Chin. Ann. of Math.
7B (1) 1986

THE CONTRACTIONS ON SPACE Π

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Abstract

The theory of the contractions on Hilbert space is now well known. Halmos[1] and Sz-Nagy[2] discussed their u-dilation. Afterwards, Foias and Sz-Nagy established the theory of the harmonic analysis of the operator on Hilbert space[3]. The aim of the present paper is to discuss the constractions on Krein space Π.

§ 0. Introduction

Throughout this paper, $(\cdot, \cdot)$ denotes the indefinite inner product of Krein space Π. The decomposition $\Pi = H_- \oplus H_+$ is called regular[5], if

(1) $H_+ \perp H_-$, i. e. for any

$$x_\pm \in H_\pm, \ (x_+, x_-) = 0,$$

and

(2) H_+, H_- become Hilbert spaces, when the inner product is taken as $(\cdot, \cdot)$, $-(\cdot, \cdot)$ respectively.

The subspace L of Π is called non-positive, positive, non-negative or negative, if the set $\{(x, x) \mid x \in L, x \neq 0\}$ is non-positive, positive, non-negative or negative respectively.

The following facts are used hereafter.

(1) For every regular decomposition $\Pi = H_- \oplus H_+$ we can introduce a new inner product on Π as follows

$$[x_- + x_+, y_- + y_+]_{H_-} = -(x_-, y_-) + (x_+, y_+), \ x_\pm, y_\pm \in H_\pm,$$

$(\Pi, [\cdot, \cdot]_{H_-})$ becomes a Hilbert space, $\|\cdot\|_{H_-}$ denotes the norm. $P_\pm$ denote the projections from Hilbert space $(\Pi, [\cdot, \cdot]_{H_-})$ on subspaces $H_\pm$ respectively. Define $J = P_+ - P_-$, then for any x, $y \in \Pi$, we have

$$[x, y]_{H_-} = (Jx, y), \ (x, y) = [Jx, y]_{H_-}.$$

When there is no possibility of confusion, symbols $(\cdot, \cdot]_{H_-}$, $\|\cdot\|_{H_-}$, P_{H_-} will be simplified to $[\cdot, \cdot]$, $\|\cdot\|$, $P_\pm$, respectively.

(2) All topologies of Π induced by the various regular decompositions of Π are

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Manuscript received December 22, 1983.

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equivalent to each other.

(3) Any unitary operator U on Krein space is bounded[4].

§ 1. Subspace L_A

In this section, we discuss the construction of a class of subspaces of Π.

Definition 1.1. *Suppose $\Pi = H_- \oplus H_+$ is a regular decomposition, and A is a densely defined linear operator from Hilbert space H_- into Hilbert space H_+. Then, the subspace $L_A = \{\{x, Ax\} \mid x \in d(A)\}$ of Π is called the subspace induced by A, where $\{x, Ax\}$ is the vector $x + Ax$ in Π.*

Lemma 1.2. *The following equivalent relations are true:*

a) *L_A is a non-positive subspace, iff A is a contraction in the usual sense of Hilbert space.*

b) *L_A is a closed subspace, iff A is a closed operator.*

c) *L_A is a maximal non-positive, iff A is a contraction with $d(A) = H_-$.*

d) *L_A is a closed maximal negative subspace, iff A is a strict contraction (i. e. $\|Ax\| < \|x\|$ for any $0 \neq x \in d(A)$) with $\mathscr{D}(A) = H_-$.*

e) *L_A is a degenerate subspace (i. e. $L_A \cap L_A^\perp \neq \{0\}$, where*

$$L_A^\perp = \{y \mid (y, x) = 0, x \in L_A, y \in \Pi\}),$$

iff $1 \in \sigma_p(A^*A)$, where $\sigma_p(B)$ denotes the point spectrum of B.

f) *L_A is a closed subspace and $\Pi = L_A \oplus L_A^\perp$, iff A is a closed operator and*

$$1 \in \rho(AA^*) \cap \rho(A^*A),$$

where $\rho(B)$ denotes the resolvent set of B.

g) *L_A is a maximal negative subspace and $\Pi = L_A \oplus L_A^\perp$, iff $\mathscr{D}(A) = H_-$ and $\|A\| \leqslant \alpha < 1$, where α is a constant.*

h) *L is a maximal non-positive (or closed maximal negative subspace), iff there exists a linear operator A from H_- into H_+ such that $\mathscr{D}(A) = H_-$, $L = L_A$ and*

$$\|Ax\| \leqslant \|x\| \quad (\text{or } \|Ax\| < \|x\|,\ x \neq 0).$$

Proof The proofs of a)—d) are trivial, and will be omitted.

e) Let G_A be the graph of A, it is clear, $L_A = G_A$ and $L_A^\perp = \{\{A^*z, z\} \mid z \in \mathscr{D}(A^*)\}$, where A^* is the adjoint operator of A in the usual sense of Hilbert space. Therefore, L_A is a degenerate subspace, iff there are two nonzero vectors $x \in \mathscr{D}(A)$ and $z \in \mathscr{D}(A^*)$ such that

$$x = A^*z,\ Ax = z. \tag{1.1}$$

Obviously, the equation system (1.1) is equivalent to that there is a nonzero vector y such that $y = A^*Ay$ (i. e. $1 \in \sigma_p(A^*A)$).

f) If L_A is a closed subspace and $\Pi = L_A \oplus L_A^\perp$, by b), then we only must prove $1 \in \rho(A^*A) \cap \rho(AA^*)$. Since $\Pi = L_A \oplus L_A^\perp$, so that for any vector $\{x_-, 0\}$, there exist

$x\in\mathscr{D}(A)$ and $z\in\mathscr{D}(A^*)$ such that

$$x_-=x+A^*z,\ 0=Ax+z. \tag{1.2}$$

Thus it can be seen, for every $x_-\in H_-$, the equation $(I-A^*A)x=x_-$ is solvable in $\mathscr{D}(A)$. By the theory of spectral decomposition of self-adjoint operator, $1\in\rho(A^*A)$. Replacing $\{x_-, 0\}$ with $\{0, x_+\}$, similarly, we will obtain $1\in\rho(AA^*)$ too.

The proof of the converse proposition is also easy.

g) By d) and f), the proposition g) is trivial

h) Obviously, every vector x in Π may be expanded uniquely $x=x_-+x_+$ relative to the regular decomposition $H=H_-\oplus H_+$. According to that the subspace L is linear and non-positive (or negative) it is easy to prove that there exists unique linear operator A from H_- into H_+ such that $L=L_A$, and by the maximality $\overline{\mathscr{D}(A)}=H_-$. The proof of remainder of h) is obvious.

We immediately give a simple proof of the following corollary, which has been obtained by other author[4].

Corollary 1.3. *Suppose that L is a closed maximal negative subspace. Then $\Pi=L\oplus L^\perp$ is a regular decomposition iff L becomes a Hilbert space relative to the inner product $(\cdot,\cdot)$.*

Proof The necessity is obvious, so we have only to prove sufficiency.

