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GLOBAL EXISTENCE FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract

This paper is concerned with the global existence of Cauchy problem for nonlinear parabolic equations. The sharp results concerning the space dimension have been obtained which improve the corresponding results obtained by S. Klainerman.

§ 1. Introduction

It is well known that the global existence or the blowing up in a finite time of solution for the initial value problem of nonlinear parabolic equation, similar to the case of nonlinear wave equation, is tightly connected with the dimension n of space variables[1-6].

Fujita, Kobayashi et al., Weissler[3-5] considered the initial value problem

$$\begin{cases} u_t - \Delta u = u^p, \\ u(0, x) = \phi(x) \end{cases}$$

and proved that if $p \leqslant 1 + \frac{2}{n}$, then for any smooth initial data $\phi \geqslant 0$, $\phi \not\equiv 0$ the solution u must blow up in a finite time no matter how small ϕ is. They also proved the global existence with small initial data when $p > 1 + \frac{2}{n}$. Klainerman[2] considered the following initial value problem for nonlinear parabolic equation

$$u_t - \Delta u = F(u, D_x u, D_x^2 u), \tag{1.1}$$

$$u(0, x) = \phi(x), \tag{1.2}$$

where $x = (x_1, \cdots, x_n) \in R^n$, $t \in R^+$, $D_x u = (u_{x_i})$, $D_x^2 u = (u_{x_i x_j})$ $(i, j = 1, \cdots, n)$, and F: $R \times R^n \times R^{n^2} \to R$ is a C function, $\phi \in H^\infty(R^n)$. Using the same technique as in [1], he proved that under the assumption

$$F(\lambda) \triangleq F(\lambda_0, \lambda_i, \lambda_{ij}) = O(|\lambda|^{p+1}), \text{ near } \lambda = 0, \tag{1.3}$$

where p is an integer, and

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$$\frac{1}{p}\left(1+\frac{1}{p}\right)<\frac{n}{2}, \tag{1.4}$$

there exists an integer $N_0>0$ and a small $\delta>0$ such that if

$$\|\phi\|_{W^{N_0,1}}\leqslant\delta, \quad \|\phi\|_{W^{N_0,2}}\leqslant\delta, \tag{1.5}$$

then the problem admits a unique global smooth solution u. Moreover, the solution has the following asymptotic behavior:

$$|u(t)|_{L^\infty}=O\left(t^{-\frac{1+\varepsilon}{p}}\right), \text{ as } t\to\infty, \tag{1.6}$$

$$\|u(t)\|_{L^2}=O(1), \text{ as } t\to+\infty. \tag{1.7}$$

In this paper we consider the same problem (1.1), (1.2) and get the following

Main Theorem.

(i) *Under the assmptions*

$$F(\lambda)=O(|\lambda|^2), \quad \text{near } \lambda=0 \tag{1.8}$$

and $n\geqslant3$, the problem (1.1), (1.2) has a unique global solution u provided ϕ satisfies (1.5). Moreover, we have the decay estimates:

$$|\Lambda u(t)|_{L^\infty}=O\left(t^{-\frac{3n}{8}+\varepsilon_1}\right), \quad t\to+\infty, \tag{1.9}$$

$$\|\Lambda u(t)\|_{L^2}=O\left(t^{-\frac{n}{4}}\right), \quad t\to+\infty, \tag{1.10}$$

where $\varepsilon_1>0$ is a sufficiently small number and

$$\Lambda u=(u, D_x u, D_t u, D_x^2 u). \tag{1.11}$$

(ii) *Under the assumptions*

$$F(\lambda)=O(|\lambda|^3), \text{ near } \lambda=0 \tag{1.12}$$

and $n\geqslant2$, or under the assumptions

$$F(\lambda)=O(|\lambda|^4), \text{ near } \lambda=0 \tag{1.13}$$

and $n\geqslant1$, the problem (1.1), (1.2) has a unique global smooth solution u provided ϕ satisfies (1.5). Moreover, in both cases we have the same decay estimates as in the case of linear heat equation:

$$|\Lambda u(t)|_{L^\infty}=O\left(t^{-\frac{n}{2}}\right), \quad t\to+\infty, \tag{1.14}$$

$$\|\Lambda u(t)\|_{L^2}=O\left(t^{-\frac{n}{4}}\right), \quad t\to+\infty. \tag{1.15}$$

(iii) *If F satisfies (1.8) and is independent of u (u, $D_x u$,* respectively*), then in (i) the assumption $n\geqslant3$ can be replaced by $n\geqslant2$ ($n\geqslant1$, respectively). Moreover, we have the decay estimates:*

$$|\Lambda_1 u(t)|_{L^\infty}=O\left(t^{-\frac{3(n+1)}{8}+\varepsilon_1}\right), \quad n\geqslant2, \tag{1.16}$$

$$\|\Lambda_1 u(t)\|_{L^2}=O\left(t^{-\frac{n+1}{4}}\right), \quad n\geqslant2, \tag{1.17}$$

$$|u(t)|_{L^\infty}=\begin{cases} O\left(t^{-\frac{3n}{8}+\varepsilon_1}\right), & n\geqslant3,\\ O\left(t^{-\frac{3(n+1)}{8}+1+\varepsilon_1}\right), & n=2,\end{cases} \tag{1.18}$$

$$\|u(t)\|_{L^2}=\begin{cases} O\left(t^{-\frac{n}{4}}\right), & \\ O\left(t^{-\frac{3(n+1)}{8}+1+\varepsilon_1}\right), & n=2,\end{cases} \tag{1.19}$$

where $\Lambda_1 u = (D_x u, D_t u, D_x^2 u)$ (*and we have*

$$|\Lambda_2 u(t)|_{L^\infty} = O(t^{-\frac{3(n+2)}{8}+\varepsilon_1}), \quad n \geqslant 1, \tag{1.20}$$

$$\|\Lambda_2 u(t)\|_{L^2} = O(t^{-\frac{n+2}{4}}), \quad n \geqslant 1, \tag{1.21}$$

where $\Lambda_2 u = (D_t u, D_x^2 u)$, *and*

$$|u(t)|_{L^\infty} = \begin{cases} O(t^{-\frac{3n}{8}+\varepsilon_1}), & n \geqslant 3, \\ O(t^{-\frac{3(n+2)}{8}+1+\varepsilon_1}), & 1 \leqslant n \leqslant 2, \end{cases} \tag{1.22}$$

$$|D_x u(t)|_{L^\infty} = \begin{cases} O(t^{-\frac{3(n+1)}{8}+\varepsilon_1}), & n \geqslant 2, \\ O(t^{-\frac{3(n+2)}{8}+1+\varepsilon_1}), & n = 1, \end{cases} \tag{1.23}$$

$$\|u(t)\|_{L^2} = \begin{cases} O(t^{-\frac{n}{4}}), & n \geqslant 3, \\ O(t^{-\frac{3(n+2)}{8}+1+\varepsilon_1}), & 1 \leqslant n \leqslant 2, \end{cases} \tag{1.24}$$

$$\|D_x u(t)\|_{L^2} = \begin{cases} O(t^{-\frac{n+1}{4}}), & n \geqslant 2, \\ O(t^{-\frac{3(n+2)}{8}+1+\varepsilon_1}), & n = 1, \end{cases} \tag{1.25}$$

respectively).

Remark 1. We emphasize that by Fujita, Kobayashi, Weissler's results mentioned above, under the assumption (1.8) ((1.12), (1.13), respectively) our result $n \geqslant 3$ ($n \geqslant 2$, $n \geqslant 1$, respectively) can not be improved.

Remark 2. The technique we adopt in this paper is Nash–Moser–Hörmander iterative scheme combined with the decay estimates of solution for the linearized parabolic equation. This technique was firstly used by Klainerman[1] in solving nonlinear wave equation,and later on in solving many kinds of nonlinear evolution equations, including nonlinear parabolic equation [2].

The difference between the results in this paper and in [2] is: under (1.8) we get $n \geqslant 3$ instead of $n \geqslant 5$; under (1.12) and (1.13) we have tha same solvable range for n but finer decay estimates. To get these results we effectively use the fact that for the solution of linear heat equation we not only have the decay estimates $|u|_{L^\infty} = O(t^{-\frac{n}{2}})$, but also have $\|u\|_{L^2} = O(t^{-\frac{n}{4}})$. Thus by means of more delicate estimates we are able to prove our Main Theorem.

Remark 3. If F involves a term qu ($q<0$), which is so-called dissipation term, then the global existence with small initial data will hold without any restriction on n, and the proof is simpler [7, 8].

