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MULTI-DIMENSIONAL Q-PROCESSES*

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Abstract

In this paper, the authors propose a method which reduces the multi-dimentional problem to one-dimensional ones. By keeping the idea in mind, some sufficient conditions which are much more practical for the uniqueness, recurrency and ergodicity of multi-dimensional Q-processes are obtained.

The conditions are effective not only for the models in non-equilibrium systems, but also for their couplings and others.

§ 1. Introduction

Some stochastic models for linear Master equations of several variables have been introduced in the studies of non-equilibrium systems[1,2,3]. In probability language, the models correspond to some Q-processes which satisfy the forward Kolmogorov equation. Thus, one would like to know the uniqueness, the recurrency, and the ergodicity for the Q-processes. It is known that there are some general results about the above problems (cf. [4]). But these results are not effective to those models studied in [3]. As we know, there is only one paper[5] which studies directly the properties mentioned above for two-dimensional Q-processes. In this paper, we will propose a method which reduces the multi-dimensional problems to one-dimensional ones. By keeping the idea in mind, we obtain some sufficient conditions which are much more practical for the uniqueness, recurrency and ergodicity of multi-dimensional Q-processes. These conditions are effective not only for the models in [3], but also for their couplings and others.

Now, we are going to state the main results in this paper.

Let E be a countable set and $(q(\eta, \zeta): \eta, \zeta \in E)$ be a Q-matrix on $E \times E$. Throughout the paper, we will assume that the Q-matrices are totally stable and conservative. Let $\{E_k \subset E: E_k \neq \phi, k \geqslant 0\}$ be a disjoint countable partion of E, and put

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$$q_{kj}\equiv\begin{cases}\sup\{\sum_{\zeta\in E_j}q(\eta,\zeta):\eta\in E_k\},\ j>k,\\ \inf\{\sum_{\zeta\in E_j}q(\eta,\zeta):\eta\in E_k\},\ j<k.\end{cases}\tag{1.1}$$

We say that a Q-matrix is regular if it determines at most one Q-process.

The following two results are on the uniqueness for Q-processes.

Theorem 1. *Suppose that* $(q(\eta,\zeta):\eta,\ \zeta\in E)$ *satisfies the following two conditions:*

$$q(\eta,\zeta)>0,\ \eta\in E_k\Rightarrow\zeta\in\bigcup_{l=0}^{k+1}E_l,\tag{1.2}$$

$$C_k=\sup\{q(\eta):\eta\in E_k\}<\infty,\quad k\geqslant 0.\tag{1.3}$$

Then, $(q(n,\zeta))$ *is regular if so is* (q_{ij}).

Theorem 2. *For each* $k\in Z_+\equiv\{0,1,2,\cdots\}$, *let* B_k *be a non-empty subset of* E_k. *Suppose that* $(q(\eta,\zeta))$ *satisfies* (1.2), (1.3) *and the following conditions:*

$$\eta\in E_k,\ \zeta\in E_{k+1},\ q(\eta,\zeta)>0\Rightarrow\zeta\in B_{k+1},\tag{1.4}$$

$$k\in Z_+,\ \eta\in E_k\backslash B_k,\ q(\eta,\zeta)>0\Rightarrow\zeta\in B_k\cup B_{k+1}.\tag{1.5}$$

Define

$$\tilde q_{kj}=\begin{cases}\inf\Big\{\sum_{\zeta\in B_j}\Big[q(\eta,\zeta)+\sum_{\zeta\in E_j\backslash B_j}\frac{q(\eta,\zeta)q(\xi,\zeta)}{q(\zeta)}\Big]:\ \eta\in B_k\Big\},\ j<k,\\ \sup\Big\{\sum_{\zeta\in B_{k+1}}\Big[q(\eta,\zeta)+\sum_{\xi\in E_k\backslash B_k}\frac{q(\eta,\xi)q(\xi,\zeta)}{q(\xi)}\Big]:\ \eta\in B_k\Big\},\ j=k+1,\\ 0,\qquad j>k+1.\end{cases}\tag{1.6}$$

Then, $(q(n,\zeta))$ *is regular if so is* $(\tilde q_{kj})$.

It is clear that Theorem 2 reduces to Theorem 1 in the case of $B_k=E_k(k\in Z_+)$.

Let S be a finite or countable set, and set $X=Z_+^S$. For $\eta=\{\eta_u:\ u\in S\}\in X$, we put

$$|\eta|=\sum_{u\in S}\eta_u,\ X_0=\{\eta\in X:\ |\eta|<\infty\}.$$

Clearly, $X_0=X$ if S is finite. We use θ to denote the element in X_0 so that $|\theta|=0$.

If $E=X_0$ in Theorems 1 and 2, we see that $(q(\eta,\zeta))$ is a multi-dimensional (even infinite dimensional) Q-matrix. What the above theorems mean is reducing the uniqueness problem in multi-dimensions to the one in one-dimension by choosing an appropriate partition $\{E_k\}_0^\infty$ of E. This idea is very useful since the Q-matrix (q_{ij}) $((\tilde q_{ij}))$ in (1.1) ((1.6)) is a generalized birth-death Q-matrix, for which we have the following uniqueness criterion.

Theorem 3.

(i) *Suppose that*

$$q_{kj}=0,\ j>k+1;\ q_{k,k+1}>0,\ k,\ j\in Z_+.\tag{1.7}$$

Then (q_{kj}) *is regular iff*

$$R\equiv\sum_{k=0}^\infty m_k=\infty,\tag{1.8}$$

where

$$\begin{cases} m_k = \sum_{i=0}^{k} F_k^{(i)}/q_{i,i+1}, \ k\in Z_+, \\ F_k^{(k)}=1, \ k\in Z_+, \\ F_k^{(i)} = q_{k,k+1}^{-1} \sum_{j=i}^{k-1} q_k^{(j)} F_j^{(i)}, \ 0\leqslant i<k, \\ q_k^{(i)} = \sum_{j=0}^{i} q_{kj}, \ 0\leqslant i<k. \end{cases} \tag{1.9}$$

(ii) *Suppose that* (1.7) *holds except* $q_{01}=0$. *Then* (q_{kj}) *is regular iff*

$$R'=\infty, \tag{1.10}$$

where R' *can be obtained from* (1.8) *and* (1.9) *when* 0 *is replaced by* 1 *and* Z_+ *is replaced by* $Z_+\setminus\{0\}$.

The above three theorems will be proved in § 2. Theorem 3 is an extension of [6, § 3, Corollary 1], for which our proof is much more simple.

It is clear that both R and R' are computable, so they are very convenient in the practice. As their applications, in § 3, we will discuss the uniqueness for the following models[3]:

An autocatalytic production of a chemical

$$X: A+X \xrightarrow{\lambda_1} 2X, \quad X+X \xrightarrow{\lambda_2} B.$$

Its Q-matrix is

$$q(\eta,\zeta)=\begin{cases} \lambda_1 a_u \eta_u, \ \zeta=\eta+e_u, \\ \lambda_2 \binom{\eta_u}{2}, \ \zeta=\eta-2e_u, \\ \eta_u p(u,v), \ \zeta=\eta-e_u+e_v, \ u\neq v, \\ 0, \ \text{other } \zeta\neq\eta, \end{cases} \quad \eta,\zeta\in X_0, \tag{1.11}$$

where S is the set of seats, $u, v\in S$, $e_u=\{\delta_{uv}: v\in S\}$, a_u and $\eta_u (u\in S)$ are the numbers of A-particles and X-particles, respectively, $p(u,v)$ is the transition rate of an X-particle from u to v. We will assume that

$$\sum_v p(u,v)\leqslant C<\infty, \ u\in S.$$

Schlögl model:

$$A+2X \underset{\lambda_2}{\overset{\lambda_1}{\rightleftarrows}} 3X, \quad X \underset{\lambda_4}{\overset{\lambda_3}{\rightleftarrows}} B.$$