By h) in Lemma 1.2, there is a strict contraction A with $\mathscr{D}(A)=H_-$ such that $L=L_A$. According to f) in Lemma 1.2, we only need to prove $1\in\rho(A^*A)$.

Obviously, if $1\in\sigma(A^*A)$, then there exists a vector $x\in H_-$, such that $\|f_n\|\to\infty$ $(n\to\infty)$, where

$$f_n=(I-A^*A)^{-1/2}E_{\Delta_n}x,\ \Delta_n=\left[0,\ 1-\frac{1}{n}\right)$$

and $A^*A=\int\lambda\, dE_\lambda$ is the spectral decomposition.

Since $\{\{f_n, Af_n\}\}\subset L$, and $-(\{f_n-f_m,\ A(f_n-f_m)\},\ \{f_n-f_m\},\ A\{f_n-f_m)\})$
$=-(\{(I-A^*A)(f_n-f_m),\ (f_n-f_m)\})=-((E_{\Delta_n}-E_{\Delta_m})^2x, x)$,

so $\{\{f_n, Af_n\}\}$ is a Cauchy sequence in Hilbert space L, henceforth, there exists $\{f,Af\}\in L$ $(f\in H_-)$ such that

$$\lim -((I-A^*A)(f_n-f), f_n-f)=0. \tag{1.3}$$

On the other hand, when $n\geqslant m$, $E_{\Delta_n}f_m=f_m$. And from (3.3), we have $E_{\Delta_m}f=(I-A^*A)^{-1/2}E_{\Delta_m}x=f_m$ $(m=1, 2, \cdots)$. Obviously, this contradicts that $\lim\|f_m\|=\infty$, so $1\in\rho(A^*A)$. Therefore, we have $\|A\|=\|A^*\|\leqslant\alpha<1$. Thus it can be seen that $\Pi=L\oplus L^1$ is a regular decomposition.

Lemma 1.4. *Let T be a bounded linear operator on a Hilbert space H.*

a) *If E^+ and $E^-(E'^+, E'^-)$ are positive and negative spectral subspaces of $(I-T^*T)$ $(I-TT^*))$ respectively, then for any $x\in E'^\pm(x\in E^\pm)$, $T^*x\in E^\pm(Tx\in E^{1\pm})$.*

b) *For any Borel measurable function $f(t)$, if $f(t)$ is bounded in a certain* *neighborhood of point $t=1$, then*

$$T^*f(I-TT^*)=f(I-T^*T)T^*. \tag{1.4}$$

The proof of Lemma 1.4 has no essential difficulty, and will be omitted.

§ 2. Regular Contraction

In this section, we discuss the construction of the contraction and regular contraction on Π.

Definition 2.1. *A linear operator T on Π is called a contraction, if for any $x\in d(T)$,*

$$(Tx, Tx)\leqslant(x, x). \tag{2.1}$$

We know that a contraction on Hilbert space must be bounded, but a contraction on Krein space can be unbounded even if its domain of definition is whole space (example of this kind is easy given). Thus, in general, the contraction is not a "good" operator. Hence, we introduce the following more special contraction.

Definition 2.2. *Suppose that T is a densely defined contraction, if there exists a regular decomposition $\Pi=H_-\oplus H_+$, such that $H_-\subset\mathscr{D}(T)$ and $\Pi=\overline{TH_-}\oplus(TH_-)^\perp$ is a regular decomposition, where $\overline{TH_-}$ is the closure of TH_-, then T is called a regular contraction.*

In Definition 2.2, we may replace $\overline{TH_-}$ with TH_-. In fact, because $H_-\subset\mathscr{D}(T)$ and $(Tx, Tx)\leqslant(x, x)$, subspace TH_- must be negative and closed, i. e. $\overline{TH_-}=TH_-$.

Later, we will prove that the regularity of a contraction is independent of the selection of regular decomposition $\Pi=H_-\oplus H_+$ (see Theroem 2.7). Now we prove that the regular contraction is bounded.

Theorem 2.3. *If T is regular, then T is bounded on $\mathscr{D}(T)$, T has unique extension on the whole space Π, and this extension is also a regular contraction*

Proof Let $[\cdot, \cdot]_1$, and $\|\cdot\|_1$ be the inner product and norm induced by the regular decomposition $\Pi=TH_-\oplus(TH_-)^\perp$ respectively. Set $T_{H-}=P_{TH-}\,T\,P_{H-}$,

$$T_1=P_{TH-}TP_{H+} \quad\text{and}\quad T_2=P_{TH-}\,TP_{H+}.$$

obviously, we have to prove that T_{H-}, T_1 and T_2 are all bounded operators from Hilbert space $(\Pi, [\cdot, \cdot]$ into Hilbert space $(\Pi, [\cdot, \cdot]_1)$ in three steps.

a) Since $\|T_{H-}x\|_1\geqslant\|x\|$ $(x\in H_-)$, T_{H-} is bijective. But T_{H-}^{-1} is a bounded operator, and $d(T_{H-}^{-1})=TH_-$. By the converse operator theorem, T_{H-} must be a bounded operator.

b) If T_1 is unbounded, then there exists a sequence $\{y_n\}\subset H_+$ such that $\|y_n\|\to0$ $(n\to\infty)$ and $\|T_1y_n\|_1=1$ $(n=1, 2, \cdots)$. We may assume that $\lim\ \|T_{H}^{-}T_1\ y_n\|=\alpha\neq0$ (otherwise, choose a subsequence of $\{y_n\}$). Putting

$$z_n = -T_{H_-}^{-1}T_1y_n + y_n \quad (n=1, 2, \cdots),$$

obviously

$$\lim (z_n, z_n) = \lim[(T_{H_-}^{-1}T_1y_n, T_{H_-}^{-1}T_1y_n) + (y_n, y_n)]. \tag{2.2}$$

And yet

$$\underline{\lim}(Tz_n, Tz_n) = \underline{\lim}(T_2y_n, T_2y_n) \geqslant 0 \tag{2.3}$$

by the assumption of contractibility, the inequality (2.3) is in contradiction with the equality (2.2). Henceforth, the operator T_1 must be bounded.

c) If T_2 is unbounded, then there exists a normed sequence $\{Y_n\} \subset H_+$, and $\|T_2y_n\|_1 \to \infty (n\to\infty)$. Similarly, define $z_n = -T_{H_-}^{-1}T_1y_n + y_n (n=1, 2, \cdots)$, and from the boundness of T_{H_-}, $T_{H_-}^{-1}$ and T_1, we immediately get the $\{(z_n, z_n)\}$ is a bounded sequence.

On the other hand, we have

$$(z_n, z_n) \geqslant (Tz_n, Tz_n) = (T_2y_n, T_2y_n) = \|T_2y_n\|_1^2 \to \infty. \tag{2.4}$$

Obviously, the boundness of $\{(z_n, z_n)\}$ is in contradiction with (2.4), hence T_2 must be a bounded operator too.

The proof of remainder of Theorem 2.4 is obvious.

Without loss of generality, we may assume that the domain of a regular contraction is the whole space Π hereafter.