The sections 2 and 3 will be devoted to the decay estimates for linear parabolic equation. The section 4 will be devoted to the proof of part (i) of the Main Theorem, and the section 5 to the proof of part (ii). The proof of part (iii) can easily follow from sections 2, 3, 4.

Throughout this paper we use the following notations.

$$|u(t)|_L=\sum_{|\alpha|\le L}|D_x^\alpha u(t,x)|_{L^\infty(R^n)},$$
$$\|u(t)\|_L=\sum_{|\alpha|\le L}\|D_x^\alpha u(t,x)\|_{L^2(R^n)}, \tag{1.26}$$
$$|\!|\!|u(t)|\!|\!|_L=\sum_{|\alpha|\le L}\|D_x^\alpha u(t,x)\|_{L^1(R^n)},$$

and

$$|u|_{k,L}=\sup_{t\ge 0}(1+t)^k|u(t)|_L,$$
$$\|u\|_{k,L}=\sup_{t\ge 0}(1+t)^k\|u(t)\|_L, \tag{1.27}$$
$$|\!|\!|u|\!|\!|_{k,L}=\sup_{t\ge 0}(1+t)^k|\!|\!|u(t)|\!|\!|_L.$$

§ 2. Decay Rates for the Heat Equation

In this section we are going to derive the decay estimates of the L^∞ and L^2 norms of solutions for the heat equation.

Consider the Cauchy problem of the heat equation:

$$\begin{cases} u_t-\Delta u=0, \\ u(0,x)=\phi(x), \end{cases} \tag{2.1}$$

where we assume that ϕ is a smooth function and the norms of ϕ appearing below are bounded.

By the Poisson formula

$$u=\frac{1}{(2\sqrt{\pi t})^n}\int_{R^n}e^{-\frac{|x-\xi|^2}{4t}}\phi(\xi)d\xi, \tag{2.2}$$

and noticing

$$\zeta^N e^{-\zeta^2}\le C$$

for any integer $N\ge 0$, we can easily get the following propositions.

Proposition 1. *For the solution u of (2.1), the following decay estimates hold:*

$$|D_x^N u|\le C|\!|\!|\phi|\!|\!|(1+t)^{-\frac{n+N}{2}},\quad t>1, \tag{2.3}$$

$$|u(t)|_L\le C|\!|\!|\phi|\!|\!|(1+t)^{-\frac{n}{2}},\quad t>1, \tag{2.4}$$

$$|D_x^N u|_L\le C|\!|\!|\phi|\!|\!|_{L+N+n}(1+t)^{-\frac{n+N}{2}},\quad t>0, \tag{2.5}$$

$$|u(t)|_L\le C|\!|\!|\phi|\!|\!|_{L+n}(1+t)^{-\frac{n}{2}},\quad t>0, \tag{2.6}$$

here and hereafter $L\ge 0$, $N\ge 0$ are integers.

Proposition 2. *For the solution u of (2.1), the following decay estimates hold:*

$$\|D_x^N u(t)\|_L\le C|\!|\!|\phi|\!|\!|_{L+N}(1+t)^{-\frac{n+N}{4}},\quad t>1, \tag{2.7}$$

$$\|D_x^N u(t)\|_L\le C|\!|\!|\phi|\!|\!|_{L+N+n}(1+t)^{-\frac{n+N}{4}},\quad t>0, \tag{2.8}$$

§ 3. Decay Estimates for Second-Order Parabolic Equation

In this section we are going to derive the a priori estimates for $|u|_{\frac{n}{2},L}$ and $\|u\|_{k,L}$ $\left(k\leqslant\frac{n}{4}\right)$ of the solution u for the following Cauchy problem of second-order parabolic equation:

$$\begin{cases} u_t-\Delta u-\sum_{i,j} b_{ij}(t, x)u_{x_ix_j}-\sum_{i=1}^{n} b_i(t, x)u_{x_i}-b_0(t, x)u=g(t, x), \\ u(0, x)=0, \end{cases} \tag{3.1}$$

where g and $b=(b_0, b_i, b_{ij})$ are smooth functions. Moreover, we assume that there exist positive constants m, M, $\theta\geqslant 1$ such that

$$b(t, x)\equiv 0, \ g(t, x)\equiv 0, \quad t\geqslant\theta, \tag{3.2}$$

$$m|\xi|^2\leqslant \sum_{i,j=1}^{n}(\delta_{ij}+b_{ij})\xi_i\xi_j\leqslant M|\xi|^2, \text{ for } \xi\in R^n. \tag{3.3}$$

First consider the Cauchy problem

$$\begin{cases} u_t-\Delta u=g, \\ u(0, x)=0, \end{cases} \tag{3.4}$$

where $g\equiv 0$ as $t\geqslant\theta$.

By Duhamel principle we have

$$u(t, x)=\int_0^t w(t-\tau, x)\,d\tau, \tag{3.5}$$

where w is the solution of

$$\begin{cases} w_t-\Delta w=0, \\ w(t=\tau, x)=g(\tau, x). \end{cases} \tag{3.6}$$

Thus by (2.6), for the solution u of (3.4) we obtain

$$\begin{aligned} |D_x^k u| &\leqslant\int_0^t |D_x^k w|\,d\tau\leqslant\int_0^{\frac{t}{2}} |D_x^k w|\,d\tau+\int_{\frac{t}{2}}^t |D_x^k w|\,d\tau \\ &\leqslant C_k\int_0^{\frac{t}{2}}(1+(t-\tau))^{-\frac{n}{2}}\|g\|_{k+n}\,d\tau+\int_{\frac{t}{2}}^t |D_x^k g|\,d\tau \\ &\leqslant C_k\left((1+t)^{-\frac{n}{2}}\|g\|_{1+\varepsilon,k+n}+\int_{\frac{t}{2}}^t (1+\tau)^{-\frac{n}{2}-1}d\tau\|g\|_{\frac{n}{2}+1,k+n}\right) \\ &\leqslant C_k(1+t)^{-\frac{n}{2}}(\|g\|_{1+\varepsilon,k+n}+\|g\|_{\frac{n}{2}+1,k+n}), \end{aligned} \tag{3.7}$$

where $\varepsilon>\!>0$ is a sufficiently small constant. (3.7) leads to

$$|\tilde{\Lambda} u|_{\frac{n}{2},L}\leqslant C_L(\|g\|_{1+\varepsilon,L+n+2}+\|g\|_{\frac{n}{2}+1,L+n+2}), \tag{3.8}$$

where $\tilde{\Lambda} u=(u, D_x u, D_x^2 u)$. Thus we get

Lemma 1. *For the solution of* (3.4) *the following estimate holds:*

$$|\Lambda u|_{\frac{n}{2},L}\leqslant C_L\tilde{R}_{L+n+2}(g), \tag{3.9}$$

where

$$\tilde{R}_L(g)=\|g\|_{1+\varepsilon,L}+\|g\|_{\frac{n}{2}+1,L}. \tag{3.10}$$

From Lemma 1 follows the following estimate for the solution u of (3.1).

$$|\Lambda u|_{\frac{n}{2},L}\leqslant C_L(\tilde{R}_{L+n+2}(g)+\tilde{R}_{L+n+2}(Lu)), \tag{3.11}$$

where

$$Lu\triangleq\sum_{i,j=1}^{n}b_{ij}u_{x_ix_j}+\sum_{i=1}^{n}b_iu_{x_i}+b_0u. \tag{3.12}$$

Set

$$\alpha=\min\left(\frac{n}{4},\ 1+\varepsilon\right). \tag{3.13}$$

Since

$$\|Lu\|_{1+\varepsilon,L+n+2}\leqslant C_L(\|b\|_{1+\varepsilon-\alpha,0}\|\tilde{\Lambda}u\|_{\alpha,L+n+2}+\|b\|_{1+\varepsilon-\alpha,L+n+2}\|\tilde{\Lambda}u\|_{\alpha,0}), \tag{3.14}$$

$$\|Lu\|_{\frac{n}{2}+1,L+n+2}\leqslant C_L(|b|_{\frac{n}{4}+1,0}\|\tilde{\Lambda}u\|_{\frac{n}{4},L+n+2}+|b|_{\frac{n}{4}+1,L+n+2}\|\tilde{\Lambda}u\|_{\frac{n}{4},0}), \tag{3.15}$$

we thus get

Proposition 3. *For the solution u of* (3.1) *the following a priori estimate holds:*

$$\begin{aligned}|\Lambda u|_{\frac{n}{2},L}\leqslant C_L(\tilde{R}_{L+n+2}(g)&+\|b\|_{1+\varepsilon-\alpha,0}\|\tilde{\Lambda}u\|_{\alpha,L+n+2}+\|b\|_{1+\varepsilon-\alpha,L+n+2}\|\tilde{\Lambda}u\|_{\alpha,0}\\&+|b|_{\frac{n}{4}+1,0}\|\tilde{\Lambda}u\|_{\frac{n}{4},L+n+2}+|b|_{\frac{n}{4}+1,L+n+2}\|\tilde{\Lambda}u\|_{\frac{n}{4},0}).\end{aligned} \tag{3.16}$$

