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ON DEGREES OF MINIMAL IMMERSIONS OF SYMMETRIC SPACES INTO SPHERES

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Abstract

This paper considers the degrees of minimal immersions of symmetric spaces into spheres. A practical way to count the degree of a given regular minimal immersion of a compact irreducible symmetric space into sphere S_1^n is given. As an application, degrees of regular minimal immersions of all rank one compact symmetric spaces into spheres are counted.

§ 1. Introduction

In this paper, we consider the degrees of minimal immersions of symmetric spaces into spheres.

By a theorem in [3], every compact homogeneous space can be isometrically minimally immersed into some sphere S^n for n large enugh. Takahashi[6] showed that any compact irreducible homogeneous space can be isometrcally minimally immersed into some S^n by using its spaces of eigenfunctions satisfying the equation

$$\Delta\varphi = -\lambda\varphi \tag{1.1}$$

for some constant λ. The set of those constants which insure (1.1) has nontrivial solution is called the spectrum of the Laplace operator Δ on M, denoted by Spec (M). In 1971, do Carmo and Wallach[1] considered the case when M is also a standard sphere. In 1972, Wallach[7] considered the case when M is symmetric. From [1] and [7], one can see that the higher fundamental forms play an important role in the study of the minimal immersions of compact homogeneous spaces into spheres. One natural problem about the higher fundamental forms is how to find the degree of a given minimal immersion of a compact irreducible homogeneous space into a standard sphers S^n? The case $M=S^n$ was considered in [1]. Also, in [5] the degree was counted for $M=CP^n$.

In this paper, we firt classify the class 1 representations for the symmetric pair (G, K) in Sectiion 2. Then we give two theorems in Section 3. Theorem 3.1 points out that for a regular minimal immersion of an irreducible compact homogeneous

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space into some spheres the degree of the immersion equals to the degree of the corresponding standard minimal immersion (Section 3). Theorem 3.2 reduces the problem to simple cases for all irreducible compact symmetric spaces of the inner type. By using these two theorems, we give a practical way to count the degrees of regular minimal immersions of compact symmetric spaces of the inner type into spheres, and figure out the degrees of regular minimal immersions of rank one compact symmetric spaces in Section 4.

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§ 2. Class 1 Representations of Symmetric Pair (G, K)

In this section, we deal with some special representations of Lie groups, which give the most important minimal immersions of the compact irreducible homogeneous spaces, and give the classification result of these representations for those Lie group pairs (G, K) which correspond to symmetric spaces.

Let $G/K=M$ be a homogeneous space, where G is a Lie group, K is a closed subgroup of G.

Definition 2.1. *Let (ρ, V) be a finite dimensional irreducible representation of G, where V is either a real vector space or a complex vector space. If there is a nonzero vector $v\in V$ such that $\rho(k)v=v$ for all $k\in K$, then (ρ, V) is said to be a real (or complex) class 1 representation of (G, K). Or equivalently, we call V a real (complex) class 1 (G, K)-module.*

If V is a real (complex) class 1 (G,K)-module and G is compact, then by a well known theorem of H. Weyl, one can define a G-invariant inner product (or Hermitian form) on V so that V becomes an orthogonal (or Hermitian) space.

Now, we assume (G, K) is a compact symmetric pair, i.e. G/K is a compact symmetric space. Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a fixed Cartan decomposition of $\mathfrak{g}$. Select a maximal Abelian subspace $\mathfrak{h}_\mathfrak{p}$ of $\mathfrak{p}$ and extend it to a Cartan subalgebra $\mathfrak{h}=\mathfrak{h}_\mathfrak{k}+\mathfrak{h}_\mathfrak{p}$ of $\mathfrak{g}$ as usual. Introduce compatible orderings into the dual spaces of $\mathfrak{h}_\mathfrak{p}$ and $\mathfrak{h}_\mathfrak{k}+\sqrt{-1}\mathfrak{h}_\mathfrak{p}$. Denote Σ^+ (resp. Δ^+) the corresponding set of positive roots of the pair $(\mathfrak{g}, \mathfrak{h}_\mathfrak{p})$ (resp. $(\mathfrak{g}^c, \mathfrak{h}^c)$). Let $\pi=\{\alpha_1, \cdots, \alpha_n\}\in\Delta$ be a simple roots system of $(\mathfrak{g}^c, \mathfrak{h}^c)$. Let λ_i be defined by

$$2\langle\lambda_i,\alpha_j\rangle/\langle\alpha_j, \alpha_j\rangle=\delta_{ij}, \quad 1\leqslant i, j\leqslant n.$$

Then λ_i, $1\leqslant i\leqslant n$, are the fundamental dominant weights of $\mathfrak{g}^c$. If $\Lambda=\Sigma n_i\lambda_i$, n_i are nonnegative integers, then Λ is a dominant weight of $\mathfrak{g}^c$. The following theorem gives a necessary and sufficient condition for Λ to ensure the irreducible complex G-moudle V determined by Λ is a class 1 module (cf. [8]).

Theorem 2.1 (Cartan-Helgason). *Let Λ be a dominant weight. Let (G, K) be a symmetric pair. Then Λ determines a complex class 1 (G, K)-module iff:*

(i) $\Lambda|_{\mathfrak{h}_{\mathfrak{k}}}\equiv 0$, *and*

(ii) $\langle\Lambda, \lambda\rangle/\langle\lambda, \lambda\rangle$ *is a nonnegative integer for all* $\lambda\in\Sigma^{+}$.

We want to express Cartan-Helgason theorem in a more practical way by means of Satake diagram. For this purpose, we need three lemmas.