Suppose that $\Pi = H_- \oplus H_+$ and $\Pi = H'_+ \oplus H'_+$ are the regular decomposition of Π. If T_{H_-}, T_1 and T_2 are three bounded lincar operators, T_{H_-}: $H_- \to H'_-$; T_1: $H_+ \to H'_-$; and T_2: $H_+ \to H'_+$, then the linear operator T on Π is bounded, which is gencrated by the equations

$$\begin{cases} Tx = T_{H_-}\, x,\ x \in H_-; \\ Ty = T_1y + T_2y,\ y \in H_+. \end{cases}$$

In this case, the operator T is denoted by $\{T_{H_-}, T_1, T_2\}$. We consider the polar decomposition $T_{H_-} = VR$, where $R = (T_{H_-}^*T_{H_-})^{1/2}$ and V is a partially isometric operator.

Theroem 2.4. *The operator* $T = \{T_{H_-}, T_1, T_2\}$ (*relative to the regular decomposition* $\Pi = H_- \oplus H_+$ *and* $\Pi = H_-^1 \oplus H_+^1$) *is a contraction, iff*

a) $R^2 \geqslant I$;

b) *for any* $y \in H_+$,

$$\|(R^2 - I)^{-1/2}\, V^* T_1 y\| \leqslant \|(I + T_1^* P T_1 - T_2^* T_2)^{1/2} y\|,$$

where P *is the projection from Hilbert space* H_-^1 *onto subspace* $H_-^1 \ominus \mathscr{R}(V)$.

Proof We first prove the necessity.

If T is a contraction on Π, then a) is trivial. We prove b) as follows.

Obviously, the contractibility of T is equivalent to that for any $x_+ \in H_+$, $x_- \in H_-$

$$-[(T_{H_-}^*T_{H_-}-I)x_-, x_-]-[T_{H_-}x_-, T_1x_+]'-[T_1x_+, T_{H_-}x_-]'$$
$$-[T_1^*T_1x_+, x_+]\leqslant[(I-T_2^*T_2)x_+, x_+], \qquad (2.5)$$

where $[\cdot, \cdot]'$ denotes the inner product induced by $\Pi=H_-^1\oplus H_+^1$.

We first prove $V^*T_1H_+\subset\mathscr{D}((R^2-I)^{-1/2})$ by the reduction to absurdity.

$$R=\int_1^{\|R\|}\lambda\, dE_\lambda$$

denotes the spectral decomposition of R. If there is a certain $x_+\in H_+$ such that $V^*T_1x_+\overline{\in}\mathscr{D}((R^2-I)^{-1/2})$, then there exists a positive number $\varepsilon(x_+)$ such that

$$\int_{1+\varepsilon(x_+)}^{\|R\|}(\lambda^2-I)^{-1}\|dE_\lambda y_1\|^2>M_0+[T_1^*T_1x_+, x_+], \qquad (2.6)$$

where $y_1=(E_{\|R\|}-E_{1+\varepsilon(x_+)})V^*T_1^*x_+$ and $M_0=[(I-T_2^*T_2)x_+, x_+]$. On the other hand, by (2.5), we have

$$\sup_{x_-\in H_-}I(x_+, x_-)\leqslant M_0, \qquad (2.7)$$

where

$$I(x_+, x_-)=-[(R^2-I)x_-, x_-]-[Rx_-, V^*T_1x_+]-[V^*T_1x_+, x_-]-[T_1^*T_1x_+, x_+].$$

Particularly, we choose $x_-=-R(R^2-I)^{-1}y_1$ in $I(x_+,x_-)$, then from (2.6)

$$I(x_+, -R(R^2-I)^{-1}y_1)=\|R(R^2-I)^{-1/2}y_1\|^2-[T_1^*T_1x_+, x_+]>M_0, \qquad (2.8)$$

the inequlity (2.8) is contrary to (2.7), and hence $V^*T_1H_+\subset\mathscr{D}((R^2-I)^{-1/2})$.

Now we prove the inequality in b). Since $V^*T_1H_+\subset\mathscr{D}((R^2-I)^{-1/2})$, so that for any $x_\pm\in H_\pm$, we have

$$I(x_+, x_-)=-\|(R^2-I)^{1/2}x_-+R(R^2-I)^{-1/2}V^*T_1x_+\|^2+\|(R^2-I)^{-1/2}V^*T_1x_+\|^2$$
$$-[T_1^*PT_1x_+, x_+]\leqslant[(I-T_2^*T_2)x_+, x_+], \qquad (2.9)$$

but the set $(R^2-I)^{1/2}H_-$ is dense in the closed subspace $\overline{R(R^2-I)^{-1/2}V^*T_1H_+}$ of H_-, thus it can be seen, the proposition b) is true.

On the contrary, if a) and b) hold, then we immediately obtain the inequality (2.9), thus, (2.5) is true too, and hence $T=\{T_{H_-}, T_1, T_2\}$ is a contraction.

The following corollaries are obvious.

Corollary 2.5. *$T=\{T_{H_-}, T_1, T_2\}$ is a regular contraction (relative to the regular decomposition $\Pi=H_-\oplus H_+$), iff*

a) *$R^2\geqslant I$, and V is a unitary operator;*

b) *T_2 is a contraction, and for any $y\in H_+$,*

$$\|(R^2-I)^{-1/2}V^*T_1y\|\leqslant\|(I-T_2^*T_2)^{1/2}y\|.$$

Corollary 2.6. Suppose $T=\{T_{H_-}, T_1, T_2\}$ is a regular contraction (relative to the regular decomposition $\Pi=H_-\oplus H_+$), then $K=(R^2-I)^{-1/2}V^*T_1(I-T_2^*T_2)^{-1/2}$ is a contraction on Hilbert space $\overline{(I-T_2^*T_2)H_+}$ into Hilbert space $\overline{(R^2-I)H}$ too.

The proofs of these corollaries are straightforward, and will be omitted.

Theorem 2.7. *Suppose that $T=\{T_{H_-}, T_1, T_2\}$ is a regular contraction related to the regular decomposition $\Pi=H_-\oplus H_+$. Then for any regular decomposition*

$$\Pi = H'_- \oplus H'_+, \quad \Pi = TH'_- \oplus (TH')$$

is also a regular decomposition (i. e. T is also a regular contraction related to $\Pi = H^1_- \oplus H^1_+$).

Proof Obviously, we only need to prove TH^1_- is a maximal negative subspace and $\Pi = TH^1_- \oplus (TH^1_-)^\perp$.

By the definition of regular contraction, $\Pi = TH_- \oplus (TH_-)$ is a regular decomposition. On the basis of g) and h) in Lemma 1.2, there exists unique operator A from H_- into H_+ such that $\mathscr{D}(A) = H_-$ $\|A\| \leqslant \alpha < 1$ and $L_A = H_-$. Thus

$$TH'_- = T(\{\{x_-, Ax_-\} \mid x_- \in H_-\} = \{\{T_{H_-}x_- + T_1Ax_-, T_2Ax_-\}_1 \mid x_- \in H_-\},$$

where $\{\cdot, \cdot\}_1$ is the graph representation related to $\Pi = TH_- \oplus (TH_-)^\perp$.