Its Q-matrix is

$$q(\eta,\zeta)=\begin{cases} \lambda_1 a_u \binom{\eta_u}{2} + \lambda_4 b_u, \ \zeta=\eta+e_u, \\ \lambda_2 \binom{\eta_u}{3} + \lambda_3 \eta_u, \ \zeta=\eta-e_u, \\ \eta_u p(u,v), \ \zeta=\eta-e_u+e_v, \ u\neq v, \\ 0, \ \text{other } \zeta\neq\eta. \end{cases} \tag{1.12}$$

Lotka-Voterra model:

$$A+X_1 \xrightarrow{\lambda_1} 2X_1, \quad X_1+X_2 \xrightarrow{\lambda_2} 2X_2. \ B+X_2 \xrightarrow{\lambda_3} D+B.$$

Its state space should be $X=(Z_+^2)^S$. For each $\eta\in X$, $n=\{\eta_{ui}: u\in S, i=1, 2\}$, set $|\eta|=\sum_{u\in S}\sum_{i=1}^{2} n_{ui}$. Then $X_0=\{n\in X: |n|<\infty\}$. n_{ui} denotes the numbers of X_i-particles in u, and we use $p_i(u, v)$ to denote the transition rate of an X_i-particle from u to v. Now, the Q-matrix for the model can be written as follows:

$$q(\eta, \zeta)=\begin{cases} \lambda_1 a_u \eta_{u1}, & \zeta=\eta+e_{u1}, \\ \lambda_3 b_u \eta_{u2}, & \zeta=\eta-e_{u2}, \\ \lambda_2 \eta_{u1}\eta_{u2}, & \zeta=\eta-e_{u1}+e_{u2}, \\ \eta_{ui} p_i(u, v), & \zeta=\eta-e_{ui}+e_{vi}, \ i=1, 2, \ u\neq v, \\ 0, & \text{other } \zeta\neq\eta, \end{cases} \tag{1.13}$$

where $e_{ui}=\{\delta_{uv}\delta_{ij}: v\in S, j=1, 2\}$.

Brusselator:

$$A \xrightarrow{\lambda_1} X_1, \quad B+X_1 \xrightarrow{\lambda_2} X_2+D$$

$$2X_1+X_2 \xrightarrow{\lambda_3} 3X_1, \quad X_1 \xrightarrow{\lambda_4} C.$$

Again, the state space is $X=(Z_+^2)^S$. Its Q-matrix is

$$q(\eta, \zeta)=\begin{cases} \lambda_1 a_u, & \zeta=\eta+e_{u1}; \\ \lambda_4 \eta_{u1}, & \zeta=\eta-e_{u1}; \\ \lambda_2 b_u \eta_{u1}, & \zeta=\eta-e_{u1}+e_{u2}, \\ \lambda_3 \binom{\eta_{u1}}{2} \eta_{u2}, & \zeta=\eta-e_{u2}+e_{u1}, \\ \eta_{ui} p_i(u, v), & \zeta=\eta-e_{ui}+e_{vi}, \ i=1, 2, \ u\neq v, \\ 0, & \text{other } \zeta\neq\eta. \end{cases} \tag{1.14}$$

The point of applying our results to the above models is simply choosing

$$E_k=\{\eta\in X_0: |\eta|=k\}.$$

As another application of Theoerm 1 and Theorem 3, we will discuss the uniqueness for coupling Q-processes. To state this, notice that there is a one-to-one mapping between Q-matrix $(q(\eta, \zeta))$ and the following operator:

$$\Omega f(\eta)=\sum_{\eta\neq\zeta\in X_0} q(\eta, \zeta)(f(\zeta)-f(\eta)), \ \eta\in X_0, \tag{1.15}$$

where f is a bounded real function on X_0. So we can also use the term "Ω-process" instead of Q-process.

Theorem 5. *Let S be finite, $X=Z_+^S$ and*

$$(\Omega f)(\eta)=\sum_u \beta(u, \eta_u)(f(\eta+e_u)-f(\eta))+\sum_u \delta(u, \eta_u)(f(\eta-e_u)-f(\eta))$$
$$+\sum_{u,v}\gamma(u, v, \eta_u, \eta_v)(f(\eta-e_u+e_v)-f(\eta)), \tag{1.16}$$

where β, δ and γ are non-negative and $\delta(u, 0)=0$. Suppose that $\beta(u, k)$ and $\gamma(u, v, k, l)$ are increasing in k and l, and define the coupling operetor $\tilde{\Omega}$ as follows:

$$\begin{aligned}\tilde{\Omega} f(\eta, \zeta)= \sum_{u:\eta_u\neq\zeta_u} \{&\beta(u, \eta_u)[f(\eta+e_u, \zeta)-f(\eta, \zeta)]+\delta(u, \eta_u)[f(\eta-e_u, \zeta)-f(\eta, \zeta)]\\ &+\beta(u, \zeta_u)[f(\eta, \zeta+e_u)-f(\eta, \zeta)]+\delta(u, \zeta_u)[f(\eta, \zeta-e_u)-f(\eta, \zeta)]\\ &+\sum_{u:\zeta_u=\eta_u}\{\beta(u, \eta_u)\wedge\beta(u, \zeta_u)[f(\eta+e_u, \zeta+e_u)-f(\eta, \zeta)]\\ &+\delta(u, \eta_u)\wedge\delta(u, \zeta_u)[f(\eta-e_u, \zeta-e_u)-f(\eta, \zeta)]\}\\ &+\sum_{u,v}(\gamma(u, v, \eta_u, \eta_v)-\gamma(u, v, \zeta_u, \zeta_v))^+\\ &\cdot[f(\eta-e_u+e_v\cdot\zeta)-f(\eta\cdot\zeta)]\\ &+\sum_{u,v}(\gamma(u, v, \zeta_u, \zeta_v)-\gamma(u, v, \eta_u, \eta_v))^+\\ &\cdot[f(\eta, \zeta-e_u+e_v)-f(\eta, \zeta)]\\ &+\sum_{u,v}\gamma(u, v, \eta_u, \eta_v)\wedge\gamma(u, v, \zeta_u, \zeta_v)\\ &\cdot[f(\eta-e_u+e_v, \zeta-e_u+e_v)-f(\eta, \zeta)],\end{aligned}$$

where f is a bounded real function on $X\times X$. Set

$$\begin{cases} r_k=\max\{\sum_u \beta(u, \eta_u): |\eta|=k\},\\ s_k=\min\{\sum_u \delta(u, \eta_u): |\eta|=k\}, k\in Z_+,\\ \bar{R}=\begin{cases}\sum_{k=1}^{\infty}\left[\frac{1}{2r_k}+\cdots+\frac{s_k\cdots s_2}{2^k r_k\cdots r_1}\right], & r_0=0,\ r_k>0,\ k\geqslant1,\\ \sum_{k=0}^{\infty}\left[\frac{1}{2r_k}+\cdots+\frac{s_k\cdots s_1}{2^{k+1}r_k\cdots r_0}\right], & r_k>0,\ k\in Z_+.\end{cases}\end{cases} \tag{1.18}$$

If

$$\bar{R}=+\infty, \tag{1.19}$$

then, both Ω-process and $\tilde{\Omega}$-process are unique. Moreover

$$\sum_{\xi\in X}\tilde{p}(t, (\eta, \zeta), (\tilde{\eta}, \tilde{\xi}))=p(t, \eta, \tilde{\eta}) \tag{1.20}$$

$$\sum_{\tilde{\eta}\in X}\tilde{p}(t, (\eta, \zeta), (\tilde{\eta}, \tilde{\zeta}))=p(t, \zeta, \tilde{\zeta}), \tag{1.21}$$

and for each $\eta\leqslant\zeta$ (i.e. $\eta_u\leqslant\zeta_u$, $u\in S$), we have

where $p(t, \eta, \zeta)$ $(\tilde{p}(t, (\eta, \zeta), (\tilde{\eta}, \tilde{\zeta})))$ is the Ω-process ($\tilde{\Omega}$-process).