Keep the above notations. Let $\mathfrak{g}_0=\mathfrak{k}+\sqrt{-1}\,\mathfrak{p}$. Let σ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}_0$, τ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. Put $\theta=\tau\sigma$. Let Δ denote the set of roots of $(\mathfrak{g}^c, \mathfrak{h}^c)$. For $\alpha\in\Delta$, let the linear functions $\alpha^\tau, \alpha^\sigma$ and α^θ be defined by

$$\alpha^\tau(H)=\overline{\alpha(\tau H)},\ \alpha^\sigma(H)=\overline{\alpha(\sigma H)},\ \alpha^\theta(H)=\overline{\alpha(\theta H)},\ H\in\mathfrak{h}^c.$$

Denote $\Delta_0=\{\alpha\in\Delta;\ \alpha|_{\mathfrak{h}_{\mathfrak{p}}}=0\}$. Then for $\alpha\in\Delta^+-\Delta_0$, $\alpha^\sigma\in\Delta^+-\Delta_0$, $\alpha^\tau=-\alpha$, $-\alpha^\theta\in\Delta^+-\Delta_0$. For $\alpha\in\Delta_0^+=\Delta^+\cap\Delta_0$, $\beta^\sigma=-\beta$, $\beta^\theta=\beta$, $\beta^\tau=-\beta$. We have the following two lemmas (cf. [2]).

Lemma 2.1. *If $\alpha\in\Delta$, then $\alpha^\sigma-\alpha\notin\Delta$.*

Lemma 2.2. *If $\alpha\in\Delta-\Delta_0$, then the restriction $\tilde{\alpha}$ of α to $\mathfrak{h}^c_{\mathfrak{p}}$ satisfies $\langle\alpha, \alpha\rangle=m\langle\tilde{\alpha}, \tilde{\alpha}\rangle$, where $m=1$, 2 or 4. And $2\tilde{\alpha}\in\Sigma$* (restricted roots), *iff $m=4$, iff $\langle\alpha, \alpha^\sigma\rangle<0$.*

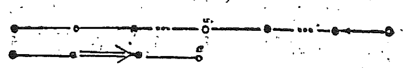
Let $\pi_0=\pi\cap\Delta_0$. In the Dynkin diagram of π, we mark the elements of π_0 with black dots. If $\alpha_i, \alpha_j\in\pi-\pi_0$, and their restrictions $\tilde{\alpha}_i$, $\tilde{\alpha}_j$ to $\mathfrak{h}^c_{\mathfrak{p}}$ equal, then we join α_i and α_j with ⌒. Such a diagram is called a Satake diagram. In the following lemma, we assume $\pi_0\neq\phi$.

Lemma 2.3. *Let $\alpha\in\pi-\pi_0$. Then*

i) *if there is no $\beta\in\pi-\pi_0$ so that $\tilde{\beta}=\tilde{\alpha}$ other than α, then $\alpha^\sigma=\alpha$ for $\langle\alpha, \pi_0\rangle=0$; $\alpha^\sigma\neq\alpha$* and *$\langle\alpha, \alpha^\sigma\rangle\leqslant 0$ for $\langle\alpha, \pi_0\rangle\neq 0$; moreover $\langle\alpha, \alpha^\sigma\rangle<0$ iff α is one of the cases as marked in the following diagrams;*

CII

FII



ii) *if there is $\alpha\neq\beta\in\pi-\pi_0$ such that $\tilde{\beta}=\tilde{\alpha}$, then $\alpha^\sigma\neq\alpha$; furthermore, $\alpha^\sigma=\beta$ and $\langle\alpha^\sigma, \alpha\rangle=0$ for $\langle\alpha, \pi_0\rangle=0$, $\alpha^\sigma\neq\beta$ and $\langle\alpha^\sigma, \alpha\rangle<0$* for *$\langle\alpha, \pi_0\rangle\neq 0$.*

Proof For convenience, let $\pi_0=\{\alpha_{l+1}, \cdots, \alpha_n\}$, $\pi-\pi_0=\{\alpha_1, \cdots, \alpha_l\}$. We first prove, for $\alpha_i\in\pi-\pi_0$, there is $\alpha_{i'}\in\pi-\pi_0$ so that

$$\alpha_i^\sigma=\alpha_{i'}+\sum_{j\geq l+1}n_j^i\alpha_j, \tag{2.1}$$

where n_j^i are integers. Let

$$\alpha_i^\sigma=\sum_{j=1}^{n}n_j^i\alpha_j. \tag{2.2}$$

Since $\alpha_i^\sigma\in\Delta^+-\Delta_0$ as it was pointed out above, there must be $i'\in\{1, \cdots, l\}$ such that $n_{i'}^i>0$. Now

$$\alpha_i=(\alpha_i^\sigma)^\sigma=\sum_{j=1}^{l}\sum_{k=1}^{n} n_j^i n_k^j \alpha_k-\sum_{j=l+1}^{n} n_j^i \alpha_j$$
$$=\sum_{k=1}^{n}\left(\sum_{j=1}^{l} n_j^i n_k^j\right)\alpha_k-\sum_{j=l+1}^{n} n_j^i \alpha_j.$$

Thus $\sum_{j=1}^{l} n_j^i n_i^j=1$, $\sum_{j=1}^{l} n_j^i n_k^j=0$ for $k\neq i$. Since every summand in these two equalities is $\geqslant 0$ and for $1\leqslant j\leqslant l$, there is $1\leqslant s\leqslant l$ so that $n_s^j>0$ by the definition of n_j^i. we see that if there is $n_j^i\neq 0$, there must be n_s^j $(1\leqslant s\leqslant l)$ such that $n_j^i n_s^j\neq 0$. This implies $s=i$. Now $n_{i'}^i>0$, so $n_{i'}^i=1$, $n_i^{i'}=1$ and $n_j^i=0(1\leqslant j\leqslant l,\ j\neq i')$. So (2.1) is proved.