Now we prove that TH'_- is a maximal negative and $\Pi = TH'_- \oplus (TH'_-)^\perp$ under the regular decomposition $\Pi = TH_- \oplus (UH_-)^\perp$, by g) and h) in Lemma 1.2, this is to prove

a) operator $B = (T_{H_-} + T_1A)$ is one-to-one map from H_- onto TH_-;

b) $\|T_2AB^{-1}\| \leqslant \beta < 1$, wher β is a certain constant. By the contractibility of T, it is clear that B is the one-to-one map from H_- into TH_-. And by the Corollaries 2.5—2.6 we have $\|T_{H_-}\| \geqslant 1$, $T_{H_-}H_- = TH_-$ and

$$\|T_{H_-}^{-1}T_1A\| = \|R^{-1}V^*T_1A\| = \|R^{-1}(R^2 - I)^{1/2}(R^2 - I)^{-1/2}V^*T_1A\|$$
$$= \|R^{-1}(R^2 - I)^{1/2}K(I - T_2^*T_2)^{1/2}A\| \leqslant \|A\| \leqslant \alpha < 1.$$

Thus it can be seen that $B = T_{H_-}[I + T_{H_-}^{-1}T_1A]$ is the one-to-one map from H_- onto TH_-, i.e. a) is true.

The graph representation of $TH'_-(= TL_A)$ related to $\Pi = TH_- \oplus (TH_-)^\perp$ is

$$L_{T_2AB^{-1}} = \{\{y, T_2AB^{-1}\}_1; \ y \in TH_-\}.$$

By the contractibility of T, for any $x_- \in H_-$,

$$\|Bx_-\|_1^2 - \|T_2Ax_-\|^2 \geqslant \|x_-\|^2 - \|Ax_-\|^2,$$

i.e. for any $y \in TH_-$,

$$\|y\|_1^2 - \|T_2AB^{-1}y\|_1^2 \geqslant \|B^{-1}y\|^2 - \|AB^{-1}y\|^2 = \|(I - A^*A)^{1/2}B^{-1}y\|^2. \quad (2.10)$$

From $\|A\| \leqslant \alpha < 1$ and the fact (2), there exists a positive number r such that

$$\|(I - A^*A)^{1/2}B^{-1}y\| \geqslant r\|y\|_1. \quad (2.11)$$

No loss in generality, we may assume $r < 1$. Thus it can be seen that for any $y \in TH_-$,

$$\|T_2AB^{-1}y\|_1 \leqslant (1 - r^2)^{1/2}\|y\|_1.$$

Therefore, b) is true, and $\beta = (1 - r^2)^{1/2}$.

Obviously, a bounded contraction may be not regular.

Example Let $\Pi = H_- \oplus H_+$, $H_+ = l^2 = H_-$. Suppose that $\{e_i\}$ $(i = -1, -2, \cdots)$ $\{e_i\}$ $(i = 0, 1, 2, \cdots)$ are the thorogonormal bases of H_- and H_+ respectively, and T is the left shift on Π (i.e. $Te_i = e_{i-1}$, $i = 0, \pm 1, \pm 2, \cdots$). Evidently, T is a bounded contraction, but TH_- is not a maximal negative subspace, and hence T can't be a regular contraction.

§ 3. U-Dilation of Regular Contraction

Definition 3.1. *Suppose that T is a contraction, and H_1, H_2 are the Hilbert space. If there exists a unitary operator (which is unitary with respect to the indefinite inner product) U from $\Pi\oplus H_1$ onto $\Pi\oplus H_2$ such that*

$$T=P^2U|_{P_1},$$

where $P^i(i=1,2)$ are the projections from $\Pi\oplus H^i$ onto Π respectively, then (U, H_1, H_2) is called the u-dilation of T in the Halmos's sense. If $H_1=H_2$ and (U, H_1, H_1) statisfies the following equations

$$T^n=P^iU^n|_{P^i},\ n=0,\ 1,\ 2,\ \cdots,$$

then (U, H_1, H_1) is called the u-dilation of T in the Nagy's sense.

In this section, we shall find out all u-dilations of a regular contraction in the Halmos's or Nagy's sense.

Lemma 3.2. (1) *If T is a contraction (or regular contraction), then for any unitary U on Π, the operator UT is also a contraction (or regular contraction).*

(2) *If $T=\{T_{H_-}, T_1, T_2\}$ is a regular contraction, then there exists a unitary $\tilde{U}$ on Π such that*

a) $\tilde{U}TH_-=H_-$;

b) *If (U, H_1, H_2) is a u-dilation of T in the Halmos's sense, then $(U'U, H_1, H_2)$ is also a u-dilation of $\tilde{U}T$ in the Halmos's sense, where $U'=\tilde{U}\oplus I_{H_2}$;*

c) *If (U, H_1, H_2) is a u-dilation of $\tilde{U}T$ in the Halmos's sense, then $(U'^{-1}U, H_1, H_2)$ is also a u-dilation of T in the Halmos's sense.*

The proof of Lemma 3.2 is straightforward, and will be omitted.

Now first give the general form of u-dilation of a regular contraction T with $TH_-=H_-$ as follows.

Theorem 3.3. *If $T=\{T_{H_-}, T_1, T_2\}$ is a regular contraction, and $TH_-=H_-$, Then the general form of u-dilation (U, H_1, H_2) of T in the Halmos's sense is*

$$U=\begin{pmatrix} T_{H_-} & T_1 & Y \\ 0 & T_2 & (I-T_2T_2^*)^{1/2}U_2^* \\ U_4(R^2-I)^{1/2} & X & Z \end{pmatrix},$$

$$X=U_4(R^2-I)^{-1/2}RV^*T_1+V_5(I-K^*K)^{1/2}(I-T_2^*T_2)^{1/2},$$

$$Y=-T_1T_2^*(I-T_2T_2^*)^{-1/2}U_2^*+V(R^2-I)^{1/2}(I-KK^*)V_1^*,$$

$$Z=-[U_4RK+V_5(I-K^*K)^{1/2}]T_2^*U_2^*+U_4R(I-KK^*)^{1/2}-V_5K^*V_1^*+W,$$

and H_1 and H_2 must be expanded

$$H_1=\mathscr{R}(U_2)\oplus\mathscr{R}(V_1)\oplus(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1))^\perp,$$

$$H_2=\mathscr{R}(U_4)\oplus\mathscr{R}(V_5)\oplus(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))^\perp,$$

where V, R and K are the same as in Theorem 2.4 and Corollary 2.6, U_4, V_5, U_2 and

V_1 are all unitary operators:

U_4: $\overline{\mathscr{R}(R^2-I)}\to\mathscr{R}(U_4)$; V_5: $\overline{\mathscr{R}(I-K^*K)^{1/2}(I-T_2^*T_2)^{1/2}}\to\mathscr{R}(V_5)$;

U_2: $\overline{\mathscr{R}(I-T_2T_2^*)}\to\mathscr{R}(U_2)$; V_1: $\overline{\mathscr{R}(I-KK^*)^{1/2}(R^2-I)^{1/2}}\to\mathscr{R}(V_1)$;

and W is a unitary operator from Hilbert space $(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1))^\perp$ onto Hilbert space $(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))^\perp$.