In what follows we derive the estimate of $\|\Lambda u\|_{k,L}\left(k\leqslant\frac{n}{4}\right)$ for the solution of (3.1). First taking the inner product with u on the both sides of (3.1) and noticing (3.3), we get

$$\frac{d}{dt}\|u(t)\|\leqslant C_m(|D_xb|^2+|b|^2+|b|)\|u(t)\|+\|g(t)\|. \tag{3.17}$$

Applying Gronwall inequality, we obtain

$$\begin{aligned}\|u(t)\|&\leqslant C_m\int_0^t\|g(\tau)\|d\tau\cdot e^{C_m\int_0^t(|D_xb|^2+|b|^2+|b|)d\tau}\\&\leqslant C_m\|g\|_{1+\varepsilon,0}\cdot e^{C_m(|b|^2_{1+\varepsilon,1}+|b|_{1+\varepsilon,0})}.\end{aligned} \tag{3.18}$$

Now we are going to derive the estimates for the higher order norms. Acting D_x^N on the both sides of (3.1) $(N=1,\cdots,L)$, we obtain

$$\begin{cases}(D_x^Nu)_t-\Delta(D_x^Nu)-\sum\limits_{i,j=1}^{n}b_{ij}D_x^Nu_{x_ix_j}=g_N,\\ D_x^Nu(0,x)=0,\end{cases} \tag{3.19}$$

where

$$g_N=D_x^N\left(\sum_{i,j=1}^{n}b_{ij}u_{x_ix_j}\right)-\sum_{i,j=1}^{n}b_{ij}D_x^Nu_{x_ix_j}+D_x^N\left(\sum_{i=1}^{n}b_iu_{x_i}\right)+D_x^N(b_0u)+D_x^Ng. \tag{3.20}$$

In a similar way to the above we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_L^2+\|D_xu(t)\|_L^2\leqslant C_{m,L}(|D_xb|^2\|u(t)\|_L^2+\sum_{N=0}^{L}\|g\|_N\|u(t)\|_L). \tag{3.21}$$

Since

$$\|g_N\| \leqslant \left\| D_x^N\left(\sum_{i,j=1}^n b_{ij}u_{x_ix_j}\right) - \sum_{i,j=1}^n b_{ij}D^N u_{x_ix_j}\right\|$$
$$+\left\| D_x^N\left(\sum_{i=1}^n b_i u_{x_i}\right)\right\| + \|D_x^N(b_0u)\| + \|D_x^N g\|$$
$$\leqslant C_N(|D_x b|\,\|D_x^2 u\|_{N-1} + |b|_N\|D_x^2u\| + |b|\,\|D_xu\|_N$$
$$+|b|_N\|D_xu\| + |b|\,\|u\|_N + |b|_N\|u\| + \|D_x^N g\|), \tag{3.22}$$

we get

$$\|u(t)\|_L \leqslant C_{L,m}\int_0^t (\|g\|_L + |b|_L\|u\|$$
$$+|b|_L\|u\|_1 + |b|_L\|u\|_2)d\tau\cdot e^{C_{L,m}\int_0^t(|b|_1^2+|b|)d\tau}, \tag{3.23}$$

which leads to

$$\|u\|_L \leqslant C_{L,m}(\|g\|_{1+\varepsilon,L} + |b|_{1+\varepsilon,L}(\|g\|_{1+\varepsilon,2} + \|g\|_{1+\varepsilon,1}(1+|b|_{1+\varepsilon,2})$$
$$+\|g\|_{1+\varepsilon,0}(1+|b|_{1+\varepsilon,1}+|b|^2_{1+\varepsilon,2})))e^{C_m(|b|^2_{1+\varepsilon,1}+|b|_{1+\varepsilon,0})}. \tag{3.24}$$

Hence, we have the following

Proposition 4. *For the solution u of* (3.1) *the following a priori estimate holds:*

$$\|\Delta u\|_L \leqslant C_{L,m}(\|g\|_{1+\varepsilon,L+2} + |b|_{1+\varepsilon,L+2}(\|g\|_{1+\varepsilon,2} + \|g\|_{1+\varepsilon,1}(1+|b|_{1+\varepsilon,2})$$
$$+\|g\|_{1+\varepsilon,0}(1+|b|_{1+\varepsilon,1}+|b|^2_{1+\varepsilon,2})))\,e^{C_m(|b|^2_{1+\varepsilon,1}+|b|_{1+\varepsilon,0})}. \tag{3.25}$$

In what follows we further derive the estimates for

$$\|\Delta u\|_{k,L} = \sup_{t\geqslant 0}(1+t)^k\|\Delta u(t)\|_L \quad \left(k\leqslant \frac{n}{4}\right).$$

For $0\leqslant t\leqslant 2\theta$, we have

$$(1+t)^k\|\Delta u(t)\|_L \leqslant C\theta^k\|\Delta u(t)\|_L. \tag{3.26}$$

On the other hand, when $t\geqslant\theta$, by (3.2) the problem (3.1) is reduced to

$$\begin{cases} u_t - \Delta u = 0,\ t>\theta, \\ u|_{t=\theta} = u(\theta, x), \end{cases} \tag{3.27}$$

If $t\geqslant 2\theta$, that is, $t-\theta\geqslant\theta\geqslant 1$, by proposition 2 we have for $k\leqslant\frac{n}{4}$

$$(1+t)^k\|\Delta u(t)\|_L \leqslant (1+t)^{k-\frac{n}{4}}\ (1+t)^{\frac{n}{4}}\|\Delta u(t)\|_L$$
$$\leqslant C\|\tilde{\Delta} u(\theta)\|_L\theta^{k-\frac{n}{4}},\ t\geqslant 2\theta. \tag{3.28}$$

By Duhamel principle and (2.11), and noticing (3.12) we have

$$\|\tilde{\Delta} u(t)\|_L \leqslant \int_0^t \|Lu+g\|_{L+2}\,d\tau \leqslant \int_0^t \|Lu\|_{L+2}d\tau + \int_0^t \|g\|_{L+2}\,d\tau$$
$$\leqslant C\int_0^t (\|b\|_0\|\Delta u\|_{L+2} + \|b\|_{L+2}\|\Delta u\|_0)\,d\tau + \int_0^t \|g\|_{L+2}\,d\tau. \tag{3.29}$$

Since $\int_0^\theta (1+\tau)^{-1-\varepsilon}d\tau \leqslant C < +\infty$, we have

$$\|\tilde{\Delta} u(\theta)\|_L \leqslant C\int_0^\theta (\|b\|_0\|\Delta u\|_{L+2} + \|b\|_{L+2}\|\Delta u\|_0)\,d\tau + \int_0^\theta \|g\|_{L+2}\,d\tau$$
$$\leqslant C(\theta^{1+\varepsilon-\alpha}(\|b\|_{\alpha,0}\|\Delta u\|_{0,L+2} + \|b\|_{\alpha,L+2}\|\Delta u\|_{0,0}) + \|g\|_{1+\varepsilon,L+2}). \tag{3.30}$$

Thus for $k\leqslant\frac{n}{4}$, $t\geqslant 2\theta$ we have

$$(1+t)^k\|\Delta u(t)\|_L\leqslant C\theta^{k-\frac{n}{4}}(\theta^{1+\varepsilon-\alpha}(\|b\|_{\alpha,0}\|\Delta u\|_{0,L+2}+\|b\|_{\alpha,L+2}\|\Delta u\|_{0,0}) + \|g\|_{1+\varepsilon,L+2}). \tag{3.31}$$

Combining (3.26) with (3.31), we finally get

Proposition 5. *For the solution u of (3.1) the following estimate holds.*

$$\|\Delta u\|_{k,L}\leqslant C\max\{\theta^k\|\Delta u\|_{0,L},\ \theta^{k-\frac{n}{4}}(\theta^{1+\varepsilon-\alpha}(\|b\|_{\alpha,0}\|\Delta u\|_{0,L+2} + \|b\|_{\alpha,L+2}\|\Delta u\|_{0,0})+\|g\|_{1+\varepsilon,L+2})\}. \tag{3.32}$$

§ 4. Proof of the Part (i) of the Main Theorem

In this section we are going to apply the decay estimates obtained in sections 2—3 and the Nash–Moser–Hörmander iterative scheme (see [1]) to prove the part (i) of our Main Theorem.