Now, we assume there is no $\beta\in\pi-\pi_0$ so that $\tilde{\beta}=\tilde{\alpha}_i$ other then α. Than by (2.1) and $\tilde{\alpha}_i^\sigma=\tilde{\alpha}_i$, we get $\alpha_{i'}=\alpha_i$. Furthermore, if $\langle\alpha_i,\pi_0\rangle=0$, then $\alpha_i+\sum_{j=l+1}^{n} n_j^i\alpha_j$ is not a root unless all $n_j^i=0$. Thus by (2.1), $\alpha_i^\sigma=\alpha_i$. If $\langle\alpha_i,\pi_0\rangle\neq 0$, let π_i be the maximum element among all those subsets π' of π which satisfy $\pi'\cap\pi-\pi_0=\{\alpha_i\}$, then $\pi_i-\{\alpha_i\}\neq\phi$. Let β be the highest one among all roots

$$\beta'=\alpha_i+\sum_{\pi_i\cap\pi_0} c'_k\alpha_k,$$

then

$$\beta=\alpha_i+\sum_{\pi_i\cap\pi_0} c_k\alpha_k\in\Delta^+-\Delta_0,$$

and there is at least one $c_k\neq 0$. Thus

$$\beta^\sigma=\alpha_i^\sigma-\sum_{\pi_i\cap\pi_0} c_k\alpha_k\in\Delta^+-\Delta_0 \tag{2.3}$$

and $\alpha_i^\sigma\neq\alpha_i$. By Lemma 2.1, $\alpha_i^\sigma-\alpha_i$ is not a root of $\mathfrak{g}^c$, so $\langle\alpha_i,\alpha_i^\sigma\rangle\leqslant 0$. By (2.3) one sees that $\sum_{\pi_i\cap\pi_0} c_k\alpha_k$ is a summand in the expression of α_i^σ expressed by elements in π, and the coefficient of α_i is 1. Thus $\alpha_i^\sigma=\beta$. Under the condition of i), β is the highest root generated by π_i except for CII and FII. By checking the Satake diagrams, one sees that $\langle\alpha_i^\sigma,\alpha_i\rangle=\langle\beta,\alpha_i\rangle=0$. In the cases CII and FII, $\langle\alpha_i^\sigma,\alpha_i\rangle<0$ holds only for the roots that are marked on the diagrams. Thus i) is proved.

We now prove ii). Let $\alpha\neq\beta\in\pi-\pi_0$ so that $\tilde{\alpha}=\tilde{\beta}$. If $\langle\alpha,\pi_0\rangle=0$, then by $\tilde{\alpha}^\sigma=\tilde{\alpha}$ we see $\alpha^\sigma=\alpha$ or $\alpha^\sigma=\beta$. If $\alpha^\sigma=\alpha$, then $\tilde{\alpha}=\alpha\in\Sigma$, and $\beta^\sigma=\beta\in\Sigma$, contradicting $\tilde{\alpha}=\tilde{\beta}$. Thus $\alpha^\sigma=\beta$. By checking Satake diagrams, we get $\langle\alpha^\sigma,\alpha\rangle=0$. If $\langle\alpha,\pi_0\rangle\neq 0$, then $2\tilde{\alpha}\in\Sigma$. By Lemma 2.2, $\langle\alpha^\sigma,\alpha\rangle<0$. Thus ii) is proved.

Q. E. D.

From Lemma 2.3, we know that if $\alpha\in\pi-\pi_0$, then $\langle\alpha,\alpha^\sigma\rangle>0$ iff $\alpha=\alpha^\sigma$.

Suppose $\Lambda=\Sigma n_i\lambda_i$ is dominant and satisfys the conditions i) and ii) in Theorem 2.1. Let $\alpha_i\in\pi-\pi_0$, $\lambda=\tilde{\alpha}_i$. If $\alpha_i^\sigma=\alpha_i$, then $\lambda=\alpha_i$. Thus

$$\langle\Lambda,\lambda\rangle/\langle\lambda,\lambda\rangle=\langle\Lambda,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle=\frac{1}{2}n_i \tag{2.4}$$

If $\alpha^\sigma\neq\alpha$, then by Lemma 2.2

$$\langle\Lambda,\lambda\rangle/\langle\lambda,\lambda\rangle=\begin{cases}\frac{1}{2}\langle\Lambda,\tilde{\alpha}_i+\tilde{\alpha}_i^\sigma\rangle\Big/\frac{1}{2}\langle\alpha_i,\alpha_i\rangle, & \langle\alpha_i,\alpha_i^\sigma\rangle=0;\\ \frac{1}{2}\langle\Lambda,\tilde{\alpha}_i+\tilde{\alpha}_i^\sigma\rangle\Big/\frac{1}{4}\langle\alpha_i,\alpha_i\rangle, & \langle\alpha_i,\alpha_i^\sigma\rangle<0,\end{cases}$$

$$=\begin{cases}\langle\Lambda,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle+\langle\Lambda,\alpha_i^\sigma\rangle/\langle\alpha_i^\sigma,\alpha_i^\sigma\rangle;\\ 2\langle\Lambda,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle+2\langle\Lambda,\alpha_i^\sigma\rangle/\langle\alpha_i^\sigma,\alpha_i^\sigma\rangle,\end{cases}$$

$$=\begin{cases}\frac{1}{2}n_i+\frac{1}{2}n_{i'}, & \langle\alpha_i,\alpha_i^\sigma\rangle=0;\\ n_i+n_{i'}, & \langle\alpha_i,\alpha_i^\sigma\rangle<0.\end{cases} \tag{2.5}$$

where i' is defined by (2.1). Furthermore, if $\langle\alpha,\alpha^\sigma\rangle<0$, then $2\lambda\in\Sigma^+$, and by Lemma 2.2

$$\langle\Lambda,2\lambda\rangle/\langle2\lambda,2\lambda\rangle=\langle\Lambda,\alpha_i\rangle/\langle\alpha_i,\alpha_i\rangle+\langle\Lambda,\alpha_i^\sigma\rangle/\langle\alpha_i^\sigma,\alpha_i^\sigma\rangle=\frac{1}{2}(n_i+n_{i'}). \tag{2.6}$$

Now combine (2.4), (2.5), (2.6) and Lemma 2.3, we get

Theorem 2.2. *Let $\Lambda=\Sigma n_i\lambda_i$ be a dominant weight of $\mathfrak{g}^c$, where λ_i are fundamental dominant weights with respect to $\pi=\{\alpha_1,\cdots,\alpha_n\}$. Then, Λ determines a class 1 (G, K)-module ((G, K) is a symmetric pair) iff*

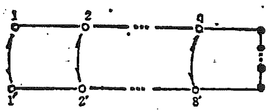
i) *for $\alpha_i\in\pi_0$, $n_i=0$; for $\alpha_i\in\pi-\pi_0$, $\langle\Lambda,\alpha_i\rangle=\langle\Lambda,\alpha_{i'}\rangle$,* i. e. *$n_i=n_{i'}$, $\alpha_{i'}$ is defined by* (2.1); *and*

ii) *if $\alpha_i\in\pi-\pi_0$ and there is no other $\alpha_j\in\pi-\pi_0$ such that $\tilde{\alpha}_i=\tilde{\alpha}_j$, then $n_i\equiv0$ (mod 2) if $\langle\alpha_i,\pi_0\rangle=0$ and n_i is arbitrary if $\langle\alpha_i,\pi_0\rangle\neq0$; if there is $\alpha_j\neq\alpha_i$ such that $\tilde{\alpha}_j=\tilde{\alpha}_i$, then $n_i+n_j\equiv0$ (mod 2).*

Remark. The above result was also obtained by sugiura.