Of course, the approach used to prove Theorem 5 can be generalized, but we will not do it in this paper. It is easy to check (see § 4) that the Q-matrices defined in (1.12), (1.13) and (1.14) satisfy the assumptions of Theorem 5. Hence, this theorem works for the above models.

The recurrence for Q-processes will be studied in § 5. The main result is the following

Theorem 6. *Let $E=X_0$ and let $(q(\eta,\zeta))$ satisfy (1.2) and (1.3). Let $E_0=\{\theta\}$, E_k $(k\geqslant1)$ be finite. Suppose that the Q-matrices $(q(\eta, \zeta))$ and (q_{ij}) defined by (1.1)* *is irreducible and regular. Then, the $(q(\eta, \zeta))$-process is recurrent if so is the (q_{ij})-process. Moreover, the (q_{ij})-process is recurrent iff*

$$\sum_{k=0}^{\infty} F_k^{(0)} = \infty,$$

where $F_k^{(0)}$ is defined in (1.9).

The positive recurrency and ergodicity for $(q(\eta, \zeta))$-processes will be studied in § 6. The main results are the following

Theorem 7. *Let (q_{ij}) be an arbitrary irreducible and regular Q-matrix on Z_+^2. Then the (q_{ij})-process is positive recurrent iff there exists a non-negative solution $(x_i$: $i \in Z_+)$ to the following inequaties:*

$$\sum_j q_{ij} x_j + 1 \leqslant 0, \quad i \neq i_0, \ |\sum_j q_{i_0 j} x_j| < \infty, \tag{1.22}$$

for some $i_0 \in Z_+$ (equivalently, for any $i_0 \in Z_+$).

Theorem 8. *Let S be finite, and let $(q(\eta, \zeta))$ satisfy (1.2) and (1.3) with $E_0 = \{\theta\}$. Suppose that both $(q(\eta, \zeta))$ and (q_{ij}) defined by (1.1) are irreducible. If (q_{ij}) is regular and for (1.22), there exists a nonnegative solution (u_i) for $i_0 = 0$, which is increasing in i, then $(q(\eta, \zeta))$-process is positive recurrent. In fact, it is ergodic, i.e. there exists a probability measure $\{\mu(\eta)\}$ on X such that*

$$\lim_{t\to\infty} p(t, \xi, \eta) = \mu(\eta), \quad \xi, \eta \in X, \tag{1.23}$$

where $p(t, \xi, \eta)$ is the Q-process corresponding to $(q(\eta, \zeta))$.

Theorem 9. *Let (q_{ij}) be a Q-matrix satisfying (1.7). Then, there exists a non-negative-and increasing solution (x_i) to (1.22) with $i_0 = 0$, iff*

$$d \equiv \sup_{k \in Z_+} d_k / F_k^{(0)} < \infty, \tag{1.24}$$

where $F_k^{(0)}$ is defined by (1.9) and

$$d_0 = 0, \ d_k = q_{k,k+1}^{-1} \Big(1 + \sum_{s=0}^{k-1} q_k^{(s)} d_s\Big), \quad k > 0. \tag{1.25}$$

Moreover, if (1.24) holds, then the function $(u_i$: $i \in Z_+)$ defined by

$$u_0 = 0, \ u_1 \geqslant d, \ u_{k+1} = u_k + F_k^{(0)} u_1 - d_k, \ k \geqslant 1 \tag{1.26}$$

is a non-negative and increasing solution to (1.22) with $i_0 = 0$.

As an application, we will show in § 6 that Schlögl model is ergodic. For the models defined in (1.11) and (1.13), we have nothing to do since θ is an absorbing state for these models. However, for the Brusselator model defined in (1.14), our conditions do not work, this is a remainder problem.

One may ask whether the condition (1.24) is equivalent to the positive recurrence for the (q_{ij})-process or not. The answer is negative. To compare the two properties, we have

Theorem 10. *Let (q_{ij}) be a Q-matrix satisfying (1.7). Then there exists a non-negative solution to (1.22) with $i_0 = 0$ iff*

$$\hat{d}\equiv\sup\left\{\left(\sum_{l=0}^{k}d_l\right)\Big/\left(\sum_{l=0}^{k}F_l^{(0)}\right):\ k\in Z_+\right\}<\infty. \tag{1.27}$$

If the Q-matrix (q_{ij}) is also irreducible and regular, then the condition (1.27) is equivalent to the positive recurrency of the (q_{ij})-process.

It is obvious that $\hat{d}\leqslant d$. For birth-death processes, it is not difficult to show that $d<\infty$ iff $\hat{d}<\infty$.

§ 2. Uniqueness

Lemma 1. *Let (q_{ij}): $i, j\in E)$ be a Q-matrix. Then*

$$(\lambda+q_i)u_i\leqslant\sum_{j\neq i}q_{ij}u_j,\quad 0\leqslant u_i\leqslant 1,\ i\in E,\lambda>0 \tag{2.1}$$

has only zero solution iff

$$(\lambda+q_i)u_i=\sum_{j\neq i}q_{ij}u_j,\quad 0\leqslant u_i\leqslant 1,\ i\in E,\lambda>0 \tag{2.2}$$

has only zero solution.

Proof It is well know that the maximal solution (u_i^*) to (2.2) can be obtained by the following procedure: define

$$u_i^{(0)}\equiv 1,\quad i\in E,$$
$$u_i^{(n+1)}=\sum_{j\neq i}q_{ij}u_j^{(n)}/(\lambda+q_i),\quad n\geqslant 0, i\in E,$$

then

$$u_i^{(n)}\searrow u_i^*\text{ as }n\to\infty\text{ for each }i\in E.$$

Now, suppose that (v_i) is a non-zero solution to (2.1). By induction, it is easy to show that $v_i\leqslant u_i^{(n)}$ for each $n\geqslant 1$ and $i\in E$. Hence $u_i^*\geqslant v_i$ $(i\in E)$. This is a contradiction.

Proof of Theorem 1.

By [8, § 4.3, Corollary 1] or [9, § 5.4, Theorem 1], it suffices to prove that for some $\lambda>0$ (or equivalently, for each $\lambda>0$),

$$(\lambda+q(\eta))u(\eta)=\sum_{\zeta\neq\eta}q(\eta,\zeta)u(\zeta),\quad 0\leqslant u(\zeta)\leqslant 1,\eta\in E \tag{2.3}$$

has only zero solution. Suppose that there exists a non-zero solution $\{u(\eta):\ \eta\in E\}$ for some $\lambda>0$. Set

$$u_k=\sup\{u(\eta):\ \eta\in E_k\},\quad k\in Z_+. \tag{2.4}$$

Then $(u_k$: $k\in Z_+)$ is non-zero. For each $k\in Z_+$, choose $\varepsilon_k>0$ and $\eta^{(k)}\in E_k$ so that

$$\varepsilon_k(\lambda+c_k)<\frac{\lambda}{2}\text{ and }u(\eta^{(k)})\geqslant(1-\varepsilon_k)u_k. \tag{2.5}$$