For the problem (1.1), (1.2) we adopt the following Nash–Moser–Hörmander iterative scheme. Let u_0 be the solution of

$$\begin{cases} u_t-\Delta u=0, \\ u(0,x)=\phi(x). \end{cases} \tag{4.1}$$

Set

$$\phi(u)=u_t-\Delta u-F(u,D_xu,D_x^2u), \tag{4.2}$$

$$g_0=-S_0\phi(u_0), \tag{4.3}$$

$$b_0=F'_\lambda(S_0\tilde{\Lambda}u_0)=(F'_{\lambda_0}(S_0\tilde{\Lambda}u_0),\ F'_{\lambda_i}(S_0\tilde{\Lambda}u_0),\ F'_{\lambda_{ij}}(S_0\tilde{\Lambda}u_0)), \tag{4.4}$$

and denoting by $\dot{u}_0$ the solution of

$$\begin{cases} v_t-\Delta v-\sum_{i,j=1}^n F'_{\lambda_{ij}}(S_0\tilde{\Lambda}u_0)\,v_{x_ix_j}-\sum_{i=1}^n F'_{\lambda_i}(S_0\tilde{\Lambda}u_0)\,v_{x_i}-F'_{\lambda_0}(S_0\tilde{\Lambda}u_0)\,v=g_0, \\ v(0,x)=0, \end{cases} \tag{4.5}$$

we obtain the first approximation

$$u_1=\dot{u}_0+u_0. \tag{4.6}$$

In general, for any integer $p\geqslant 0$ let

$$b_p=F'_\lambda(S_p\tilde{\Lambda}u_p) \tag{4.7}$$

and let L_p be the operator defined as follows:

$$\begin{aligned} L_pv&=v_t-\Delta v-F'_\lambda(S_p\tilde{\Lambda}u_p)\tilde{\Lambda}v \\ &=v_t-\Delta v-\sum_{i,j=1}^n F'_{\lambda_{ij}}(S_p\tilde{\Lambda}u_p)\,v_{x_ix_j}-\sum_{i=1}^n F'_{\lambda_i}(S_p\tilde{\Lambda}u_p)\,v_{x_i}-F'_{\lambda_0}(S_p\tilde{\Lambda}u_p)\,v. \end{aligned} \tag{4.8}$$

Set

$$\begin{aligned} e'_p&=(\phi'(u_p)-L_p)\dot{u}_p, \\ e''_p&=\phi(u_{p+1})-\phi(u_p)-\phi'(u_p)\dot{u}_p, \\ e_p&=e'_p+e''_p, \\ E_p&=\sum_{j=0}^{p-1}e_j, \end{aligned} \tag{4.9}$$

for $p\geqslant 1$, and

$$g_p=-(S_p-S_{p-1})E_{p-1}-S_p e_{p-1}-(S_p-S_{p-1})\phi(u_0), \tag{4.10}$$

where $E_0=0$.

Denoting by $\dot{u}_p$ the solution of

$$\begin{cases} L_p v=g_p, \\ v(0, x)=0, \end{cases} \tag{4.11}$$

we obtain the $(p+1)$-th approximation of the iteration

$$u_{p+1}=u_p+\dot{u}_p. \tag{4.12}$$

In the above, $S_p(p=0, 1, \cdots)$ are the smooth operators defined in [1].

From [1], we have

Lemma 2. *For the smooth operators $S_p(p=0, 1, \cdots)$ as $0\leqslant s\leqslant k$, $0\leqslant M\leqslant N$, we have*

$$|S_p u|_{k,N}\leqslant C\theta_p^{k-s}\,\theta_p^{\bar{\varepsilon}(N-M)}|u|_{s,M}, \tag{4.13}$$

and

$$|(I-S_p)u|_{s,M}\leqslant C(\theta_p^{-(k-s)}|u|_{k,M}+\theta_p^{-\bar{\varepsilon}(N-M)}|u|_{s,N}). \tag{4.14}$$

The same conclusion holds for the norm $\|\cdot\|$ and the norm $|||\cdot|||$.

Lemma 3. *Suppose u is a smooth function such that*

$$|u|_{0,0}\leqslant C\theta^{-\beta}, \tag{4.15}$$

$$|u|_{k,L}\leqslant C\theta^{k+\lambda L-\beta},\ k+\lambda L-\beta\geqslant\bar{\varepsilon},\ 0\leqslant k\leqslant\tilde{k},\ 0\leqslant L\leqslant\tilde{L}, \tag{4.16}$$

where $\bar{\varepsilon}>0$, $\beta>\bar{\varepsilon}$ and

$$\tilde{k}-\beta\geqslant\bar{\varepsilon},\quad \lambda\tilde{L}-\beta\geqslant\bar{\varepsilon}, \tag{4.17}$$

then

$$|u|_{k,L}\leqslant(C_{\tilde{k},\tilde{L}}C)\theta^{k+\lambda L-\beta},\ 0\leqslant k\leqslant\tilde{k},\ 0\leqslant L\leqslant\tilde{L}. \tag{4.18}$$

The same conclusion holds for $\|u\|_{k,L}$ and $|||u|||_{k,L}$.

Proof See Lemma 6.1 in [1].

We now choose the positive constant β and the sufficiently small constant ε and $\bar{\varepsilon}$ such that

$$\begin{gathered}\frac{n}{2}-\bar{\varepsilon}\geqslant\beta\geqslant\max\left(\frac{n}{4}+\bar{\varepsilon}(n+3),\ 1+\varepsilon+\bar{\varepsilon}(n+2)\right),\\ 2\beta+\bar{\varepsilon}\leqslant\frac{3n}{4}.\end{gathered} \tag{4.19}$$

It is easy to see that when $n\geqslant 3$, such choice is always possible.

Let $\tilde{L}>0$ be an integer such that

$$\tilde{L}\geqslant\frac{2\beta}{\bar{\varepsilon}}+(n+2). \tag{4.20}$$

By the results in section 2 it is easy to see that we have

$$|\Delta u_0|_{\frac{n}{2},\tilde{L}}\leqslant\delta,\quad \|\Delta u_0\|_{\frac{n}{4},\tilde{L}}\leqslant\delta \tag{4.21}$$

provided that $\|\phi\|_{\tilde{L}+n}$ is sufficiently small, where $\delta>0$ is a sufficiently small constant.

The proof of the part (i) *of the Main Theorem.*

It is sufficient to prove that for $j\geqslant 0$

$$(P_1, j):\ \|\Delta\dot{u}_j\|_{k,L}\leqslant\delta\,\theta_j^{k-\beta+\bar{\varepsilon}L},\quad 0\leqslant k\leqslant\frac{n}{4},\ 0\leqslant L\leqslant\tilde{L},\tag{4.22}$$

$$(P_2, j):\ |\Delta\dot{u}_j|_{k,L}\leqslant\delta\,\theta_j^{k-\beta+\bar{\varepsilon}L},\quad 0\leqslant k\leqslant\frac{n}{2},\ 0\leqslant L\leqslant\tilde{L},\tag{4.23}$$

$$(P_3, j):\ |\Delta u_j|_{1+\varepsilon,0}\leqslant C\delta\leqslant 1,\quad \|\Delta u_j\|_{0,0}\leqslant C\delta\leqslant 1.\tag{4.24}$$

In fact, once we have proved (4.22)—(4.24), similarly to [1], we may conclude that the limit function of u_p is a global smooth solution of (4.1).

It is easy to verify that (4.22)—(4.24) hold for $j=0$ which we will prove later. Assuming now that (4.22)—(4.24) hold for $j=p$, we prove that these are valid for $j=p+1$.

In fact, we may prove in the same way as in [1].