By using Theorem 2.2 and Satake diagrams, one can write down all class 1 dominant weights for any symmetric pair (G, K) easily.

Example. Let $M=SU(p+q)/S(U_p\times U_q)$ $p>q$. The corresponding Satake diagram is



Then, a dominant weight $\Lambda=\Sigma n_i\lambda_i$ is class 1 iff $n_i=n_{i'}$ for $1\leqslant i\leqslant q$, and $n_i=0$ otherwise. Thus every class 1 dominant weight is given by nonnegative integer linear combination of the following dominant weights

$$\Lambda_{i+i'}=\lambda_i+\lambda_{i'},\quad 1\leqslant i\leqslant q.$$

We would like to call a dominant weights set that gives all class 1 weights for (G, K) by nonnegative integer linear combination and whose elements are linear independent a basic class 1 dominant weights set of the pair (G, K).

§ 3. Higher Fundamental Forms and Degrees

In this section, following [4], we define the higher fundamental forms and the osculationg spaces of an immersion of a Riemannian manifold into a constant curvature Riemannian manifold, and give two theorems on the degree of a minimal isometric immersion of an irreducible compact homogeneous space into sphere.

Let $\overline{M}$ be a Riemennian manifold with constant curvature K, M be a Riemannian manifold. Let x: $M \to \overline{M}$ be an isometric immersion. Since x is locally an imbedding, for convenience, we may identify M with its image in $\overline{M}$. Then for $p \in M$, $T_p(M) \subset T_p(\overline{M})$.

Let $p \in M$ and let B_{2p}; $T_p(M) \times T_p(M) \to N_p(M) = \{v \in T_p(\overline{M}) : \langle v, T_p(M) \rangle_p = 0\}$ ($\langle \ , \ \rangle$ is the Riemannian metric on M) be the second fundamantal form of x. We set $O_p^2(M)$ equal to the linear span of the image of B_{2p}. We say that $p \in M$ is degree 2 regular if $O_p^2(M)$ is of maximal dimension. Let $R_2 \subset M$ be the space of all degree 2 regular elements of M. Then R_2 is open in M. Let $p \in R_2$. Let N_2 be the normal projection in $N_p(M)$ relative to $N_p(M) = O_p^2 \oplus O_p^{2\perp}$ i. e. N_2: $v \to v^{N_2} \in O_p^{\perp}$. Define B_{3p} $(x_1, x_2 x_3) = (\overline{\nabla}_{x_{1p}} B_2(x_2, x_3))^{N_2}$ for x_1, x_2, $x_3 \in T_p(M)$ arbitrarily extended to vector fields on M, where $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$. Let O_p^3 be the linear span of the image of B_{3p}. $p \in R_2$ is said to be degree 3 regular if dim O_p^3 is maximal. We define B_{ip}, O_p^i for $i = 2, 3, \cdots$ by recursion as above on the space R_{i-1} of all degree $i-1$ regular points of M. Clearly the above process must eventually stop since dim $(T_p(M) + O_p^2 + O_p^3 + \cdots + O_p^m) \leqslant \dim T_p(\overline{M})$. Let m be the first integer $\geqslant 1$ such that $B_m \neq 0$ but $B_{m+1} \equiv 0$. Then we call m the degree of x. Since R_m is open in M_1, $R_m \neq \phi$. Let, for each nonnegative integer k, $S^k(T_p(M))$ be the k-fold symmetric power of $T_p(M)$. The universal property of $S^k(T_p(M))$ says that for $p \in R_{k-1}$, B_k induces a linear map of $S^k(T_p(M)) \to O_k^p$. Let O_p: $S^+(T_p(M)) \to T_p(\overline{M})$ for $p \in R_m$ be defined by $Q_p = x_{*p} + B_{2p} + \cdots + B_{mp}$, where $S^+(T_p(M)) = \sum_{k=1}^{\infty} S^k(T_p(M))$. Then Q_p is call the higher fundamental form of x at p, B_{ip} is called the i-th fundamental form of x at p (call x_* the first fundamental form of x). The integer m is also called the degree of Q_p.

Now we assume that $M = G/K$ is an isotropic compact connected homogeneous space, where G is a compact connected Lie group, K its closed connected subgroup.

Let V be a nontrivial real class 1 (G, K)-module with K-fixed unit vector v. Then by Prop. 8.1 in [4], the map x: $M \to S_1$ (the unit sphere of V) defined by $x(gK) = g \cdot v$ is a minimal isometric immersion of a multiple of the G-invariant metric of M. We call minimal immersions of M into spheres defined by real class 1 (G, K)-modules standard minimal immersions.

Let x: $M \to S_1^q \subset E^{q+1}$ be a full isometric minimal immersion, that is, $x(M)$ is not contained in any great sphere of S_1^q. Then according to Theorem 9.1 in [4], there is a class 1 (G, K)-module $V = E^{p_\lambda+1}$ and the corresponding standard minimal immersion $x_\lambda: M \to S_1^{p_\lambda}$, a linear isometric injection $A: E^{q+1} \to E^{p_\lambda+1}$ such that $A \circ x = B \circ x_\lambda$ for a linear map $B: E^{p_\lambda+1} \to E^{p_\lambda+1}$.