Replacing η in (2.3) with $\eta^{(k)}$, it follows from (1.2), (1.3), (2.4) and (2.5) that

$$\frac{\lambda}{2}u_k+\left[\sum_{j=0}^{k-1}\sum_{\zeta\in E_j}q(\eta^{(k)},\zeta)+\sum_{\eta^{(k)}\neq\zeta\in E_k}q(\eta^{(k)},\zeta)+\sum_{\zeta\in E_{k+1}}q(\eta^{(k)},\zeta)\right]u_k$$
$$\leqslant u_k(\lambda-\varepsilon_k(\lambda+c_k))+q(\eta^{(k)})u_k\leqslant(\lambda+q(\eta^{(k)}))(1-\varepsilon_k)u_k$$

$$\leqslant(\lambda+q(\eta^{(k)}))u(\eta^{(k)})=\sum_{\zeta\neq\eta}q(\eta^{(k)},\zeta)u(\zeta)$$

$$\leqslant\sum_{j=0}^{k-1}\sum_{\zeta\in E_j}q(\eta^{(k)},\zeta)u_j+\sum_{\eta^{(k)}\neq\zeta\in E_k}q(\eta^{(k)},\zeta)u_k+\sum_{\zeta\in E_{k+1}}q(\eta^{(k)},\zeta)u_{k+1},$$

i.e.,

$$\frac{\lambda}{2}u_k+\sum_{j=0}^{k-1}\Big(\sum_{\zeta\in E_j}q(\eta^{(k)},\zeta)\Big)(u_k-u_j)\leqslant\sum_{\zeta\in E_{k+1}}q(\eta^{(k)},\zeta)(u_{k+1}-u_k). \tag{2.6}$$

Clearly, u_k is increasing. By (1.1), we now get

$$\frac{\lambda}{2}u_k+\sum_{j=0}^{k-1}q_{kj}(u_k-u_j)\leqslant q_{k,k+1}(u_{k+1}-u_k),\ k\in Z_+. \tag{2.7}$$

By (1.1) and (1.2), this means that

$$\Big(\frac{\lambda}{2}+q_k\Big)u_k\leqslant\sum_{j\neq k}q_{kj}u_j,\quad 0\leqslant u_k\leqslant 1,\ k\in Z_+. \tag{2.8}$$

But this is impossible by Lemma 1 and the regularity for (q_{ij}).

Proof of Theorem 2. The proof is similar to the proof of Theorem 1. We leave it to the reader as an exercise.

Proof of Theorem 3. (1). Suppose that (q_{ij}) satisfies (1.7). Then there exists only one increasing solution to

$$(\lambda+q_i)u_i=\sum_{j\neq i}q_{ij}u_j,\quad u_0=1,\ i\in Z_+ \tag{2.9}$$

for each $\lambda>0$. In fact, we have

$$u_0=1, \tag{2.10}$$

$$u_{i+1}=\Big[(\lambda+q_i)u_i-\sum_{j=0}^{i-1}q_{ij}u_j\Big]\Big/q_{i,i+1},\ i\geqslant 0,$$

$$u_{i+1}-u_i=\Big[\sum_{j=0}^{i-1}q_{ij}(u_i-u_j)+\lambda u_i\Big]\Big/q_{i,i+1},\ i\geqslant 0. \tag{2.11}$$

We now prove that

$$\lambda u_0m_k\leqslant u_{k+1}-u_k\leqslant(u_1-u_0)F_k^{(0)}+\lambda u_km_k,\ k\in Z_+. \tag{2.12}$$

When $k=1$, (2.12) holds since

$$u_1-u_0=\lambda u_0/q_{01}=\lambda u_0m_0.$$

Suppose that it holds for $k<n$. Then, by using

$$\sum_{i=0}^{k-1}q_{ki}(u_k-u_i)=\sum_{i=0}^{k-1}q_k^{(i)}(u_{i+1}-u_i),\ k\in Z_+, \tag{2.13}$$

(2.11), (1.9) and the increasing property of u_k, it follows that

$$u_{n+1}-u_n=\frac{1}{q_{n,n+1}}\Big[\sum_{k=0}^{n-1}q_n^{(k)}(u_{k+1}-u_k)+\lambda u_n\Big]$$

$$\leqslant q_{n,n+1}^{-1}\Big[(u_1-u_0)\sum_{k=0}^{n-1}q_n^{(k)}F_k^{(0)}+\lambda u_n\Big(\sum_{k=0}^{n-1}q_n^{(k)}m_k+1\Big)\Big]$$

$$=(u_1-u_0)F_n^{(0)}+\lambda u_n\Big[\sum_{j=0}^{n-1}q_{j,j+1}^{-1}q_{n,n+1}^{-1}\sum_{k=j}^{n-1}q_n^{(k)}F_k^{(j)}+q_{n,n+1}^{-1}F_n^{(n)}\Big]$$

$$=(u_1-u_0)F_n^{(0)}+\lambda u_nm_n,$$

$$u_{n+1}-u_n\geqslant q_{n,n+1}^{-1}\Big[\sum_{k=0}^{n-1}\lambda u_0q_n^{(k)}m_k+\lambda u_n\Big]\geqslant\lambda u_0q_{n,n+1}^{-1}\Big(\sum_{k=0}^{n-1}q_n^{(k)}m_k+1\Big)=\lambda u_0m_n.$$

By induction, this proves (2.12).

(2). To prove (i) of Theorem 3, it is enough to show that (2.2) has only zero solution iff (1.8) holds. Suppose that $R<\infty$ and (u_i) is the solution to (2.9) constructed by (2.10). By (2.12) and (1.9), we get

$$u_{k+1}u_k^{-1}-1\leqslant(u_1-u_0)u_k^{-1}F_k^{(0)}+\lambda m_k\leqslant[\lambda+(u_1-u_0)q_{01}]m_k.$$

Hence $u_{k+1}u_k^{-1}-1<1/2$, and so $\log(u_{k+1}u_k^{-1})\leqslant 2(u_{k+1}u_k^{-1}-1)$ for k large enough. Therefore, there exists a constant $C>0$ such that

$$\log u_k=\sum_{i=0}^{k-1}\log(u_{i+1}u_i^{-1})\leqslant CR<\infty.$$

This implies that

$$u_\infty=\lim_{k\to\infty}u_k<\infty.$$

Now, $\hat{u}_k\equiv u_k u_\infty^{-1}\in[0,\ 1]$ is a non-zero solution to (2.2). Conversely, if (2.2) has a non-zero solution, it is easy to show that $R<\infty$.

As for (ii) of Theorem 3, it is enough to notice that (2.2) has only zero solution iff

$$(\lambda+q_k)u_k=\sum_{j\neq 0,k}q_{kj}u_j,\quad 0\leqslant u_k\leqslant 1,\ k\geqslant 1$$

has only zero solution in the case of $q_{01}=0$.

§ 3. Uniqueness for Some Processes of Non-equilibrium Systems

As applications of the results in § 1, we will prove in this section that the Q-matrices defined by (1.11)—(1.14) are regular, i. e. they determine uniquely the Q-processes.

Theorem 4.

(i) *The Q-matrix* $(q(\eta,\zeta):\eta,\zeta\in X_0)$ *defined in* (1.11) *is regular if*

$$a\equiv\sup\{a_u:u\in S\}<\infty;\tag{3.1}$$

(ii) *Let S be finite, then the Q-matrix* $(q(\eta,\zeta):\eta,\ \zeta\in X)$ *defined by* (1.12) *is regular;*

(iii) *The Q-matrix* $(q(\eta,\zeta):\eta,\ \zeta\in X_0)$ *defined by* (1.13) *is regular if* (3.1) *holds;*

(iv) *The Q-matrix* $(q(\eta,\zeta):\eta,\zeta\in X_0)$ *defined by* (1.14) *is regular if*

$$\tilde{a}\equiv\sum_{u\in S}a_u<\infty.\tag{3.2}$$

Proof Take $E_k=\{\eta\in X_0:\ |\eta|=k\}$, $k\in Z_+$ in the four cases. Then (1.2) and (1.3) hold.