In particular, when $H_1=\mathscr{R}(U_2)\oplus\mathscr{R}(V_1)$ and $H_2=\mathscr{R}(U_4)\oplus\mathscr{R}(V_5)$ (i. e. W vanith), (U, H_1, H_2) is the minimal u-dilation of T in the Halmos's sense.

Proof The operator U is represented by the 3×3 matrix

$$U=\begin{pmatrix} T_{H_-} & T_1 & A_1 \\ 0 & T_2 & A_2 \\ A_4 & A_5 & A_3 \end{pmatrix},$$

where A_1: $H_1\to H_-$; A_2: $H_1\to H_+$; A_3: $H_1\to H_2$; A_4: $H_-\to H_2$ and A_5: $H_+\to H_2$. Evidently, U is a unitary operator from $\Pi\oplus H_1$ onto $\Pi\oplus H_2$, iff $A_i (i=1, 2, \cdots, 5)$ are all bounded, and satisfy the following equations:

$$T_{H_-}^*T_{H_-}-A_4^*A_4=I_{H_-}, \quad (1) \qquad -T_{H_-}T_{H_-}^*+T_1T_1^*+A_1A_1^*=-I_{H_-}, \quad (1')$$

$$-T_1^*T_1+T_2^*T_2+A_5^*A_5=I_{H_+}, \quad (2) \qquad T_2T_2^*+A_2A_2^*=I_{H_+}, \quad (2')$$

$$-A_1^*A_1+A_2^*A_2+A_3^*A_3=I_{H_1}, \quad (3) \qquad -A_4A_4^*+A_5A_5^*+A_3A_3^*=I_{H_2}, \quad (3')$$

$$-T_{H_-}^*T_1+A_4^*A_5=0, \quad (4) \qquad T_1T_2^*+A_1A_2^*=0, \quad (4')$$

$$-T_{H_-}^*A_1+A_4^*A_3=0, \quad (5) \qquad -T_{H_-}A_4^*+T_1A_5^*+A_1A_3^*=0, \quad (5')$$

$$-T_1^*A_1+T_2^*A_2+A_5^*A_3=0, \quad (6) \qquad T_2A_5^*+A_2A_3^*=0. \quad (6')$$

In order to find all sloutions of the equations (1)—(6) and (1′)—(6′), we shall divide into several steps, and when there is no posibility of confusion, the below indexes of all idential operator (for example I_{H_-}, I_{H_+}, I_{H_1}, I_{H_2}, etc) are omitted.

a) Solving (1). $T_{H_-}=VR$ is the polar decomposition, $R^2\geqslant I$, and V is a unitary operator on H_-. From the equation (1), we have

$$A_4=U_4(R^2-I)^{1/2}, \tag{3.1}$$

where U_4 is a unitary operator from $\overline{\mathscr{R}(R^2-I)}$ onto $\mathscr{R}(U_4)\subset H_2$.

b) Solving (4). By (3.1), the equation (4) is reduced to

$$-RV^*T_1+(R^2-I)^{1/2}U_4^*A_5=0.$$

Using Theorem 2.4, we obtain

$$-R(R^2-I)^{-1/2}V^*T_1+U_4^*A_5=0.$$

Therefore, the general form of A_5 is

$$A_5=A_5^0+A_5^1,\ \mathscr{R}(A_5^0)\subset\mathscr{N}(U_4^*)^{1)},\ \mathscr{R}(A_5^1)\subset\mathscr{N}(U_4^*)^\perp,$$

$$A_5^1=U_4R(R^2-I)^{-1/2}V^*T_1. \tag{3.2}$$

Since $K_1=(R^2-I)^{-1/2}V^*T_1$ is bounded, A_5^1 is also bounded and

$$A_5^{1*}=K_1^*RU_4^*.$$

1) $\mathscr{N}(A)$ denotes the null subspace of A.

c) Solving (2). Since $\mathscr{R}(A_5^0)\perp\mathscr{R}(A_5^1)$, we have $A_5^*A_5=A_5^{0*}A_5^0+A_5^{1*}A_5^1$. But

$$T_1^*T_1=T_1^*VV^*T_1=K_1^*(R^2-I)K_1=A_5^{1*}A_5^1-K_1^*K_1$$

and hence (2) is reduced to

$$A_5^{0*}A_5^0=(I-T_2^*T_2)-K_1^*K_1.$$

Thus, for any $x\in H_+$,

$$\|A_5^0x\|^2=\|(I-K^*K)^{1/2}(I-T_2^*T_2)^{1/2}x\|^2,$$

i.e. there is a unitary operator V_5 from $\overline{\mathscr{R}((I-K^*K)^{1/2}(I-T_2^*T_2)^{1/2})}$ onto $\overline{\mathscr{R}(A_5^0)}$ such that

$$A_5^0=V_5(I-K^*K)^{1/2}(I-T_2^*T_2)^{1/2},\ \mathscr{R}(V_5)\subset\mathscr{N}(U_4^*). \tag{3.3}$$

d) Solving (2′). Similar to step a), $A_2^*=U_2(I-T_2T_2^*)^{1/2}$, where U_2 is an isometric operator from $\overline{\mathscr{R}(I-T_2T_2^*)}$ into H_1.

e) Solving (4′). If $x\in\mathscr{N}(A_2^*)$ (i.e. $(I-T_2T_2^*)x=0$), then $(I-T_2^*T_2)T_2^*x=0$. And by Theorem 2.4, we obtain

$$\|(R^2-I)^{-1/2}V^*T_1T_2^*x\|=0,\quad x\in\mathscr{N}(A_2^*)$$

and hence $T_1T_2^*x=0$. Thus it can be seen, that the equation $T_1T_2^*+A_1A_2^*=0$ may be divided by A_2^*, so we have

$$A_1=A_1^0+A_1^1,\ A_1^1=-T_1T_2^*(I-T_2T_2^*)^{-1/2}U_2^*(=-T_1T_2^*A_2^{*-1}),$$

$$\overline{\mathscr{D}(A_1^1)}=\mathscr{R}(A_2^*)=\mathscr{R}(U_2)\subset H_1,\ \mathscr{D}(A_1^0)=\mathscr{R}(U_2)^\perp\subset H_1 \tag{3.4}$$