Thus, every isometric minimal immersion of M can be obtained from a standard minimal immersion by a linear transformation. By virtue of Section 12 in the same paper, if $x_\lambda: M \to S_1^{p_\lambda} \subset E^{p_\lambda+1}$ is a standard minimal immersion, then there is a compact convex body $L_\lambda \subset S^2(E^{p_\lambda+1})$ such that L_λ parametrizes smoothly the set of all inequivalent (up to orthogonal transformation) minimal isometric immersions x: $M \to S_1^q$ (we allow $L_\lambda = \{0\}$). We sketch out this results as follows:

First, we identify the space of all symmetric mappings of $E^{p_\lambda+1}$ with $S^2(E^{p_\lambda+1})$: Let u, $v \in E^{p_\lambda+1}$, denote their symmetric product by uv, then $uv \in S^2(E^{p_\lambda+1})$. If $t \in E^{p_\lambda+1}$, we set

$$uv(t) = \frac{1}{2}\{\langle u, t\rangle v + \langle v, t\rangle u\},$$

where $\langle\ ,\ \rangle$ is the inner product on $E^{p_\lambda+1}$. Under this identification, the inner product on $S^2(E^{p_\lambda+1})$ is given by

$$(A, B) = \operatorname{tr} AB.$$

We identify $T_p(S_1^{p_\lambda})$ with $\{v \in E^{p_\lambda+1}: \langle v, p\rangle = 0\}$. Let, for each $p \in M$, $S^2(x_*(T_p(M)))$ be the symmetric square of $x_*(T_p(M))$ in $S^2(E^{p_\lambda+1})$. Let W be the subspace of $S^2(E^{p_\lambda+1})$ spanned by $\bigcup_{p \in M} S^2(x_*(T_p(M)))$. Let W_1 be the orthogonal complement of W in $S^2(E^{p_\lambda+1})$, $L_\lambda = \{C \in W_1: C + I \geqslant 0\}$.

Now we can give the parametrization. If $C \in L_\lambda$, then the corresponding minimal isometric immersion is given by $y_c = (C+I)^{\frac{1}{2}} \circ x_\lambda$. Under this correspondence, the interior points of L_λ correspond to minimal immersions $\sqrt{C+I}\ x_\lambda$ with $\sqrt{C+I} > 0$, which are full immersions of M into $S_1^{p_\lambda}$. We call them regular minimal immersions. For a boundary point, some eigenvalues of $\sqrt{C+I}$ are zero and the immersion is full into some S_1^h for some $h < P_\lambda$. We have

Theorem 3.1. *Let $G/K = M$ be an irreducible compact homogeneous space. For any interior point c of L_λ, the degree of the minimal immersion $y_c = \sqrt{C+I}\, x_\lambda$ defined above is equal to the degree of standard minimal immersion x_λ.*

In order to prove this theorem, we need a lemma.

Lemma 3.2. *Let x: $M \to S_1^v$ be a full isometric minimal immersion. Let T be an orthogonal transformation of S_1^v. Then x and $T_\circ x$ have the same degree.*

Proof Let m be the degree of x, R_m be the degree m regular points set of M, $p \in R_m$. Let $y = T \circ x$. Let $Q^1(Q^2)$ be the fundamental form of $x(y)$. Let $p_1 = x(p)$, $p_2 = y(p)$. We identify $T_p(M)$ with $T_{p_1}(M)$ or $T_{p_2}(M)$. Then

$$Q_{p_1}^1(S^+(T_p(M))) = R_{p_1} + x_*(T_p(M)) + O_{p_1}^2 + \cdots + O_{p_1}^{s_1}, \tag{3.1}$$

$$Q_{p_2}^2(S^+(T_p(M))) = R_{p_2} + y_*(T_p(M)) + \bar{O}_{p_2}^2 + \cdots + \bar{O}_{p_2}^{s_2}. \tag{3.2}$$

We prove $s_1 = s_2$. For this purpose, we need only to prove the i-th fundamental form

B_i^2 of y is the image of the i-th fundamental form B_i^1 of x under T.

Since $y_*(T_p(M))=T\circ x_*(T_p(M))$, the statement is true for $i=1$. Supposing the statement is true for $1\leqslant i\leqslant k-1$, we prove $B_k^2=T\circ B_k^1$. Let $X_{1p_2}, \cdots, X_{kp_2}\in T_{p_2}(M)$ and be arbitrarily extended to vector fields $X_1, \cdots, X_k$ on a neighborhood U of p_2 in M. Then there are vector fields $Y_1, \cdots, Y_k$ on $T^{-1}U$ such that $T(Y_i)=X_i$. Thus

$$\begin{aligned} B_{kp_2}^2(X_1, \cdots, X_k) &= (\overline{\nabla}_{x_{1p_2}}B^2, (X_2,\cdots, X_k))^{N_{k-1}^2} \\ &= (\overline{\nabla}_{(TY_1)p_2}B_{k-1}^2(TY_2, \cdots, TY_k))^{N_{k-1}^2} \\ &= (\overline{\nabla}_{(TY_1)p_2}(T\cdot B_{k-1}^1(Y_2, \cdots, Y_k)))^{N_{k-1}^2} \\ &= (T\overline{\nabla}_{Y_1p_1}B_{k-1}^1(Y_2, \cdots, Y_k))^{N_{k-1}^2}=T(\overline{\nabla}_{Y_1p_1}B_{k-1}^1(Y_2,\cdots, Y_k))^{N_{k-1}^1} \\ &= T\cdot B_{kp_1}^1(Y_1, \cdots, Y_k). \end{aligned}$$

The third step and 5-th step hold since T is orthogonal and $T\cdot O_{p_1}^i=O_{p_2}^i$ for $i\leqslant k-1$. Thus $B_i^2=T\circ B_i^1$ by induction. This implies $s_1=s_2$. Q. E. D.