I(i). $|S_{p+1}\Delta U_{p+1}|_{k,L}\leqslant C_{k,L}\delta\,\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{n}{2},\quad k-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon},\quad L\geqslant 0;\tag{4.25}$$

where

$$u_{p+1}=\sum_{j=0}^{p}\dot{u}_j;\tag{4.26}$$

(ii). $\|S_{p+1}\Delta U_{p+1}\|_{k,L}\leqslant C_{k,L}\delta\,\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{n}{4},\ k-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon},\ L\geqslant 0;\tag{4.27}$$

II(i). $|S_{p+1}\Delta U_{p+1}|_{k,L}\leqslant C\delta$,

$$0\leqslant k\leqslant\frac{n}{2},\ k-\beta+\bar{\varepsilon}L\leqslant-\bar{\varepsilon},\ L\geqslant 0;\tag{4.28}$$

(ii). $\|S_{p+1}\Delta U_{p+1}\|_{k,L}\leqslant C\delta$,

$$0\leqslant k\leqslant\frac{n}{4},\ k-\beta+\bar{\varepsilon}L\leqslant-\bar{\varepsilon},\ L\geqslant 0;\tag{4.29}$$

III. $|(I-S_{p+1})\Delta U_{p+1}|_{k,L}\leqslant C\delta\,\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{n}{2},\ 0\leqslant L\leqslant\tilde{L}.\tag{4.30}$$

It follows from $u_{p+1}=U_{p+1}+u_0$ that

I′(i). $|S_{p+1}\Delta u_{p+1}|_{k,L}\leqslant C_{k,L}\delta\,\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{n}{2},\ k-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon},\ L\geqslant 0;\tag{4.31}$$

(ii). $\|S_{p+1}\Delta u_{p+1}\|_{k,L}\leqslant C_{k,L}\delta\,\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{n}{4},\ k-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon},\ L\geqslant 0;\tag{4.32}$$

II′(i). $|S_{p+1}\Delta u_{p+1}|_{k,L}\leqslant C\delta$,

$$0\leqslant k\leqslant\frac{n}{2},\ k-\beta+\bar{\varepsilon}L\leqslant-\bar{\varepsilon},\ L\geqslant 0;\tag{4.33}$$

(ii). $\|S_{p+1}\Delta u_{p+1}\|_{k,L}\leqslant C\delta$,

$$0\leqslant k\leqslant\frac{n}{4},\ k-\beta+\bar{\varepsilon}L\leqslant-\bar{\varepsilon},\ L\geqslant 0; \tag{4.34}$$

III′(i). $|(I-S_{p+1})\Delta u_{p+1}|_{k,L}\leqslant C\delta\,\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{n}{2},\ 0\leqslant L\leqslant\tilde{L}. \tag{4.35}$$

we now prove

(ii). $\|(I-S_p)\Delta u_p\|_{k,L}\leqslant C_{k,L}\delta\theta_p^{k-\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{n}{4},\ -\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon},\ 0\leqslant L\leqslant\tilde{L};$$

(iii). $\|(I-S_p)\Delta u_p\|_{k,0}\leqslant C_k\delta\,\theta_p^{k-\frac{n}{4}}$,

$$0\leqslant k\leqslant\frac{n}{4}. \tag{4.36}$$

In fact by the property (4.14) of the smooth operator, for $0\leqslant L\leqslant\tilde{L}$, $0\leqslant k\leqslant\frac{n}{4}$, we have

$$\|(I-S_p)\Delta u_p\|_{k,L}\leqslant C_{k,L}\big(\theta_p^{-\left(\frac{n}{4}-k\right)}\|\Delta u_p\|_{\frac{n}{4},L}+\theta_p^{\bar{\varepsilon}(L-\tilde{L})}\|\Delta u_p\|_{k,\tilde{L}}\big). \tag{4.37}$$

By (4.21) and (P_1, j), $0\leqslant j\leqslant p$, as $-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon}$ we get

$$\|\Delta u_p\|_{\frac{n}{4},L}\leqslant\|\Delta u_0\|_{\frac{n}{4},L}+\sum_{j=0}^{p-1}\|\Delta\dot{u}_j\|_{\frac{n}{4},L}\leqslant C\delta\,\theta_p^{\frac{n}{4}-\beta+\bar{\varepsilon}L}. \tag{4.38}$$

Similarly, it follows from (4.19) that

$$\|\Delta u_p\|_{k,\tilde{L}}\leqslant\|\Delta u_0\|_{k,\tilde{L}}+\sum_{j=0}^{p-1}\|\Delta\dot{u}_j\|_{k,\tilde{L}}\leqslant C\delta\,\theta_p^{k-\beta+\bar{\varepsilon}\tilde{L}}. \tag{4.39}$$

Substituting (4.38) and (4.39) into (4.37), as $-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon}$ we obtain

$$\|(I-S_p)\Delta u_p\|_{k,L}\leqslant C\delta\theta_p^{k-\beta+\bar{\varepsilon}L}.$$

Similarly, it follows from (4.19) that for $k\leqslant\frac{n}{4}$

$$\|(I-S_p)\Delta u_p\|_{k,0}\leqslant C_k\big(\theta_p^{-\left(\frac{n}{4}-k\right)}\|\Delta u_p\|_{\frac{n}{4},0}+\theta_p^{-\bar{\varepsilon}\tilde{L}}\|\Delta u_p\|_{k,\tilde{L}}\big)\leqslant C_k\delta\,\theta_p^{k-\frac{n}{4}}. \tag{4.40}$$

We now prove

IV(i). $\|e_p\|_{k,L}\leqslant C_{k,L}\delta^2\,\theta_p^{k-2\beta+\bar{\varepsilon}L}$,

$$0\leqslant k\leqslant\frac{3n}{4},\ 0\leqslant L\leqslant\tilde{L}; \tag{4.41}$$

(ii). $\|e_p\|_{k,L}\leqslant C_{k,L}\delta^2\,\theta_p^{k-\beta-\frac{n}{4}+\bar{\varepsilon}L^p}$,

$$0\leqslant k\leqslant\frac{n}{2},\ 0\leqslant L\leqslant\tilde{L}. \tag{4.42}$$

Writing k as $k=k_1+k_2$, $k_1\leqslant\frac{n}{2}$, $k_2\leqslant\frac{n}{4}$, by (4.8) we have

$$\begin{aligned}\|e''_p\|_{k,L}\leqslant C_{k,L}\big(&|\Delta\dot{u}_p|_{k_1,L}\|\Delta\dot{u}_p\|_{k_2,0}\\&+(1+|\Delta u_{p+1}|_{0,L}+|\Delta u_p|_{0,L})\,|\Delta\dot{u}_p|_{k_1,0}\|\Delta\dot{u}_p\|_{k_2,0}\big),\end{aligned} \tag{4.43}$$

$$\begin{aligned}\|e'_p\|_{k,L}\leqslant C_{k,L}\big(&|(I-S_p)\Delta u_p|_{k_1,L}\|\Delta\dot{u}_p\|_{k_2,0}\\&+|(I-S_p)\Delta u_p|_{k_1,0}(\|\Delta\dot{u}_p\|_{k_2,L}+(1+|\Delta u_p|_{0,L})\|\Delta\dot{u}_p\|_{k_2,0})\big).\end{aligned} \tag{4.44}$$

Applying (P_1, p), (P_2, p) and I′, III′, in the same way as in [1], we obtain as $-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon}$, $0\leqslant k\leqslant\frac{3n}{4}$,

$$\|e''_p\|_{k,L}\leqslant C_{k,L}\delta^2\,\theta_p^{k-2\beta+\bar{\varepsilon}L}, \tag{4.45}$$

$$\|e''_p\|_{k,0}\leqslant C_k\delta^2\,\theta_p^{k-2\beta}. \tag{4.46}$$

By Lemma 3, we get

$$\|e''_p\|_{k,L}\leqslant C_{k,L}\delta^2\,\theta_p^{k-2\beta+\bar{\varepsilon}L},$$
$$0\leqslant k\leqslant\frac{3n}{4},\ 0\leqslant L\leqslant\tilde{L}. \tag{4.47}$$

Similarly, we have

$$\|e'_p\|_{k,L}\leqslant C_{k,L}\delta^2\,\theta_p^{k-2\beta+\bar{\varepsilon}L},$$
$$0\leqslant k\leqslant\frac{3n}{4},\ 0\leqslant L\leqslant\tilde{L}. \tag{4.48}$$

This completes the proof of (4.41). We now prove (4.42). For $k\leqslant\frac{n}{2}$, writing it as $k=k_1+k_2$, $k_1\leqslant\frac{n}{4}$, $k_2\leqslant\frac{n}{4}$, as $-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon}$, we have

$$\begin{aligned}\|e''_p\|_{k,L}\leqslant{}& C_{k,L}(\|\Delta\dot{u}_p\|_{k_1,L}\|\Delta\dot{u}_p\|_{k_2,0}+(1+|\Delta u_{p+1}|_{0,L}\\&+|\Delta u_p|_{0,L})\|\Delta\dot{u}_p\|_{k_1,0}\|\Delta\dot{u}_p\|_{k_2,0})\\\leqslant{}& C_{k,L}\delta^2(\theta_p^{k_1-\beta+\bar{\varepsilon}L}\,\theta_p^{k_2-\beta}+\theta_p^{-\beta+\bar{\varepsilon}L}\,\theta_p^{k-2\beta})\leqslant C_{k,L}\delta^2\theta_p^{k-2\beta+\bar{\varepsilon}L}.\end{aligned} \tag{4.49}$$