By Lemma 1.3 and Corollary 2.5, we obtain

$$A_1^1=-T_1(I-T_2^*T_2)^{-1/2}T_2^*U_2^*=-VK_2T_2^*U_2^*, \tag{3.4′}$$

where $K_2=V^*T_1(I-T_2^*T_2)^{-1/2}$ is a bounded operator

f) Solving (1′). Since $\mathscr{D}(A_1^0)\perp\mathscr{D}(A_1^1)$, we have $A_1A_1^*=A_1^1A_1^{1*}+A_1^0A_1^{0*}$. Since

$$\mathscr{N}(I-T_2^*T_2)\subset\mathscr{N}(V^*T_1),\ \mathscr{R}(V^*T_1)\subset\mathscr{R}((R^2-I)^{-1/2})$$

and $V^*T_1(I-T_2^*T_2)^{-1/2}$ is a bounded operator from $\overline{\mathscr{R}(I-T_2^*T_2)}$ into $\overline{\mathscr{R}((R^2-I)^{-1/2})}$, it is easy to prove that $(I-T_2^*T_2)^{-1/2}T_1^*V$ is also a bounded operator from $\overline{\mathscr{R}((R^2-I)^{-1/2})}$ into $\overline{\mathscr{R}(I-T_2^*T_2)}$, and hence

$$T_1T_1^*=VV^*T_1T_1^*VV^*=VK_2K_2^*V^*-VK_2T_2^*T_2K_2^*V^*.$$

Note that the domain of A_1^1 is included in $\mathscr{R}(U_2)$ and

$$\mathscr{D}(U_2)=\overline{\mathscr{R}(I-T_2T_2^*)}\supset\mathscr{R}(T_2K_2^*),$$

so

$$A_1^1A_1^{1*}=VK_2T_2^*U_2^*U_2T_2K_2^*V^*=VK_2T_2^*T_2K_2^*V^*.$$

Thus, the equation (1′) is reduced to

$$VK_2K_2^*V^*+A_1^0A_1^{0*}=V(R^2-I)V^*.$$

According to Theorem 2.4, above equation is solvable, and

$$A_1^{0*}V|_{(R^2-I)}=0,\ A_1^{0*}V|_{(R^2-I)^\perp}=V_1(I-KK^*)^{1/2}(R^2-I)^{1/2},$$

$$\mathscr{R}(A_1^{0*})=\mathscr{R}(V_1)\subset\mathscr{R}(U_2)^\perp, \tag{3.5}$$

where V_1 is an isometric operator from $\overline{\mathscr{R}((I-KK^*)^{1/2}(R^2-I))^{1/2}}$ into $\mathscr{R}(U_2)^\perp$.

g) Solving (5). By the step a), (5) is reduced to

$$-RV^*A_1+(R^2-I)^{1/2}U_4^*A_3=0$$

and using (3.4′) and (3.5), we obtain

$$-R(-K_2T_2^*U_2^*+(R^2-I)^{1/2}(I-KK^*)^{1/2}V_1^*)+(R^2-I)^{1/2}U_4^*A_3=0$$

obviously, this equation may be dividedby $(R^2-I)^{1/2}$, so

$$A_3=A_3^0+A_3', A_3'=U_4R(R^2-I)^{-1/2}V^*A_1(=U_4R(I-KK^*)^{1/2}V_1^*-U_4RKT_2^*U_2^*),$$

$$\mathscr{R}(A_3^0)\subset\mathscr{N}(U_4^*), \mathscr{R}(A_3^1)\subset\mathscr{N}(U_4^*)^\perp. \quad (3.6)$$

h) Solving (6′). We apply the expressions of A_3, A_5 and A_2^* to the equation (6′), it is reduced to

$$T_2(A_5^{0*}+K_1^*RU_4^*)+(I-T_2T_2^*)^{1/2}U_2^*(A_3^{0*}+A_1^*V(R^2-I)^{-1/2}RU_4^*)=0$$

and according to the conclusion of step c), Lemma 1.4, (3.6) and

$$\mathscr{R}(T_2A_5^{0*})\subset\mathscr{R}(I-T_2T_2^*)^{1/2},$$

above equation is reduced to

$$T_2(I-K^*K)^{1/2}V_5^*+U_2^*A_3^{0*}=0.$$

Therefore, we have

$$A_3^0=A_{30}^0+A_{30}', A_{30}'=-V_5(I-K^*K)^{1/2}T_2^*U_2^*,$$

$$\mathscr{D}(A_{30}^1)\subset\mathscr{R}(U_2), \mathscr{D}(A_{30}^0)=\mathscr{R}(U_2)^\perp. \quad (3.7)$$

i) Solving (5′). Applying the expressions of A_4^*, A_5^*, A_1 and A_3^* to (5′), we get

$$T_1A_5^{0*}+A_1A_3^{0*}=0,$$

it is equivalent to the following equation

$$(A_5^0T_1^*+A_3^0A_1^*)V=0. \quad (3.8)$$

We substitute the expressions of A_5^0 and A_1^* in (3.8), thus

$$V_5(I-K^*K)^{1/2}(I-T_2^*T_2)^{1/2}T_1^*V+A_3^0(-U_2T_2K_2^*+V_1(I-KK^*)^{1/2}(R^2-I)^{1/2})$$
$$=0. \quad (3.9)$$

Owing to $\mathscr{D}(A_{30}^0)=\mathscr{R}(U_2)^\perp$, and $\mathscr{R}(V_1)\subset\mathscr{R}(A_2^*)^\perp=\mathscr{R}(U_2)^\perp$, (3.9) is reduced to

$$V_5(I-K^*K)^{1/2}(I-T_2^*T_2)^{1/2}T_1^*V+V_5(I-K^*K)^{1/2}T_2^*T_2K_2^*$$
$$+A_{30}^0V_1(I-KK^*)^{1/2}(R^2-I)^{1/2}=0$$

i. e.

$$[V_5(I-K^*K)^{1/2}K^*+A_{30}^0V_1(I-KK^*)^{1/2}](R^2-I)^{1/2}=0, \quad (3.10)$$

Since $(I-K^*K)^{1/2}K^*=K^*(I-KK^*)^{1/2}$, the equation (3.10) is solvable, and

$$A_{30}^0=A_{300}^0+A_{300}', A_{300}'=-V_5K^*V_1^*, \mathscr{R}(A_{300}')=\mathscr{R}(V_1),$$

$$\mathscr{D}(A_{300}^0)=\mathscr{R}(V_1)^\perp\cap\mathscr{R}(U_2)^\perp. \quad (3.11)$$

j) Solving (3′). Applying the expressions of A_3^0, A_3^1, A_4 and A_5 to the equation (3′), (3′) is reduced to the following two equations

$$-U_4(R^2-I)U_4^*+U_4(R^2-I)^{-1/2}T_{H_-}^*T_1T_1^*T_{H_-}(R^2-I)^{-1/2}U_4^*$$
$$+U_4(-R^2-I)^{-1/2}T_{H_-}^*A_1A_1^*T_{H_-}(R^2-I)^{-1/2}U_4^*+A_5^0A_5^{1*}$$
$$+A_3^0A_3^{1*}=I_{\mathscr{R}(U_4)}, \quad (3.12)$$

$$A_3^0A_3^{0*}+A_5^0A_5^{0*}+A_5'A_5^{0*}+A_3'A_3^{0*}=I_{\mathscr{R}(U_4)^\perp}. \quad (3.13)$$