Proof of Theorem 3.1 We use the same notations as above. Note that since x_λ is a standard minimal immersion, the fundamental form is defined entirely on M. Denote the degree of x_λ by α_x. Put $p_\lambda+1=N$. Then by § 12 in [4],

$$E^N=R_{p_1}+T_{p_1}(M)+O_{p_1}^2+\cdots+O_{p_1}^{\alpha_x}, \tag{3.3}$$

where $p\in M$, $p_1=x_1(p)$. Let $E_1, \cdots, E_N$ be an orthonormal basis of E^N such that

$$E_1\in R_{p_1},\ E_2, \cdots, E_{i_1}\in T_{p_1}(M),\ E_{i_1+1}, \cdots, E_{i_2}\in O_{p_1}^2, \cdots,$$
$$E_{i_{\alpha_x-1}+1}, \cdots, E_{i_{\alpha_x}}\in O_{p_1}^{x}\quad (i_{\alpha_x}=N).$$

Let $x_\lambda=(x_1, \cdots, x_N)$ under this basis. Since $\sqrt{C+I}$ is positive definite for an interior point $c\in L$, $\sqrt{C+I}=T\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}T'$, for some $\lambda_i>0$, $1\leqslant i\leqslant N$, and an orthogonal transformation T of E^N. By Lemma 3.2, we may assume $\sqrt{C+I}=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}$. In this case, $y_c=\sqrt{C+I}\,x_\lambda=(\lambda_1x_1, \cdots, \lambda_Nx_N)$.

Now we use the method used in [4]. Let $\sigma(t)$ be an arbitrary geodesic through p in M. Then

$$\frac{d^k}{dt^k}(x\circ\sigma)(0)\in R_{p_1}+T_{p_1}(M)+\cdots+O_{p_1}^k \tag{3.4}$$

by the definition of the higher fundamental forms. In order that

$$\frac{d^j}{dt^j}(x_i\circ\sigma)(0)\neq 0 \text{ and belongs to } O_{p_1}^{k+1}+\cdots+O_{p_1}^{\alpha_x},$$

it is necessary that $i>i_k$ and $j\geqslant k+1$. Since $y_i=\lambda_ix_i$, $\lambda_i>0 (1\leqslant i\leqslant N)$ is a constant, we have

$$\frac{d^k}{dt^k}(y_i\circ\sigma)(0)=0 \text{ iff } \frac{d^k}{dt^k}(x_i\circ\sigma)(0)=0.$$

It follows immediately from the definition of the higher fundamental forms that

$$\dim(R_{p_1}+T_{p_1}(M)+O_{p_1}^2+\cdots+O_{p_1}^i)=\dim(R_{p_2}+T_{p_2}(M)+\overline{O}_{p_2}^2+\cdots+\overline{O}_{p_2}^i)$$

for $1\leqslant i\leqslant\alpha_x$. Now by (3.3)

$$R_{p_2}+T_{p_2}(M)+\bar{O}^2_{p_2}+\cdots+\bar{O}^{\alpha_x}_{p_2}=E^N.$$

Thus the degree of y_c at p is the same as x_λ. But p is arbitrary in M, so Theorem 3.1 follows. Q, E, D

According to Theorem 3.1, in order to find the degree of the fundamental form of the minimal immersion $\sqrt{C+I}\,x_\lambda$ for an interior point of L_λ, we need only to find the degree of the corresponding standard minimal immersion x_λ. We note that if c is a boundary point of L_λ, Theorem 3.1 does not hold, for if c is a boundary point, $\sqrt{C+I}\,x_\lambda$ is a full isometric minimal immersion of M into S^q_1 for some $q<p_\lambda$. In fact, Hsiang's[2] gave some examples of minimal imbeddings of S^{n-1} into S^n. Since codimension $=1$ in these cases, the degrees of these imbeddings are$\leqslant 3$. But these imbedding correspond to the boundary points of certain L_λ for which the degree of the standard minimal immersion x_λ is>3.

Now let $M=G/K$ be an irreducible compact symmetric space. Denote the Lie algebras of G and K by$\mathfrak{g}$ and $\mathfrak{k}$ respectively. Let $\mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{h}$. Let $\alpha_1, \cdots, \alpha_r$ be a simple roots system of the pair $(\mathfrak{g}, \mathfrak{g})$. Let λ_i, $i=1, \cdots, r$ be defined by

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)}=\delta_{ij}, \quad 1\leqslant i, j\leqslant r.$$

Then $\lambda_i(1\leqslant i\leqslant r)$ are fundamental dominant weights of $\mathfrak{h}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then $T_{eK}(M)$ can be identified with $\mathfrak{p}$.

If there exists a Cartan algebra $\mathfrak{g}$ of $\mathfrak{h}$ in $\mathfrak{k}$, then we call M symmetric space of inner type. In the remains of this setion, we always assume M is of inner type. Such being the case, by [9], from Throrem 2.2 every complex class 1 (G, K)-module V is the complexification of some real class 1 (G, K)-module V_0. If Λ is the highest weight of V, then we also call Λ a highest weight of V_0.

Theorem 3.2. *Let $M=G/K$ be an inner type irreducible compact symmetric space. Let $\{\Lambda_1, \cdots, \Lambda_s\}$ be a basic class 1 dominant weights set of (G, K). Then*

i) Every $\Lambda_i\in\{\Lambda_1, \cdots, \Lambda_s\}$ corresponds to one and only one real class $1(G, K)$-module V_i up to isomorphism. Thus we may assume the degree of the standard minimal immersion defined by V_i is m_i.

ii) If $\Lambda=\Sigma k_i\Lambda_i$ is a class 1 dominant weight of (G, K), then Λ corresponds to one and only one real class 1 (G, K)-module V, the degree of the standard minimal immersion given by V is $\Sigma k_i m_i$.

Proof Since M is of inner type, if $V(\Lambda_1)$ and $V(\Lambda_2)$ are two real class 1 (G, K)-modules, then the irreducible component of the tensor product $V(\Lambda_1)\otimes V(\Lambda_2)$ with highest weight $\Lambda_1+\Lambda_2$ are real class 1 (G, K)-module. We denote it by V_0. Let

$$x_1: M\to S^{p_1}_1\subset V_1,$$

$$x_2\colon\ M\to S_1^{p_2}\subset V_2$$

be the corresponding standard minimal immersions defined by real class 1 (G, K)-modules V_1 and V_2 respectively. Assume the degree of x_1 is m_1, the degree of x_2 is m_2. Let

$$x\colon\ M\to S_1^{p}\subset V_0$$

be the standard minimal immersion defined by the real class 1 (G, K)-module V_0 defined as above. We show the degree of x is m_1+m_2.