Similarly, we have

$$\|e''_p\|_{k,0}\leqslant C_k\delta^2\theta_p^{k-2\beta}. \tag{4.50}$$

Interpolating with respect to L, we obtain for $0\leqslant k\leqslant\frac{n}{2}$, $0\leqslant L\leqslant\tilde{L}$

$$\|e''_p\|_{k,L}\leqslant C_{k,L}\delta^2\theta_p^{k-2\beta+\bar{\varepsilon}L}. \tag{4.51}$$

On the other hand

$$\begin{aligned}\|e'_p\|_{k,L}\leqslant{}& C_{k,L}(\|(I-S_p)\Delta u_p\|_{k_1,L}\\&+|(I-S_p)\Delta u_p|_{k_1,0}(1+\|\Delta u_p\|_{0,L}))\|\Delta\dot{u}_p\|_{k_2,0}\\&+C_{k,L}\|(I-S_p)\Delta u_p\|_{k_1,0}\|\Delta\dot{u}_p\|_{k_2\bar{\varepsilon}L}.\end{aligned} \tag{4.52}$$

Applying (P_1, p) and III′, as $-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon}$, we get

$$\begin{aligned}\|e'_p\|_{k,L}\leqslant{}& C_{k,L}\delta^2(\theta_p^{k-2\beta+\bar{\varepsilon}L}+\theta_p^{k-2\beta-\beta+\bar{\varepsilon}L})\\&+C_{k,L}\delta^2\,\theta_p^{k-\beta-\frac{n}{4}+\bar{\varepsilon}L}\leqslant C_{k,L}\delta^2\,\theta_p^{k-\beta-\frac{n}{4}+\bar{\varepsilon}L}\end{aligned} \tag{4.53}$$

and

$$\|e'_p\|_{k,0}\leqslant C_k\delta^2\,\theta_p^{k-\beta-\frac{n}{4}}. \tag{4.54}$$

Interpolating with respect to L, we conclude that the following holds for $0\leqslant k\leqslant\frac{n}{2}$, $0\leqslant L\leqslant\tilde{L}$

$$\|e'_p\|_{k,L}\leqslant C_{k,L}\delta^2\,\theta_p^{k-\beta-\frac{n}{4}+\bar{\varepsilon}L}. \tag{4.55}$$

Thus the proof of (4.42) is completed.

In what follows we prove

V(i). $\|g_{p+1}\|_{k,L} \leqslant C_{k,L}\,\delta^2\,\theta_{p+1}^{k-2\beta+\bar{\varepsilon}L}$,

$$k \geqslant 0,\ L \geqslant 0; \tag{4.56}$$

(ii). $\|g_{p+1}\|_{1+\varepsilon,L} \leqslant C_L \delta^2\,\theta_{p+1}^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}L}$,

$$\frac{n}{4}-\beta+\bar{\varepsilon}L \geqslant \bar{\varepsilon},\ L \geqslant 0, \tag{4.57}$$

$$\|g_{p+1}\|_{1+\varepsilon,L} \leqslant C_L \delta^2 \theta_{p+1}^{1+\varepsilon-\frac{n}{2}},$$

$$\frac{n}{4}-\beta+\bar{\varepsilon}L \leqslant -\bar{\varepsilon},\ L \geqslant 0. \tag{4.58}$$

The proof of (4.56) is similar to [1].

By (4.10) we obtain

$$\|g_{p+1}\|_{1+\varepsilon,L} \leqslant \|(S_{p+1}-S_p)E_p\|_{1+\varepsilon,L} + \|S_{p+1}e_p\|_{1+\varepsilon,L} + \|(S_{p+1}-S_p)\phi(u_0)\|_{1+\varepsilon,L}, \tag{4.59}$$

It follows from (4.42) that

$$\|S_{p+1}e_p\|_{1+\varepsilon,L} \leqslant C\delta^2\,\theta_{p+1}^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}L},\ L \geqslant 0, \tag{4.60}$$

$$\begin{aligned}
\|(S_{p+1}-S_p)E_p\|_{1+\varepsilon,L} &\leqslant \sum_{j=0}^{p-1}\left(\|(I-S_{p+1})e_j\|_{1+\varepsilon,L} + \|(I-S_p)e_j\|_{1+\varepsilon,L}\right)\\
&\leqslant C\sum_{j=0}^{p-1}\left(\theta_{p+1}^{1+\varepsilon-\frac{n}{2}}\|e_j\|_{\frac{n}{2},L} + \theta_{p+1}^{\bar{\varepsilon}(L-\tilde{L})}\|e_j\|_{1+\varepsilon,\tilde{L}}\right)\\
&\leqslant C_L\delta^2\left(\theta_{p+1}^{1+\varepsilon-\frac{n}{2}}\sum_{j=0}^{p-1}\theta_j^{\frac{n}{2}-\beta-\frac{n}{4}+\bar{\varepsilon}L} + \theta_{p+1}^{\bar{\varepsilon}(L-\tilde{L})}\sum_{j=0}^{p-1}\theta_j^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}\tilde{L}}\right)\\
&\leqslant C_L\delta^2\left(\theta_{p+1}^{1+\varepsilon-\frac{n}{2}}\sum_{j=0}^{p-1}\theta_j^{\frac{n}{4}-\beta+\bar{\varepsilon}L} + \theta_{p+1}^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}L}\right).
\end{aligned} \tag{4.61}$$

Therefore, as $\frac{n}{4}-\beta+\bar{\varepsilon}L \geqslant \bar{\varepsilon}$, $L \leqslant \tilde{L}$, we have

$$\|(S_{p+1}-S_p)E_p\|_{1+\varepsilon,L} \leqslant C_L\delta^2\,\theta_{p+1}^{1+\varepsilon-\frac{n}{4}-\beta+\bar{\varepsilon}L}. \tag{4.62}$$

On the other hand, as $\frac{n}{4}-\beta+\bar{\varepsilon}L \leqslant -\bar{\varepsilon}$, we have

$$\|(S_{p+1}-S_p)E_p\|_{1+\varepsilon,L} \leqslant C_L\delta^2\,\theta_{p+1}^{1+\varepsilon-\frac{n}{2}}. \tag{4.63}$$

Similarly, for $0 \leqslant L \leqslant \tilde{L}$, we have

$$\begin{aligned}
\|(S_{p+1}-S_p)\phi(u_0)\|_{1+\varepsilon,L} &\leqslant C\left(\theta_{p+1}^{1+\varepsilon-\frac{n}{2}}\|\phi(u_0)\|_{\frac{n}{2},L} + \theta_{p+1}^{\bar{\varepsilon}(L-\tilde{L})}\|\phi(u_0)\|_{1+\varepsilon,\tilde{L}}\right)\\
&\leqslant C\delta^2\left(\theta_{p+1}^{1+\varepsilon-\frac{n}{2}} + \theta_{p+1}^{\bar{\varepsilon}(L-\tilde{L})}\right).
\end{aligned} \tag{4.64}$$

Hence, as $\frac{n}{4}-\beta+\bar{\varepsilon}L \geqslant \bar{\varepsilon}$, $0 \leqslant L \leqslant \tilde{L}$, we have

$$\|(S_{p+1}-S_p)\phi(u_0)\|_{1+\varepsilon,L} \leqslant C\delta^2\,\theta_{p+1}^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}L}, \tag{4.65}$$

and as $\frac{n}{4}-\beta+\bar{\varepsilon}L \leqslant -\bar{\varepsilon}$, we get

$$\|(S_{p+1}-S_p)\phi(u_0)\|_{1+\varepsilon,L} \leqslant C_L\delta^2\,\theta_{p+1}^{1+\varepsilon-\frac{n}{2}}. \tag{4.66}$$

In virtue of (4.60) and the property (4.13) of the smooth operator S_p we obtain

(4.57) and (4.58) By the difinition of $\tilde{R}_{L+n+2}(g)$ in section 3 and (4.56)—(4.58) we easily obtain

VI(i) $$\tilde{R}_{\bar{L}+n+2}(g_{p+1})\leqslant C\delta^2\theta_{p+1}^{\frac{n}{2}-\beta+\bar{\varepsilon}\bar{L}},\tag{4.67}$$

(ii). $$\tilde{R}_{n+2}(g_{p+1})\leqslant C\delta^2\,\theta_{p+1}^{\frac{n}{2}-\beta}.\tag{4.68}$$

VII. For $b_{p+1}=F'_\lambda(S_{p+1}\tilde{\Lambda}u_{p+1})$ we have

(i). $|b_{p+1}|_{k,L}\leqslant C_{k,L}\delta\,\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,
$$k\geqslant 0,\ k-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon},\ L\geqslant 0,\tag{4.69}$$

(ii), $|b_{p+1}|_{k,L}\leqslant C_{k,L}\delta$,
$$k\geqslant 0,\ k-\beta+\bar{\varepsilon}L\leqslant-\bar{\varepsilon},\ L\geqslant 0;\tag{4.70}$$

(iii). $\|b_{p+1}\|_{k,L}\leqslant C_{k,L}\delta\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}$,
$$k\geqslant 0,\ k-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon},\ L\geqslant 0;\tag{4.71}$$

(iv). $\|b_{p+1}\|_{k,L}\leqslant C_{k,L}\delta$,
$$k\geqslant 0,\ k-\beta+\bar{\varepsilon}L\leqslant-\bar{\varepsilon},\ L\geqslant 0.\tag{4.72}$$

The proof is similar to [1]. We omit the detail here.

