorem 1

$$p_e(n)-p_0(n)=(-1)^n q(n).$$

Hence, we get from

$$p(n)=p_e(n)+p_0(n)$$

that

$$p(n)-q(n)=\begin{cases}2p_0(n), & \text{when n is even,}\\ 2p_e(n), & \text{when n is odd.}\end{cases}$$

i. e. $p(n)-q(n)$ is even.

The author is much indebted to professor Duan Xuefu for his critical reading of the manuscript.

References

[1] Gupta, H., Selected topics in number thoery, Abacus Press, 1980.
[2] Kerber, A., Representations of permutation groups I. Springer-Verlag, 1971.
[3] Sylvester, J. J., The collected mathematical popers. Vol. 4, Cambridge Univ. Press, 1912, reprinted by Chelsea, 1974.