Since M is homogeneous and all minimal immersions involved in the discussion are standard, we may consider the j-th fundamental form B_j of any minimal immersion y as defined in the j-th symmetric power $S^j(\mathfrak{p})$ of $\mathfrak{p}$, and interpret it in terms of derivations by elements of $\mathfrak{p}$. For instance, in the case of B_2 at $v=x(\mathrm{eK})$, we set

$$Y(Xv)=\left(\frac{d}{dt}(e^{tY}\cdot Xv)\right)_{t=0},\quad X, Y\in\mathfrak{p}.$$

Then

$$B_2(Yv, Xv)=B_2(X, Y)=(Y(Xv))^{N_1}.$$

The other cases are treated similarly and the situation at an arbitrary point is obtained by equivariance.

Let $v_i=x_i(\mathrm{eK})\,(i=1, 2)$, $v=x(\mathrm{eK})$. Then by the beginning of this section, we have

$$V_1=V_1^0+V_1^1+\cdots+V_1^{m_1},\tag{3.5}$$

$$V_2=V_2^0+V_2^1+\cdots+V_2^{m_2},\tag{3.6}$$

$$V_0=V^0+V^1+\cdots+V^q,\tag{3.7}$$

where $V_i^0=Rv_i$, $i=1, 2$; $V_i^j=O_{v_i}^j$, $V^0=Rv_0$, $V^i=O^i$. We have to show $q=m_1+m_2$.

First we show since (G, K) is of inner type, for an arbitrary complex class 1 (G, K)-module $V(\Lambda)$, if $v\in V(\Lambda)$ is a K-fixed nonzero vector, then v is a weight vector, i.e. there exists a weight subspace V_λ of $V(\Lambda)$ such that $v\in V_\lambda$. Denote the weights set of $V(\Lambda)$ by $\Pi(\Lambda)$. Then

$$v=\sum_{\mu\in\Pi(\Lambda)}v^\mu,\quad v^\mu\in V_\mu.$$

Since there exists a Cartan subalgetra $\mathfrak{g}$ of $\mathfrak{h}$ in $\mathfrak{k}$, and for any $X\in\mathfrak{k}$, $X\cdot v=0$, we have

$$v=\sum_{\Pi_0(\Lambda)}v^\mu,\tag{3.8}$$

where $\Pi_0(\Lambda)\subset\Pi(\Lambda)$ is a weights subset such that for any $H\in\mathfrak{f}$ and $\mu\in\Pi_0(\Lambda)$, $\mu(H)=0$. If $\mu\in\Pi_0(\Lambda)$, then for any $\alpha_i\in\pi=\{\alpha_1, \cdots, \alpha_s\}$ (simple roots system of $(\mathfrak{g}, \mathfrak{h})$), we have

$$\langle\lambda, \alpha_i\rangle=0,\tag{3.9}$$

where $\langle\lambda, \alpha_i\rangle=2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$. Since $\lambda=\Lambda-\Sigma k_i\alpha_i$, where k_i are nonnegative integers, by (3.9) we have

$$\langle \Lambda, \alpha_i\rangle - \Sigma k_j \langle \alpha_j, \alpha_i\rangle = 0, \quad 1 \leqslant i \leqslant s. \tag{3.10}$$

Relation (3.10) is a linear equations system satisfied by k_i. Since the coefficient matrix of this system is the Cartan matrix, there exists one and only one solution. Thus $\Pi_0(\Lambda)$ consists of only one element. Denote this element by λ. Then by (3.8) $v \in V_\lambda$.

Next we show $v_0 = cv_1 \cdot v_2$. Let $v_0 = \Sigma k_{ij}\, v^{\lambda_i} \cdot v^{\mu_j}$, where $v^{\lambda_i} \in V_1^{\lambda_i}$, $v^{\mu_j} \in V_2^{\mu_j}$ are weight vectors of V_1 and V_2 respectively. Since

$$\mathfrak{k}^c = \mathfrak{h}^c + \sum_{\Delta_k} \mathfrak{g}_\alpha,$$

if $\alpha_i \in \Delta_k$ is a simple root, $x_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, then $x_{\alpha_i} \cdot v_0 = 0$. Thus for any $v^{\lambda_l} \cdot v^{\mu_k}$ in the expression of v_0, $\langle \lambda_l, \alpha_i\rangle = \langle \mu_k, \alpha_i\rangle = 0$. Since the simple roots set of Δ_k has cardinality either r or $r-1$ ($r = \mathrm{Car}\,\pi$), it is not hard to prove $v_0 = cv_1 \cdot v$.

Since

$$X_1 \cdots X_i v_1 \in V_1^0 + \cdots + V_1^i, \tag{3.11}$$

$$X_1 \cdots X_i v_2 \in V_2^0 + \cdots + V_2^i \tag{3.12}$$

for $X_1, \cdots, X_i \in \mathfrak{p}$ by (3.5) and (3.6), we have for $X_1, \cdots, X_k \in \mathfrak{p}$,

$$X_1 \cdots X_k v_0 \in \sum_{i=1}^{k} \sum_{l+m=i} V_1^l\, V_2^m. \tag{3.13}$$

Thus $q = m_1 + m_2$.

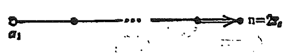
Now the theorem follows by induction. Q. E. D.

§ 4. Degrees of the Standard Minimal Immersions of Rank One Compact Symmetric Spaces into Spheres

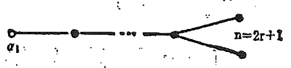
In this section, by using the results in Section 3, we figure out the degrees of all standard minimal immersions of rank one compact symmetric spaces.

For $M = S^n$, the degree of a standard minimal immersion was indicated in [7]. For $M = CP^n$, the degree was counted in [5]. For the sake of completeness, we also list the results of these two cases here.