Based on what have been obtained in the above, we now prove $(P_1, p+1)$, $(P_2, p+1)$, $(P_3, p+1)$.

By Proposition 4 we obtain

$$\begin{aligned}\|\Lambda\dot{u}_{p+1}\|_{0,L}\leqslant C_L(&\|g_{p+1}\|_{1+\varepsilon,L+2}+|b_{p+1}|_{1+\varepsilon,L}(\|g_{p+1}\|_{1+\varepsilon,2}\\&+\|g_{p+1}\|_{1+\varepsilon,1}(1+|b_{p+1}|_{1+\varepsilon,2})+\|g_{p+1}\|_{1+\varepsilon,0}(1+|b_{p+1}|_{1+\varepsilon,1}\\&+|b_{p+1}|^2_{1+\varepsilon,2})))\,e^{C_m(|b_{p+1}|^2_{1+\varepsilon,1}+|b_{p+1}|_{1+\varepsilon,0})}.\end{aligned}\tag{4.73}$$

In virtue of V, VII and (4.19) for all L satisfying $\bar{\varepsilon}(L+1)-\beta\geqslant 0$ we have

$$\|\dot{u}\Lambda_{p+1}\|_{0,L}\leqslant C_L\delta^2(\theta_{p+1}^{1+\varepsilon-2\beta+\bar{\varepsilon}(L+2)}+\theta_{p+1}^{2(1+\varepsilon)-3\beta+\bar{\varepsilon}(L+2)})\leqslant C_L\delta^2\theta_{p+1}^{-\beta+\bar{\varepsilon}L-\bar{\varepsilon}n},\tag{4.74}$$

For $L=0$ wa have

$$\|\Lambda\dot{u}_{p+1}\|_{0,0}\leqslant C\delta^2(\theta_{p+1}^{1+\varepsilon-2\beta+2\bar{\varepsilon}}+\theta_{p+1}^{1+\varepsilon-2\beta+2\bar{\varepsilon}})\leqslant C\delta^2\theta_{p+1}^{-\beta-\bar{\varepsilon}n}.\tag{4.75}$$

Interpolating with respect to L, for all $L\geqslant 0$ we get

$$\|\Lambda\dot{u}_{p+1}\|_{0,L}\leqslant C\delta^2\theta_{p+1}^{-\beta+\bar{\varepsilon}L-\bar{\varepsilon}n}\tag{4.76}$$

We now apply proposition 5 to estimate

$$\|\Lambda\dot{u}_{p+1}\|_{k,L}\left(k\leqslant\frac{n}{4}\right).$$

For $L\geqslant 0$ and $\alpha-\beta+\bar{\varepsilon}(L+2)\geqslant\bar{\varepsilon}$, it follows from VI and VII that

$$\|b_{p+1}\|_{\alpha,L+2}\leqslant C_L\delta\,\theta_{p+1}^{\alpha-\beta+\bar{\varepsilon}(L+2)},\tag{4.77}$$

$$\|g_{p+1}\|_{1+\varepsilon,L+2}\leqslant C_L\delta^2\,\theta_{p+1}^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}(L+2)}.\tag{4.78}$$

Thus by (3.32) we get

$$\begin{aligned}\|\Lambda\dot{u}_{p+1}\|_{k,L}\leqslant C_{k,L}\delta^2\max\{&\theta_{p+1}^{k-\beta+\bar{\varepsilon}L-\bar{\varepsilon}n},\ \theta_{p+1}^{k-\frac{n}{4}+1-\beta+\bar{\varepsilon}(L+2)+\varepsilon-\sigma}\\&+\theta_{p+1}^{k-\frac{n}{4}-2\beta+1+\varepsilon+\bar{\varepsilon}(L+2)}+\theta_{p+1}^{k-\frac{n}{4}+1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}(L+2)}\}\\\leqslant C_{k,L}\delta^2\theta_{p+1}^{k-\beta+\bar{\varepsilon}L-\bar{\varepsilon}n}.&\end{aligned}\tag{4.79}$$

Similarly, we have

$$\|\Lambda \dot{u}_{p+1}\|_{k,0}\leqslant C_k\delta^2\max\{\theta_{p+1}^{k-\beta-\bar{\varepsilon}n},\ \theta_{p+1}^{k-\frac{n}{4}+1+\varepsilon-\alpha-\beta+2\bar{\varepsilon}-\bar{\varepsilon}n}+\theta_{p+1}^{k+1+\varepsilon-\frac{3n}{4}}\}$$
$$\leqslant C_k\delta^2\theta_{p+1}^{k-\beta-\bar{\varepsilon}n}. \tag{4.80}$$

Interpolating with respect to L, we obtain for all $0\leqslant k\leqslant\frac{n}{4}$, $0\leqslant L$

$$\|\Lambda \dot{u}_{p+1}\|_{k,L}\leqslant C_{k,L}\delta^2\theta_{p+1}^{k-\beta+\bar{\varepsilon}L-\bar{\varepsilon}n}. \tag{4.81}$$

Choosing δ small enough so that $C_{k,L}\delta\leqslant 1\ \left(k\leqslant\frac{n}{4},\ L\leqslant\tilde{L}\right)$, we obtain $(P_1,\ p+1)$.

We now prove $(P_2,\ p+1)$. By Proposition 3, we have

$$\Lambda \dot{u}_{p+1}|_{\frac{n}{2},\tilde{L}}\leqslant C(\|g_{p+1}\|_{1+\varepsilon,\tilde{L}+n+2}+\|g_{p+1}\|_{\frac{n}{2}+1,\tilde{L}+n+2}+\|b_{p+1}\|_{1+\varepsilon-\alpha,0}\|\Lambda\dot{u}_{p+1}\|_{\alpha,\tilde{L}+n+2}$$
$$+\|b_{p+1}\|_{1+\varepsilon-\alpha,\tilde{L}+n+2}\|\Lambda\dot{u}_{p+1}\|_{\alpha,0}$$
$$+|b_{p+1}|_{\frac{n}{4}+1,0}\|\Lambda\dot{u}_{p+1}\|_{\frac{n}{4},\tilde{L}+n+2}+|b_{p+1}|_{\frac{n}{4}+1,\tilde{L}+n+2}\|\Lambda\dot{u}_p\|_{\frac{n}{4},0})$$
$$\leqslant C\delta^2(\theta_{p+1}^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}(\tilde{L}+n+2)}+\theta_{p+1}^{\frac{n}{2}+1-2\beta+\bar{\varepsilon}(\tilde{L}+n+2)}$$
$$+\theta_{p+1}^{\alpha-\beta+\bar{\varepsilon}(\tilde{L}+n+2)-\bar{\varepsilon}n}+\theta_{p+1}^{1+\varepsilon-2\beta+\bar{\varepsilon}(\tilde{L}+n+2)-\bar{\varepsilon}n}$$
$$+\theta_{p+1}^{\frac{n}{4}+1-\beta+\bar{\varepsilon}(\tilde{L}+n+2)+\frac{n}{4}-\beta-\bar{\varepsilon}n}+\theta_{p+1}^{\frac{n}{4}+1-\beta+\bar{\varepsilon}(\tilde{L}+n+2)+\frac{n}{4}-\beta-\bar{\varepsilon}n})$$
$$\leqslant C\delta^2\theta_{p+1}^{\frac{n}{2}-\beta+\bar{\varepsilon}\tilde{L}}. \tag{4.82}$$

$$|\Lambda\dot{u}_{p+1}|_{\frac{n}{2},0}\leqslant C\delta^2(\theta_{p+1}^{1+\varepsilon-\frac{n}{2}}+\theta_{p+2}^{\frac{n}{2}+1-2\beta+\bar{\varepsilon}(n+2)}+\theta_{p+1}^{\alpha-\beta+\bar{\varepsilon}(n+2)-\bar{\varepsilon}n}+\theta_{p+1}^{\alpha-\beta-\bar{\varepsilon}n}$$
$$+\theta_{p+1}^{\frac{n}{4}+1-\beta+\frac{n}{4}-\beta+\bar{\varepsilon}(n+2)-\bar{\varepsilon}n}+\theta_{p+1}^{\frac{n}{4}+1-\beta+\bar{\varepsilon}(n+2)+\frac{n}{4}-\beta-\bar{\varepsilon}n})\leqslant C\delta^2\theta_{p+1}^{\frac{n}{2}-\beta}. \tag{4.83}$$