For (i) and (iii), the conditions of Theorem 3. (ii) hold, and

$$q_{k,k+1}\leqslant\lambda_1 ak.$$

Since

$$R'=\sum_{k=1}^{\infty}\Big(\sum_{i=1}^{\infty}F_k^{(i)}/q_{i,i+1}\Big)\geqslant\sum_{k=1}^{\infty}F_k^{(k)}q_{k,k+1}^{-1}\geqslant\sum_{k=1}^{\infty}(\lambda_1ak)^{-1}=\infty,$$

now, (i) and (iii) follow from Theorem 3. (ii).

The Q-matrices (q_{ij}) corresponding to (1.12) and (1.14) are birth-death Q-matrices. By (1.8) and (1.9), it is easy to carry out that

$$R=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{s_k\cdots s_{i+1}}{r_k\cdots r_{i+1}r_i},\tag{3.3}$$

$$s_k\equiv q_{k,k-1},\ r_k=q_{k,k+1}.$$

For (ii), we have

$$r_k=\sup\Big\{\sum_{u\in S}\Big(\lambda_1a_u\binom{\eta_u}{2}+\lambda_4b_u\Big):|\eta|=k\Big\}\leqslant2^{-1}\lambda_1a(k^2-k)+\lambda_4\sum_{u\in S}b_u.\tag{3.4}$$

Here we have used

$$\sup\{\sum_{u\in S}\eta_u^2:\ |\eta|=k\}=k^2.\tag{3.5}$$

By (3.5) and

$$(\sum_u\eta_u/|S|)^3\leqslant\sum_u\eta_u^3/|S|$$

we see that

$$\begin{aligned}s_k&=\inf\Big\{\sum_{u\in S}\Big[\lambda_2\binom{\eta_u}{3}+\lambda_3\eta_u\Big]:\ |\eta|=k\Big\}\\&\geqslant6^{-1}\lambda_2[\inf_{|\eta|=k}\sum_{u\in S}\eta_u^3-3\sup_{|\eta|=k}\sum_{u\in S}\eta_u^2]+(\lambda_3+\lambda_2/3)k\\&>\frac{\lambda_2}{6}\Big[|S|\Big(\frac{k}{|S|}\Big)^3-3k^2\Big]+(\lambda_3+\lambda_2/3)k.\end{aligned}$$

Now, by (3.4) and (3.6), we get

$$\varliminf_{k\to\infty}\sum_{i=0}^{k}\frac{s_k\cdots s_{i+1}}{r_k\cdots r_{i+1}r_i}\geqslant\varliminf_{k\to\infty}\frac{s_ks_{k-1}}{r_kr_{k-1}r_{k-2}}\geqslant2\lambda_1^2/(9|S|^4(\lambda_1a)^3).$$

It follows from (3.3) that $R=\infty$. Therefore (ii) follows from Theorem 3. (i).

For (iv), we have

$$r_k=\sup\{\sum_u\lambda_1a_u:\ |\eta|=k\}=\lambda_1\tilde a,\ k\in Z_+.$$

By (3.3), we get

$$R\geqslant\sum_{k=0}^{\infty}\frac{1}{r_k}=\infty.$$

Therefore (iv) follows also from Theorem 3. (i).

To conlude this section, we discuss two special cases.

Proposition 1. *Let $(\bar q_{ij}:\ i,\ j\in Z_+)$ be a Q-matrix satisfying*

$$\bar q_{ij}=0,\ |i-j|>2\tag{3.6}$$

and

$$\begin{cases}r_k\equiv\max(\bar q_{2k,2k+2},\ \bar q_{2k+1,2k+2}+\bar q_{2k+1,2k+3})>0,\\ s_k\equiv\min(\bar q_{2k,2k-2}+\bar q_{2k,2k-1},\ \bar q_{2k+1,2k-1})>0,\end{cases}\tag{3.7}$$

$$\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{s_k\cdots s_{i+1}}{r_k\cdots r_{i+1}r_i}=\infty. \tag{3.8}$$

Then $(\bar{q}_{ij})$ *is regular.*

Proof Simply take $E_k=\{2k, 2k+1\}$, $k\in Z_+$ and apply Theorems 1 and 3.

Proposition 2. *Let* $(\bar{q}_{ij}: i, j\in Z_+)$ *be a Q-matrix satisfying*

$$\bar{q}_{ij}=0,\ i, j\geq n \text{ and } |i-j|>1, \tag{3.9}$$
$$\bar{q}_{n+k,n+k+1}>0,\ k\in Z_+,$$

where n is a given positive integer. Set $r_k=\bar{q}_{n+k,n+k+1}$, $k\geq 0$ *and* $s_k=\bar{q}_{n+k,n+k-1}$, $k\geq 1$ *Then* $(\bar{q}_{ij})$ *is regular if* (3.8) *holds.*

Proof Simply take $E_0=\{0, 1, 2, \cdots, n\}$, $E_k=\{n+k\}$, $k\geq 1$ and apply Theorems 1 and 3.

§ 4. Further Applications

In order to prove Theorem 5, we introduce a relation "$\to$" on $X\times X$ as follows:

$$(\eta,\zeta)\longrightarrow\begin{cases}(\eta+e_u,\zeta)\text{ or }(\eta,\zeta+e_u),\ \eta_u\neq\zeta_u\\ (\eta-e_u,\zeta),\ \eta_u\neq\zeta_u,\ \eta_u\geq 1,\\ (\eta,\zeta-e_u),\ \eta_u\neq\zeta_u,\ \zeta_u\geq 1,\\ (\eta+e_u,\zeta+e_u),\ \eta_u=\zeta_u,\\ (\eta-e_u,\zeta-e_u),\ \eta_u=\zeta_u\geq 1\end{cases} \tag{4.1}$$

for each $u\in S$. Then, define

$$\begin{cases}\tilde{B}_0=\{(\theta,\theta)\},\\ \tilde{B}_{k+1}=\{(\eta,\zeta): \exists(\tilde{\eta},\tilde{\zeta})\in\bigcup_{l=0}^{k}\tilde{B}_l \text{ such that } (\eta,\zeta)\to(\tilde{\eta},\tilde{\zeta})\}\cup\tilde{B}_0,\\ \tilde{E}_0=\tilde{B}_0,\ \tilde{E}_{k+1}=\tilde{B}_{k+1}\backslash\tilde{B}_k,\ k\in Z_+.\end{cases} \tag{4.2}$$

Lemma 1.

(i) *For each* $n\in Z_+$, $\tilde{B}_n\subset\tilde{B}_{n+1}$;

(ii) *For each* $n\in Z_+$,

$$\tilde{B}_n=\{(\theta,\theta)\}\cup\Big\{\Big(\sum_{l=1}^{h_m}e_{u_l},\sum_{l=h_m+1}^{m}e_{u_l}\Big): \tag{4.3}$$
$$m=1,2,\cdots,n;\ h_m=0,1,\cdots,m;\ S\ni u_1,\cdots,u_n \text{ may be repeated}\Big\}$$
$$\cup\Big\{\Big(\sum_{l=1}^{k_i}e_{u_l},\sum_{l=1}^{i}e_{u_l}+\sum_{l=k_i+1}^{n}e_{u_l}\Big): i=1,2,\cdots,n;\ k_i=i,i+1,\cdots,n;$$
$$S\ni u_1,\cdots,u_n \text{ may be repeated}\Big\},$$

where $\sum_{l=a}^{b}e_{u_l}=\theta$ *whenever* $a>b$;

(iii) $\tilde{E}_n(n\in Z_+)$ *are disjointed,* $\bigcup_{n=0}^{\infty}\tilde{E}_n=X\times X$ *and* $\{\tilde{E}_n\}$ *satifies* (1.2) *and* (1.3).