Actually, on the basis of the equation (1′), (3.12) is equivalent to

$$A_5^0 A_5^{1*} + A_3^0 A_3^{1*} = 0, \tag{3.14}$$

and using (3.8), we easily verify that (3.14) is satisfied. Owing to the equality (3.14), the equation (3.13) is reduced to

$$A_3^0 A_3^* + A_5^0 A_5^{0*} = I_{\mathscr{R}(U_4)^\perp}. \tag{3.15}$$

Note that $\mathscr{D}(A_0^1)$, $\mathscr{D}(A_{00}^1)$ and $\mathscr{D}(A_{300}^0)$ are subspace orthogonal to each other, so the equation (3.15) is easily reduced to

$$V_5 V_5^* + A_{300}^0 A_{300}^{0*} = I_{\mathscr{R}(U_4)^\perp}.$$

Thus it can be seen, A_{300}^{0*} must be an isometric operator from $\mathscr{N}(U_4) \ominus \mathscr{N}(V_5)$ into $\mathscr{R}(A_2^*)^\perp \ominus \mathscr{R}(V_1)$.

k) Solving (6). Since

$$V^* A_1 = -V^* T_1 (I - T_2^* T_2)^{-1/2} T_2^* U_2^* + (R^2 - I)^{1/2} (I - KK^*)^{1/2} V_1^*,$$

we have $\mathscr{R}(V^* A_1) \subset \mathscr{D}((R^2 - I)^{-1/2})$, and

$$\begin{aligned} T_1^* A_1 &= T_1^* V V^* A_1 = T_1^* V (R^2 - I)^{-1/2} (R^2 - I)(R^2 - I)^{-1/2} V^* A_1 \\ &= K_1^* R U_4^* U_4 (R^2 - I)^{-1/2} R V^* A_1 - K_1^* (R^2 - I)^{-1/2} V^* A_1 \\ &= A_5^{1*} A_3^1 - K_1^* (R^2 - I)^{-1/2} V^* A_1. \end{aligned} \tag{3.16}$$

On the other hand

$$T_2^* A_2 + A_5^{0*} A_3^0 = T_2^* A_5 + A_5^{0*} (-V_5 (I - K^* K)^{1/2} T_2^* U_2^* - V_5 K^* V_1^* + A_{300}^0). \tag{3.17}$$

By (3.16), (3.17) and the orthogonality of domains of the operators, the equation (6) is reduced to the following three equations.

$$A_5^{0*} A_{300}^0 = 0, \tag{3.18}$$

$$T_2^* A_2 - A_5^{0*} V_5 (I - K^* K)^{1/2} T_2 U_2^* - K_1^* K T_1^* U_2^* = 0, \tag{3.19}$$

$$-A_5^{0*} V_5 K^* V_1^* + K_1^* (I - KK^*) V_1^* = 0. \tag{3.20}$$

According to $\mathscr{D}(V_5) = \overline{\mathscr{R}(I - K^* K)^{1/2} (I - T_2^* T_2)}$, it is easy to verify that the equations (3.19)* and (3.20)* are automatically satisfied, and hence we only need to solve (3.18).

From (3.18), we have

$$\overline{\mathscr{R}(A_{300}^0)} \perp \mathscr{R}(V_5). \tag{3.21}$$

L) Solving (3). We substitute the expressions of A_1, A_2 and A_3 in the equation (3), and restrict (3) in $\mathscr{R}(U_2)$ and $\mathscr{R}(U_2)^\perp$ respectively, thus the equation (3) is divided into two equations. Then, for every equation, we continuously observe the ranges of those operators in the equation, either which belong $\overline{\mathscr{R}(A_2^*)}$ or $\mathscr{R}(A_2^*)^\perp$, thus, the equation (3) is divided further into four equations. But a pair among them is conjugate, hence the equation (3) essentially is divided into the following three equations

$$I_{\overline{\mathscr{R}(A_2^*)}} - A_1^* V (R^2 - I)^{-1/2} (R^2 - I)^{-1/2} V A_1 - A_2^* A_2 - A_{30}'^* A_{30}^1 = 0, \tag{3.22}$$

$$-A_1^{0*} V (R^2 - I)^{-1/2} (R^2 - I)^{-1/2} V^* A_1 - (A_{00}^1 + A_{00}^0)^* A_{30}' = 0, \tag{3.23}$$

$$I_{\overline{\mathscr{R}(A_2^*)^\perp}} - A_1^0 V (R^2 - I)^{-1/2} (R^2 - I)^{-1/2} V^* A_1^0 - (A_{300}' + A_{300}^0)^* (A_{300}' + A_{300}^0) = 0. \tag{3.24}$$

It is easily checked that (3.22) and (3.23) automatically hold, thus, we only need to solve the equation (3.24). Owing to

$$I_{\mathscr{R}(V_1)}-A_1^{0*}V(R^2-I)^{-1/2}(R^2-I)^{-1/2}V^*A_1^0-A_{300}^{1*}A_{300}^1=0,$$

the equation (3.24) is reduced to

$$I_{\mathscr{R}(A_2^*)^\perp\ominus\mathscr{R}(V_1)}-A_{300}'^*A_{300}^0-A_{300}^*A_{300}^0-A_{300}^{0*}A_{300}^0=0, \tag{3.25}$$

but $A'_{300}=-V_5K^*V_1^*$, $A_{300}^{0*}A'_{300}=0$ and $A'^*_{300}A^0_{30}=0$ (see (3.21)), and hence

$$I_{(A_2^*)^\perp\ominus\mathscr{R}(V_1)}=A_{300}^{0*}A_{300}^0. \tag{3.26}$$

And since A_{300}^{0*} is an isometric operator (see step j)), A_{300}^0 must be a unitary operator from $\mathscr{R}(A_2^*)^\perp\ominus\mathscr{R}(V_1)$ onto $\mathscr{R}(A_4)^\perp\ominus\mathscr{R}(V_5)$.

Thus, the all conclusions of Theroem 3.3 have been proved.

For the general regular contraction, we have the following theorem.

Theroem 3.4. *Suppose that $T=\{T_{H_-}, T_1, T_2\}$ is a regular contraction. Let $\tilde{U}$ be arbitrary unitary operator with $\tilde{U}TH_-=H_-$. Then the general form of the udilation (U, H_1, H_2) of T in the Halmos's sense is $(U'^{-1}U_1, H_1, H_2)$, where (U_1, H_1, H_2) is the u-dilation of $\tilde{U}T$ in the Halmos's sense, and $U'=\tilde{U}\oplus I_{H_2}$.*

Now we discuss the u-dilation of a regular contraction in the Nagy's sense. For convenience, we rewrite the form of U as follows

$$U=\begin{pmatrix} T & B_1 \\ B_2 & A_3 \end{pmatrix} \quad \left(\text{where } B_1=\begin{pmatrix} A_1 \\ A_2 \end{pmatrix},\ B_2=(A_4\ A_5)\right).$$

Theroem 3.5. *Suppose that $T=\{T_{H_-}, T_1, T_2\}$ is a regular contraction. Let $\tilde{U}$ be arbitrary unitary operator with $\tilde{U}TH_-=H_-$. Then the general form of the u-dilation of T in the Nagy's sense is*

$$U=\begin{pmatrix} U & B_1 \\ B_2 & A_3 \end{pmatrix}, \tag{3.27}$$

where

$$B_1=\tilde{U}^{-1}\begin{pmatrix} A_1 \\ B_2 \end{pmatrix},\quad B_2=(A_4\ A_5),$$

and the forms of $A_i(i=1, 2, \cdots, 5)$ are the same as in Theorem 3.4, which are given for the regular contraction $\tilde{U}T$. And $H_1(=H_2)$ must be expanded

$$H_1=\left[\sum_{k=1}^{\infty}\oplus W^{k*}(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1))\right]\oplus[\mathscr{R}(U_2)\oplus\mathscr{R}(V_1)]\oplus[\mathscr{R}(U_4)\oplus\mathscr{R}(V_5)]$$
$$\oplus\left[\sum_{k=1}^{\infty}\oplus W^{k}(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))\right]\oplus H,$$

and W is exactly a unitary operator on H.