Let $M = S^n$. The Satake diagram is

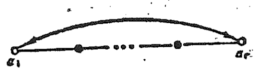


or



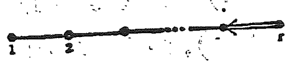
Thus, by Theorem 2.2, every class 1 dominant weight has the form $k\lambda_1$, where λ_1 is the fundamental dominant weight with respect to α_1. Let x_k be the standard minimal immersion defined by the real class 1 (G, K)-module V_k (V_k^c has highest weight $k\lambda_1$).

For $M = CP^n$, the diagram is



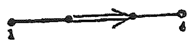
If V_0 is a real class 1 (G, K)-module, then V_0^c is a complex class 1 (G, K)-module with highest weight $k(\lambda_1+\lambda_r)$. Every complex class 1 (G,K)-module has the highest weight of the form $k(\lambda_1+\lambda_r)$and is the complexification of a real class 1 (G,K)-module. We denote the standard minimal immersion defined by real class 1 (G, K)-module V_k (V_k^c has highest weight $k(\lambda_1+\lambda_r)$) by x_k.

For $M=SP(n+1)/SP(n)\times SP(1)$, the diagram is



In this case, all class 1 dominant weights are given by $k\ \lambda_2$. We also denote the standard minimal immersion corresponding to $k\ \lambda_2$ by x_k.

For $M=FII$, the diagram is



All class 1 dominant weights are given by $k\ \lambda_4$. Denote the corresponding standard minimal immersion by x_k.

Theorem 4.1 *For S^n, the degree of x_k is k. For CP^n, $SP(n+1)/SP(n)\times SP(1)$ and FII, the degree of x_k is $2k$.*

Remark. The results were obtained by Mashimo.

Proof Since in all these four cases, real class 1 (G, K)-module V_k can be obtained from V_1 by taking the irreducible component with highest weight of V_1. Thus, from Theorem 3.2, one sees that noly the degree of x_1 must be counted.

For S^n, let x_1: $S^n\to S_1^p\subset V_1$. As a K-module, G-module$V_1=Rv\oplus V_1^n$, Rv is a 1-dimensional K-module, $V_1^n\simeq T_{ek}(S^n)$. Thus x_1 is of degree 1 by the definition of the higher fundamental form. So the degree of x_k is k.

Let $M=CP^n=SU(n+1)/S(U(1)\times U(n))$. Then A_n is the Lie algebra of $SU(n+1)$, the Lie subalgebra of $S(U(1)\times U(n)$is $R\oplus A_{n-1}$. Denote the fundamental dominant weights of A_n by $\lambda_1, \cdots, \lambda_n$ and the fundamental dominant weights of $R\oplus A_{n-1}$ by $\lambda_1', \cdots, \lambda_n'$. Let $V_k=V_0(k(\lambda_1+\lambda_n))$. Let

$$x_1:\ CP^n\to S_1^p\subset V_0(\lambda_1+\lambda_n).$$

As a K-module, $V_0(\lambda_1+\lambda_n)=Rv\oplus V_1^k\oplus V_2^k$, where Rv is a 1-dimensional K-module, $V_1^k\simeq T_{ek}(M)$. V_2^k is an irreducible K-module such that $(V_2^k)^c=V^k(\lambda_1'+\lambda_{n-1}')$. Thus, by the definition of higher fundamental form, the degree of x_1 is 2. So the degree of x_k is $2k$.

Let $M=SP(n+1)/SP(n)\times SP(1)$. The Lie algebra of $G=SP(n+1)$ is C_{n+1}, the

Lie subalgebra of $K=SP(n)\times SP(1)$ is $C_1\oplus C_n$. Let $\alpha_0=x_0-x_1$, $\alpha_1=x_1-x_2,\cdots,\alpha_n=2x_n$ be a simple roots system of C_{n+1}. Let $\alpha'_0=2x_0$ and $\alpha'_i=\alpha_i$, $1\leqslant i\leqslant n$, be a simple roots system of $C_1\oplus C_n$. Then, any dominant weight of C_{n+1} can be uniquely expressed by

$$\Lambda=k_0x_0+k_1x_1+\cdots+k_nx_n,$$

where $k_0\geqslant k_1\geqslant\cdots\geqslant k_n\geqslant 0$ are integers. Any dominant weight Λ' of $C_1\oplus C_n$ can be uniquely expressed by

$$\Lambda'=h_0x_0+h_1x_1+\cdots+h_nx_n,$$

where $h_0\geqslant 0$, $h_1\geqslant\cdots\geqslant h_n\geqslant 0$ are integers. Now let

$$x_1:\ M\to S_1^p\subset V_0(\lambda_2)=V_0(x_0+x_1).$$

As a K-module, G-module $V_0(x_0+x_1)=V^k(x_0+x_1)\oplus V^k(x_1+x_2)\oplus Rv$. Since as an irreducible K-module, $T_{eK}\simeq V^k(x_0+x_1)$, x_1 is of degree 2. Thus x_k is of degree $2k$.

Let $G=F_4$, $K=SO(9)$, $M=G/K$. Denote the Lie algebra of G by f_4 and the Lie subalgebra of K by B_4. Let $\alpha_1=x_2-x_3$, $\alpha_2=x_3-x_4$, $\alpha_3=x_4$, $\alpha_4=\frac{1}{2}(x_1-x_2-x_3-x_4)$ be a simple roots system of f_4. Let $\alpha'_1=-x_1-x_2$, $\alpha'_2=x_2-x_3$, $\alpha'_3=x_3-x_4$, $\alpha'_4=x_4$ be a simple roots system of B_4. Denote the fundamental weights of f_4 by $\lambda_i, 1\leqslant i\leqslant 4$ (and λ'_i for B_4). Let

$$x_1:\ M\to S_1^p\subset V_0(\lambda_4).$$

As a K-module, $V_0(\lambda_4)=V_0(\lambda'_1)\oplus V_0(\lambda'_4)\oplus Rv$, $T_{eK}(M)\simeq V_0(\lambda_4)$. Thus, the degree of x_1 is 2, the degree of x_k is $2k$. Q.E.D.

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