Proof (i) Clearly, $\tilde{B}_0\subset\tilde{B}_1$. For each $(\eta,\zeta)\in\tilde{B}_n$ $(n\geq 1)$, by (4.2), there exists an $(\tilde{\eta}, \tilde{\zeta}) \in \bigcup_{l=0}^{n-1} \tilde{B}_l \subset \bigcup_{l=0}^{n} \tilde{B}_l$ such that $(\eta, \zeta) \to (\tilde{\eta}, \tilde{\zeta})$. Hence $(\eta, \zeta) \in \tilde{B}_{n+1}$. By induction this proves (i).

(ii) To prove (4.3), denote the right side of (4.3) by $\bar{B}_n$. Clearly $\tilde{B}_0 = \bar{B}_0$. Suppose that $\tilde{B}_n = \bar{B}_n$ for all $n < k$. We then have to show that $\tilde{B}_n = \bar{B}_n$ for $n = k$.

First, we consider such an element (η, ζ) that has the form $\left(\sum_{l=1}^{h_m} e_{u_l}, \sum_{l=h_m+1}^{m} e_{u_l}\right)$. If $m < k$, then

$$(\eta, \zeta) = \left(\sum_{l=1}^{h_m} e_{u_l}, \sum_{l=h_m+1}^{m} e_{u_l}\right) \in \bar{B}_m = \tilde{B}_m \subset \tilde{B}_k.$$

We may now assume that $m = k$. Hence

$$(\eta, \zeta) = \left(\sum_{l=1}^{h_k} e_{u_l}, \sum_{l=h_k+1}^{k} e_{u_l}\right).$$

If $h_k = k$, then

$$(\eta, \zeta) = \left(\sum_{l=1}^{k} e_{u_l}, \theta\right) \to \left(\sum_{l=1}^{k-1} e_{u_l}, \theta\right) \in \tilde{B}_{k-1}.$$

If $h_k < k$, then there are two cases:

(a) The times of u_k appeared in $\{u_1, \cdots, u_{h_k}\}$ and in $\{u_{h_k+1}, \cdots, u_k\}$ are different. Then

$$(\eta, \zeta) \to (\eta, \zeta - e_{u_k}) = \left(\sum_{l=1}^{h_k} e_{u_l}, \sum_{l=h_k+1}^{k-1} e_{u_l}\right) \in \tilde{B}_{k-1}.$$

(b) The times of u_k appeared in $\{u_1, \cdots, u_{h_k}\}$ and in $\{u_{h_k+1}, u_k\}$ are the same. Then we may assume $u_{h_k} = u_k$. Hence

$$(\eta, \zeta) \to (\eta - e_{u_k}, \zeta - e_{u_k}) \in \tilde{B}_{k-1}.$$

Therefore, in both cases (a) and (b), we have $(\eta, \zeta) \in \tilde{B}_k$.

Next, we consider such an element (η, ζ) that has the form $\left(\sum_{l=1}^{k_i} e_{u_l}, \sum_{l=1}^{i} e_{u_l} + \sum_{l=k_i+1}^{k} e_{u_l}\right)$. There are three cases:

(a) The times of u_1 appeared in $\{u_1, \cdots, u_{k_i}\}$ and in $\{u_1, \cdots, u_i, u_{k_i+1}, \cdots, u_k\}$ are the same. Then $(\eta, \zeta) \to (\eta - e_{u_1}, \zeta - e_{u_1}) \in \tilde{B}_{k-1}$.

Now, assume that the times of u_1 appeared in the above two sets are different. Then, u_1 should appear in $\{u_{i+1}, \cdots, u_{k_i+1}, \cdots, u_k\}$.

(b) If $u_1 \in \{u_{i+1}, \cdots, u_{k_i}\}$, we may assume that $u_{k_i} = u_1$. Then

$$(\eta, \zeta) \to (\eta - e_{u_1}, \zeta) \in \tilde{B}_{k-1}.$$

(c) If $u_1 \in \{u_{k_i+1}, \cdots, u_k\}$, then

$$(\eta, \zeta) \to (\eta, \zeta - e_{u_1}) \in \tilde{B}_{k-1}.$$

In the above three cases, we always have $(\eta, \zeta) \in \tilde{B}_k$.

Combining the above discussions, we get $\bar{B}_k \subset \tilde{B}_k$.

Conversely, assume that $(\eta, \zeta) \in \tilde{B}_k$. If $(\eta, \zeta) \in \tilde{B}_{k-1} = \bar{B}_{k-1}$, then it is immediately that $(\eta, \zeta) \in \bar{B}_k$ by the definition of $\bar{B}_n$. Therefore, we may assume

$(\eta, \zeta)\notin\widetilde{B}_{k-1}$. Then, by (4.2), we can choose an

$$(\tilde{\eta}, \tilde{\zeta})\in\bigcup_{l=0}^{k-1}\widetilde{B}_l=\widetilde{B}_{k-1}=\overline{B}_{k-1}$$

such that $(\eta, \zeta)\to(\tilde{\eta}, \tilde{\zeta})$. We say that $(\tilde{\eta}, \tilde{\zeta})\neq(\eta+e_u, \zeta)$, $(\eta, \zeta+e_u)$ or $(\eta+e_u, \zeta+e_u)$. Otherwise, since $(\eta, \zeta)=(\tilde{\eta}-e_u, \tilde{\zeta})$, $(\tilde{\eta}, \tilde{\zeta}-e_u)$ or $(\tilde{\eta}-e_u, \tilde{\zeta}-e_u)$ and $(\tilde{\eta}, \tilde{\zeta})\in\overline{B}_{k-1}$, we would have $(\eta, \zeta)\in\overline{B}_{k-2}\subset\overline{B}_{k-1}=\widetilde{B}_{k-1}$, this is a contradiction. So, by (4.1), there is a $u\in S$ such that

$$(\eta, \zeta)=(\tilde{\eta}+e_u, \tilde{\zeta}),\ (\tilde{\eta}, \tilde{\zeta}+e_u)\ \text{or}\ (\tilde{\eta}+e_u, \tilde{\zeta}+e_u).$$

By the definition of $\overline{B}_{k-1}$, we see that $(\eta, \zeta)\in\overline{B}_k$. Hence $\widetilde{B}_k\subset\overline{B}_k$.

By induction, we have proved $\widetilde{B}_n=\overline{B}_n$ for each $n\geqslant0$.

(iii) Suppose that $(\eta, \zeta)\in\widetilde{E}_n\subset\widetilde{B}_n$ and $\tilde{q}((\eta, \zeta), (\tilde{\eta}, \tilde{\zeta}))>0$. Then by (1.17) $(\tilde{\eta}, \tilde{\zeta})$ should be one of $(\eta\pm e_u, \zeta)$, $(\eta, \zeta\pm e_u)$, $(\eta\pm e_u, \zeta\pm e_u)(u\in S)$, $(\eta+e_u-e_v, \zeta)$, $(\eta, \zeta+e_u-e_v)$ or $(\eta+e_u-e_v, \zeta+e_u-e_v)(u, v\in S, u\neq v)$. By (4.3),

$$(\tilde{\eta}, \tilde{\zeta})\in\widetilde{B}_{n+1}=\bigcup_{k=0}^{n+1}\widetilde{E}_k,$$

and so (1.2) is satisfied.