Interpolating with respect to L, we get for all $0\leqslant L\leqslant\tilde{L}$

$$|\Lambda\dot{u}_{p+1}|_{\frac{n}{2},L}\leqslant C_L\delta^2\theta_{p+1}^{\frac{n}{2}-\beta+\bar{\varepsilon}L}. \tag{4.84}$$

By Sobolev inequality we have

$$|\Lambda\dot{u}_{p+1}|_{0,L}\leqslant C\|\Lambda\dot{u}_{p+1}\|_{0,L+n}\leqslant C_L\delta^2\theta_{p+1}^{-\beta+\bar{\varepsilon}L}. \tag{4.85}$$

Interpolating with respect to k, we get for all $0\leqslant k\leqslant\frac{n}{2}$, $0\leqslant L\leqslant\tilde{L}$

$$|\Lambda\dot{u}_{p+1}|_{k,L}\leqslant C_{k,L}\delta^2\theta_{p+1}^{k-\beta+\bar{\varepsilon}L}. \tag{4.86}$$

Choosing δ small enough so that $C_{k,L}\delta\leqslant 1$, we get $(P_2,\ p+1)$.

It follows from $u_{p+1}=\sum_{j=0}^{p}\dot{u}_j+u_0$ that

$$|\Lambda u_{p+1}|_{1+\varepsilon,0}\leqslant|\Lambda u_0|_{1+\varepsilon,0}+\sum_{j=0}^{p}|\Lambda\dot{u}_j|_{1+\varepsilon,0}\leqslant\delta+\delta\sum_{j=0}^{p}\theta_j^{1+\varepsilon-\beta}$$
$$\leqslant C\delta\leqslant 1, \tag{4.87}$$
$$\|\Lambda u_{p+1}\|_{0,0}\leqslant\|\Lambda u_0\|_{0,0}+\sum_{j=0}^{p}\|\Lambda\dot{u}_j\|_{0,0}\leqslant C\delta\leqslant 1. \tag{4.88}$$

This completes the proof of $(P_3,\ p+1)$.

It now remains to prove $(P_1,\ 0)$, $(P_2,\ 0)$, $(P_3,\ 0)$. Since

$$g_0=-S_0\phi(v_0),\ \|\phi(u_0)\|_{\frac{3n}{4},\tilde{L}}\leqslant C\delta^2,$$

$\|\phi(u_0)\|_{\frac{n}{2},\bar{L}}\leqslant C\delta^2$ and $\theta_0=1$, by the property of the smooth operators S_θ, we have

$$\|g_0\|_{k,L}\leqslant C_{k,L}\delta^2\,\theta_0^{k-2\beta+\bar{\varepsilon}L},\ k\geqslant 0,\ L\geqslant 0,$$

$$\|g_0\|_{1+\varepsilon,L}\leqslant C'_L\delta^2\,\theta_0^{1+\varepsilon-\beta-\frac{n}{4}+\bar{\varepsilon}L},$$

$$L\geqslant 0,\ \frac{n}{4}-\beta+\bar{\varepsilon}L\geqslant\bar{\varepsilon}.$$

$$\|g_0\|_{1+\varepsilon,L}\leqslant C_L\delta^2\,\theta_0^{1+\varepsilon-\frac{n}{2}},$$

$$L\geqslant 0,\ \frac{n}{4}-\beta+\bar{\varepsilon}L\leqslant-\bar{\varepsilon}. \tag{4.89}$$

On the other hand, it is easy to see that VII holds for $b_0=F'_\lambda(S_0\Lambda u_0)$. Thus repeating the same procedure as above for $|\Lambda\dot{u}_0|_{k,L}$ and $\|\Lambda\dot{u}_0\|_{k,L}$, we conclude that $(P_1, 0)$, $(P_2, 0)$, $(P_3, 0)$ hold. Therefore, by induction we have proved that (P_1, p)—(P_3, p) hold for all $p\geqslant 0$. Thus we have

$$\sum_{j=0}^{\infty}|\Lambda(u_{j+1}-u_j)|_{1,n+2}=\sum_{j=0}^{\infty}|\Lambda\dot{u}_j|_{1,n+2}\leqslant\delta\sum_{j=0}^{\infty}\theta_j^{1-\beta+\bar{\varepsilon}(n+2)}\leqslant\delta\sum_{j=0}^{\infty}\theta_j^{-\varepsilon}\leqslant C\delta, \tag{4.90}$$

$$\sum_{j=0}^{\infty}\|\Lambda(u_{j+1}-u_j)\|_{0,n+2}\leqslant\delta\sum_{j=0}^{\infty}\theta_j^{-\varepsilon}\leqslant C\delta, \tag{4.91}$$

$$\sum_{j=0}^{\infty}\|\Lambda(u_{j+1}-u_j)\|_{\frac{n}{4},0}\leqslant\delta\sum_{j=0}^{\infty}\theta_j^{\frac{n}{4}-\beta}\leqslant C\delta. \tag{4.92}$$

It can be seen from (4.19) that β may be choosen as $\frac{3n}{8}-\frac{\bar{\varepsilon}}{2}$. Therefore, for $k<\frac{3n}{8}-\frac{\bar{\varepsilon}}{2}$ we have

$$\sum_{j=0}^{\infty}|\Lambda(u_{j+1}-u_j)|_{k,0}\leqslant\delta\sum_{j=0}^{\infty}\theta_j^{k-\beta}<\infty. \tag{4.93}$$

Thus we conclude that there exists $u\in C^{1,n+2}([0,+\infty)\times R^n)$ such that $\{\Lambda u_j\}$ converges to Λu in $|\cdot|_{1,n+2}$, $\|\cdot\|_{\frac{n}{4},n+2}$ and $|\cdot|_{k,0}$ $\left(k<\frac{3n}{8}-\frac{\bar{\varepsilon}}{2}\right)$. Similarly to [1], we can conclude that u satisfies the equation (1.1) and the initial condition (1.2).

From (4.21) and (4.92), (4.93) we immediately get the decay estimates (1.9), (1.10) for u, with $\varepsilon_1>\frac{\bar{\varepsilon}}{2}$. The uniqueness of global solution in the bounded $C^{1,n+2}([0,+\infty)\times R^n)$ class can be deduced from the uniqueness of local smooth solution [9].

Similarly to [1], we can also conclude the C^∞ smoothness of u, Thus the proof of the part (i) of Main Theorem is completed.

Using the fact that for the solution u of initial value problem for the heat equation we have the decay estimates (see (2.5), (2.8))

$$|D_xu|_L\leqslant C\|\phi\|_{L+n+1}(1+t)^{-\frac{n+1}{2}},$$

$$\|D_xu\|_L\leqslant C\|\phi\|_{L+n+1}(1+t)^{-\frac{n+1}{4}},$$

$$|D_x^2u|_L\leqslant C\|\phi\|_{L+n+2}(1+t)^{-\frac{n+2}{2}},$$

$$\|D_x^2 u\|_L \leqslant C\|\phi\|_{L+n+2}(1+t)^{-\frac{n+2}{4}}. \tag{4.94}$$

Proceeding along the same line as above, we are able to prove the part (iii) of the Main Theorem.

§ 5. Proof of the Part (ii) of the Main Theorem

In this section we briefly discribe the proof of the part (ii) of the Main Theorem. Here we effectively use the assumption of three order power (four order power, respectively) of F in λ which enable us to raise the supreme value of k in the estimates for $\|e_p\|_{k,L}$ and $\|e_p\|_{k,L}$, for example, in $F(\lambda)=O(|\lambda|^3)$ case, $k\leqslant\frac{5n}{4}$ for $\|e_p\|_{k,L}$ and $k\leqslant n$ for $\|e_p\|_{k,L}$. Moreover, we only need to estimate $\|g_p\|_{k,L}$ and $\|g_p\|_{k,L}$ for $k\geqslant 1+\varepsilon$, Finally, a slight modification is made in the decay estimate (3.32) and (3.16), in which α is changed into $\alpha=\min\left(\frac{n}{2}, 1+\varepsilon\right)$ and $\alpha=0$, respectively. Using these techniques, we are able to get the finer estimates. We omit the detail here.

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