In particular, when $H=\{0\}$, (U, H_1, H_2) is the minimal u-dilation of T in the Nagy's sense.

Proof Obuiously, the equality

$$T^2=P^1U^2/P^1$$

is equivalent to

$$B_1B_2=0. \tag{3.28}$$

Since $\mathscr{D}(A_1)=\mathscr{D}(A_2)=\mathscr{D}(A_3)=H_1$, $\mathscr{D}(A_4)=H_-\supset\mathscr{R}(A_1)$, $\mathscr{D}(A_5)=H_+\supset\mathscr{R}(A_2)$, $H_1 \supset\mathscr{R}(A_i)\,(i=3, 4, 5)$ and $\tilde{U}$ is a unitary operator on Π, it can be seen, the equation (3.28) is equivalent to

$$A_1A_4=A_2A_4=A_1A_5=A_2A_5=0.$$

Thus, $\mathscr{R}(V_1)$, $\mathscr{R}(V_5)$, $\mathscr{R}(U_4)$ and $\mathscr{R}(U_2)$ must be orthogonal to each other, i.e.

$$H_1=(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))\oplus(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1))\oplus H_2.$$

Similarly, provided that (3.28) holds, the equality

$$T^3=P^1U^3/P^1$$

is equivalent to

$$B_1WB_2=0. \tag{3.29}$$

Obviously, (3.29) is equivalent to $A_i\,W\,A_j=0$, $i=1, 2$, $j=4, 5$, i. e. the mapping W must be of the following form

$$H_1=(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1))\oplus(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))\oplus H_2,$$

$$H_1=(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))\oplus W((\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))\oplus(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1)\oplus H_3),$$

$$H_2\xrightarrow{W}\mathscr{R}(U_2)\oplus\mathscr{R}(V_1)\oplus H_3.$$

By the induction, we have $A_iWA_j=0$, $i=1, 2, j=4, 5$, $n=2, 3, \cdots$, thus

$$H_1=(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1))\oplus\Big[\sum_{k=0}^{\infty}\oplus W^k(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5)\Big]\oplus H',$$

$$H_1=(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1))\oplus\Big[\sum_{k=0}^{\infty}\oplus W^{k+1}(\mathscr{R}(U_4)\oplus\mathscr{R}(V_5))\Big]$$
$$\oplus(\mathscr{R}(U_2)\oplus\mathscr{R}(V_1)\oplus H'),$$

$$H'\xrightarrow{W}\mathscr{R}(U_2)\oplus\mathscr{R}(V_1)\oplus H'.$$

If we consider the equations $(A_iW^nA_j)^*=0\,(n=1, 2, \cdots, i=1, 2, j=4, 5)$, then it is not difficult to prove the conclusion of Theroem 3.5.

§ 4. Contraction with the U-Dilation

In this section we shall give some important applications of Theorem 3.4.

Theorem 4.1. *If T is a linear bounded operator on Π, then T has a u-dilation, iff T is a regular contraction.*

Proof The sufficiency has been proved in Theorem 3.4, we have to prove the necessity.

If (U, H_1, H_2) is a u-dilation of T, then for any $x\in\Pi$,

$$(Tx, Tx)=(P^2Ux, P^2Ux)\leqslant(Ux, Ux)=(x, x),$$

and hence T is a contraction on Π. We prove the regularityof T as follows.

Because $\Pi\oplus H_1=H_-\oplus(H_+\oplus H_1)$ is a regular decomposition of $\Pi\oplus H_1$, henceforth, $\Pi\oplus H_2=UH_-\oplus U(H_+\oplus H_1)$ is also a regular decomposition of $\Pi\oplus H_2$, thus, UH_- is a closed maximal negative subspace of $\Pi\oplus H_2$. By Lemma 1.2, there exists a contraction A from H_- into $H_+\oplus H_2$, such that $\mathscr{D}(A)=H_-$, $\|A\|\leqslant\alpha<1$ and $UH_-=L_A$. Let P_+ and P_2 be the projections from $H_+\oplus H_2$ onto H_+ and H_2 respectively, obvi ously, we have $A=A_++A_2$, where $A_+=P_+A$, $A_2=P_2A$. Since $\mathscr{R}(A_+)\perp\mathscr{R}(A_2)$, so

$$\max(\|A_+\|, \|A_2\|)\leqslant\alpha<1 \tag{3.30}$$

From (3.30) and the equality $(P^2=I-P_2)$

$$TH_-=P^2UH_-=P^2L_A=\{\{x_-, P^2Ax_-\}\,|\,x_-\in H_-\}$$
$$=\{\{x_-, A_+x_-\}\,|\,x_-\in H_-\}=L_{A_+},$$

we immediately obtain that $\Pi=(TH_-)\oplus(TH_-)^{\perp}$ is a regular decomposition, in other words, T is a regular contraction.

Theorem 4.2. *T is a regular contraction iff $T^\dagger$ is also a regular contraction.*

Proof Since $(T^\dagger)^\dagger=T$, so that we only need to prove that $T^\dagger$ is also a regular contraction provided that T is a regular contraction.

When T is a regular contraction, by Theorem 3.4, T has a u-dilation (U, H_1, H_2), i.e.

$$T=P^2U|_{P^1}$$

herefore

$$T^\dagger=P^1U^\dagger|_{P^2},$$

i.e. $(U^\dagger, H_2, H_1)$ is a u-dilation of $T^\dagger$, so that $T^\dagger$ is also a regular contraction.

Corollarx 4.3. *T is a contraction on Π_k $(k<\infty)$ iff $T^\dagger$ is also a contraction.*

References

[1] Halmos, P. R., Normal dilations and extensions of operators, *Summa Brasil, Math.*, **2** (1950).

[2] Sz-Nagy, B., Sur les contractions de l'espace de Hilbert, *Acta. Sci. Math.*, **15** (1953).

[3] Sz-Nagy, B., Foias, c. Harmonic analysis of operatorson Hilbert space, North-Holland Amesferdan (1970).

[4] Bogna'r, J., Indefinite inner product spaces, Springer-Verlag (1974).