The remains are obvious.

Lemma 2. *Let $\tilde{q}((\eta, \zeta), (\tilde{\eta}, \tilde{\zeta}))$ be the Q-matrix corresponding to $\widetilde{\Omega}$ defined in (1.17). Then, for each $(\eta, \zeta)\in\widetilde{E}_n(n\geqslant1)$ there exist $\{v_1, \cdots, v_J\}\subset S$ and $\{a_1, \cdots, a_J\}\subset Z_+$, $\sum_{j=1}^{J}a_j=n$ such that*

$$\sum_{(\tilde{\eta},\tilde{\zeta})\in\widetilde{B}_{n-1}}\tilde{q}((\eta, \zeta), (\tilde{\eta}, \tilde{\zeta}))=\sum_{j=1}^{J}\delta(v_j, a_j). \tag{4.4}$$

Moreover, $\sum_{j=1}^{J}a_je_{v_j}$ determined by the right side of (4.4) varies over the whole set $\{\eta: |\eta|=n\}$ whenever (η, ζ) varies over the whole set $\widetilde{E}_n$.

Proof (i) By (4.3), we see that $(\eta, \zeta)\in\widetilde{E}_n=\widetilde{B}_n\setminus\widetilde{B}_{n-1}$ iff it has the form

$$\left(\sum_{l=1}^{k_i}e_{u_l}, \sum_{l=1}^{i}e_{u_l}+\sum_{l=k_i+1}^{n}e_l\right), \tag{4.5}$$

where $i\in\{0, 1, \cdots, n\}$, $k_i\in\{i, i+1, \cdots, n\}$, $S\ni u_1, \cdots, u_n$ may be repeated but

$$\{u_{i+1}, \cdots, u_{k_i}\}\cap\{u_{k_i+1}, \cdots, u_n\}=\phi.$$

(ii) Denote the distinct elements of $\{u_1, \cdots, u_i\}\setminus\{u_{i+1}, \cdots, u_n\}$, $\{u_{i+1}, \cdots, u_{k_i}\}$ and $\{u_{k_i+1}, \cdots, u_n\}$ by $\{v_1, \cdots, v_{J_1}\}$, $\{v_{J_1+1}, \cdots, v_{J_2}\}$ and $\{v_{J_2+1}, \cdots, v_J\}$ respectively. Then we may write

$$\begin{cases}\sum_{l=1}^{i}e_{u_l}=\sum_{j=1}^{J_1}a_je_{v_j}+\sum_{j=J_1+1}^{J}a_j^{(1)}e_{v_j},\ a_j>0,\ a_j^{(1)}\geqslant0,\\ \sum_{l=i+1}^{k_i}e_{u_l}=\sum_{j=J_1+1}^{J_2}a_j^{(2)}e_{v_j},\\ \sum_{l=k_i+1}^{n}e_{u_l}=\sum_{j=J_2+1}^{J}a_j^{(2)}e_{v_j},\ a_j^{(2)}>0.\end{cases} \tag{4.6}$$

Set $a_j=a_j^{(1)}+a_j^{(2)}$, $j=J_1+1, \cdots, J$. It fallows from (i) and (4.6) that

$$\begin{cases} \eta=\sum_{j=1}^{J_2} a_j e_{v_j}+\sum_{j=J_2+1}^{J} a_j^{(1)} e_{v_j}, \\ \zeta=\sum_{j=1}^{J_1} a_j e_{v_j}+\sum_{j=J_2+1}^{J} a_j e_{v_j}+\sum_{j=J_1+1}^{J_2} a_j^{(1)} e_{v_j}, \end{cases} \tag{4.7}$$

and

$$\sum_{j=1}^{J} a_j=n. \tag{4.8}$$

(iii) We are going to compute $\sum_{(\tilde\eta,\tilde\zeta)\in\tilde E_{n-1}} \tilde q((\eta, \zeta), (\tilde\eta, \tilde\zeta))$ for $(\eta, \zeta)\in\tilde E_n$. What we need indeed is figure out

$$\tilde E_{(\eta,\zeta)}\equiv\{(\tilde\eta, \tilde\zeta)\in\tilde E_{n-1}: (\eta, \zeta)\to(\tilde\eta, \tilde\zeta)\}.$$

By (4.1), (4.6), (4.7) and (i), we see that

(a) $(\eta+e_{v_j}, \zeta+e_{v_j}), (\eta+e_{v_j}, \zeta), (\eta, \zeta+e_{v_j})\notin\tilde E_{n-1}$.

(b) If $j\in\{1, \cdots, J_1\}$, then $(\eta-e_{v_j}, \zeta-e_{v_j})\in\tilde E_{n-1}$; $(\eta-e_{v_j}, \zeta)$ and $(\eta, \zeta-e_{v_j})\in\tilde E_n$.

(c) If $j\in\{J_1+1, \cdots, J_2\}$, then $(\eta-e_{v_j}, \zeta)\in\tilde E_{n-1}$, either $(\eta, \zeta-e_{v_j})\in\tilde E_n$ (when $a_j^{(1)}>0$) or $(\eta, \zeta-e_{v_j})$ has no meaning (when $a_j^{(1)}=0$), and either $(\eta, \zeta)\nrightarrow(\eta-e_{v_j}, \zeta-e_{v_j})$ (when $a_j^{(1)}>0$) or $(\eta-e_{v_j}, \zeta-e_{v_j})$ has no meaning (when $a_j^{(1)}=0$).

(d) If $j\in\{J_2+1, \cdots, J\}$, then $(\eta, \zeta-e_{v_j})\in\tilde E_{n-1}$, either $(\eta-e_{v_j}, \zeta)\in\tilde E_n$ (when $a_j^{(1)}>0$) or $(\eta-e_{v_j}, \zeta)$ has no meaning (when $a_j^{(1)}=0$) and either $(\eta, \zeta)\nrightarrow(\eta-e_{v_j}, \zeta-e_{v_j})$ (when $a_j^{(1)}>0$) or $(\eta-e_{v_j}, \zeta-e_{v_j})$ has no meaning (when $a_j^{(1)}=0$).

Combninig the above discussion, we get

$$\begin{aligned}\sum_{(\tilde\eta,\tilde\zeta)\in\tilde E_{n-1}} \tilde q((\eta, \zeta), (\tilde\eta, \tilde\zeta)) &= \sum_{(\tilde\eta,\tilde\zeta)\in\tilde E_{(\eta,\zeta)}} \tilde q((\eta, \zeta), (\tilde\eta, \tilde\zeta)) \\ &=\sum_{j=1}^{J_1}\delta(v_j, \eta_{v_j})\wedge\delta(v_j, \zeta_{v_j})+\sum_{j=J_1+1}^{J_2}\delta(v_j, \eta_{v_j})+\sum_{j=J_2+1}^{J}\delta(v_j, \zeta_{v_j}) \\ &=\sum_{j=1}^{J}\delta(v_j, a_j).\end{aligned}$$

(iv) The last assertion of Lemma 2 is obvious.

Lemma 3. *Under the assumptions of Lemma 2, for each $(\eta, \zeta)\in\tilde E_n$ $(n\geqslant0)$, there exist $\{v_1, \cdots, v_J\}\subset S$ and $\{a_1, \cdots, a_J\}\subset Z_+$, $\sum_{j=1}^{J} a_j=n$ such that*

$$\sum_{j=1}^{J}\beta(v_j, a_j)\leqslant\sum_{(\tilde\eta,\tilde\zeta)\in\tilde E_{n+1}}\tilde q((\eta, \zeta), (\tilde\eta, \tilde\zeta))\leqslant2\Big[\sum_{j=1}^{J}\beta(v_j, a_j)+\sum_{u\neq v_j}\beta(u, 0)\Big]. \tag{4.9}$$

Moreover, $\sum_{j=1}^{J} a_j e_{v_j}$ varies over the whole $\{\eta: |\eta|=n\}$ whenever (η, ζ) varies over the whole $\tilde E_n$.

Proof Notice that (i) and (ii) in the proof of Lemma 2 are still available. Using the notations and the proof for (iii) given there, it follows that

$$\begin{aligned}\{(\tilde\eta, \tilde\zeta)\in\tilde E_{n+1}: (\eta, \zeta)\to(\tilde\eta, \tilde\zeta)\}&=\{(\eta+e_u, \zeta+e_u): u\in\{v_{J_1+1}, \cdots, v_J\}^c|\} \\ &\cup\{(\eta+e_{v_j}, \zeta), (\eta, \zeta+e_{v_j}): j=J_1+1, \cdots, J\}.\end{aligned}$$

By (1.17) and (4.7), we get

$$\sum_{(\tilde{\eta},\tilde{\zeta})\in\tilde{E}_{n+1}} \tilde{q}((\eta,\zeta),(\tilde{\eta},\tilde{\zeta}))$$

$$=\sum_{\substack{u\neq v_j\\ J_1<j\leqslant J}} \beta(u,\eta_u)\wedge\beta(u,\zeta_u)+\sum_{j=J_1+1}^{J}[\beta(v_j,\eta_{v_j})+\beta(v_j,\zeta_{v_j})]$$

$$=\sum_{j=1}^{J}\beta(v_j,a_j)+\sum_{j=J_1+1}^{J}\beta(v_j,a_j^{(1)})+\sum_{\substack{u\neq v_j\\ 1\leqslant j\leqslant J}}\beta(u,0).$$

Now (4.9) follows since $\beta(u,k)$ is increasing in k.

Lemma 4. *Under the assumptions of Lemma 2, for each n, $k\in Z_+$ and each $(\eta,\zeta)\in\tilde{E}_n$, we have*

$$\sum_{(\tilde{\eta},\tilde{\zeta})\in\tilde{E}_k} \tilde{q}((\eta,\zeta),(\tilde{\eta},\tilde{\zeta}))=0 \tag{4.11}$$

whenever $|n-k|>1$.

Proof Using the proof of Lemma 2, it is easy to show that

$$\{(\tilde{\eta},\tilde{\zeta})\in\tilde{E}_k:\ (\eta,\zeta)\to(\tilde{\eta},\tilde{\zeta})\}=\phi.$$

This proves (4.11).

Proof of Theorem 5. For the Q-matrix $(q(\eta,\zeta):\eta,\zeta\in X)$ defined by (1.16), take $E_k=\{\eta\in X:\ |\eta|=k\}$. Then the Q-matrix (q_{ij}) defined by (1.1) is a birth-death Q-matrix and satisfies

$$\begin{cases} q_{k,k+1}=\max\{\sum_{|\eta|=k+1}q(\eta,\zeta):\ |\eta|=k\}=r_k,\ k\in Z_+,\\ q_{k,k-1}=\min\{\sum_{|\eta|=k-1}q(\eta,\zeta):\ |\eta|=k\}=s_k,\ k\in Z_+\setminus\{0\}. \end{cases} \tag{4.12}$$

For the Q-matrix $(\tilde{q}((\eta,\zeta),(\tilde{\eta},\tilde{\zeta})):\ (\eta,\zeta),(\tilde{\eta},\tilde{\zeta})\in X\times X)$ replace E_k in (1.1) by $\tilde{E}_k$ defined by (4.2), and define a Q-matrix $(\tilde{q}_{ij})$ according to (1.1). By Lemma 4, $(\tilde{q}_{ij})$ is again a birth-death Q-matrix. By Lemma 3 and Lemma 2, we have

$$\begin{cases} \tilde{r}_k\equiv\tilde{q}_{k,k+1}\equiv\max\{\sum_{(\tilde{\eta},\tilde{\zeta})\in\tilde{E}_{k+1}}\tilde{q}((\eta,\zeta),(\tilde{\eta},\tilde{\zeta})):\ (\eta,\zeta)\in\tilde{E}_k\}\\ \quad\leqslant\max\{\sum_{u\in S}2\beta(u,\eta_u):\ |\eta|=k\}=2r_k\geqslant r_k,\ k\in Z_+,\\ \tilde{S}_k\equiv\tilde{q}_{k,k-1}\equiv\min\{\sum_{(\tilde{\eta},\tilde{\zeta})\in\tilde{E}_{k-1}}\tilde{q}((\eta,\zeta),(\tilde{\eta},\tilde{\zeta})):\ (\eta,\zeta)\in\tilde{E}_k\}\\ \quad=\min\{\sum_{u\in S}\delta(u,\eta_u):\ |\eta|=k\}=S_k,\ k\in Z_+\setminus\{0\}. \end{cases} \tag{4.13}$$

Hence, if (1.19) holds and $r_k>0$ $(k\in Z_+)$, then (4.12) and (4.13) imply that

$$\sum_{k=0}^{\infty}\Big[\frac{1}{r_k}+\frac{S_k}{r_kr_{k-1}}+\cdots+\frac{S_k\cdots S_1}{r_k\cdots r_1r_0}\Big]\geqslant\bar{R}=\infty,$$

$$\sum_{k=0}^{\infty}\Big[\frac{1}{\tilde{r}_k}+\frac{S_k}{\tilde{r}_k\tilde{r}_{k-1}}+\cdots+\frac{S_k\cdots S_1}{\tilde{r}_k\cdots\tilde{r}_1\tilde{r}_0}\Big]\geqslant\bar{R}=\infty.$$

Therefore, Ω and $\tilde{\Omega}$ are regular by Theorem 1 and Theorem 3, (i). If (1.19) holds and $r_0=0$, $r_k>0$ $(k\in Z_+\setminus\{0\})$, then the conclusion follows from Theorem 1 and Theorem 3, (ii).

Finally, the remains are easily consequences of [10].

Proposition 3. *Let S be finite. Then the Q-matrices defined by* (1.12), (1.13) *and* (1.14) *satisfy the assumptions of Theorem* 5.

Proof The calculations are elementary once we write down the Q-matrices. For (1.12)

$$\begin{cases} \beta(u,\ \eta_u)=\lambda_1 a_u \binom{\eta_u}{2}+\lambda_4 b_u,\ \delta(u,\ \eta_u)=\lambda_2\binom{\eta_u}{3}+\lambda_3\eta_u, \\ \gamma(u,\ v,\ \eta_u,\ \eta_v)=\eta_u p(u,\ v),\ u,\ v\in S,\ u\neq v. \end{cases} \tag{4.14}$$

For (1.13) and (1.14), S and u should be replaced with $S\times\{1,\ 2\}$ and $(u,\ i)$ $(u\in S,\ i=1,\ 2)$ respectively. Then, for (1.13), we have

$$\beta((u,\ i),\ \eta_{ui})=\begin{cases}\lambda_1 a_u\eta_{u1},\ i=1,\\ 0,\ i=2,\end{cases}$$

$$\delta((u,\ i),\ \eta_{ui})=\begin{cases}0,\ i=1,\\ \lambda_3 b_u\eta_{u2},\ i=2,\end{cases}$$

$$\gamma((u,\ i),\ (v,\ j),\ \eta_{ui},\ \eta_{vj})=\begin{cases}\lambda_2\eta_{u1}\eta_{u2},\ u=v,\ i=1,\ j=2,\\ \eta_{u1}p_1(u,\ v),\ u\neq v,\ i=j=1,\\ \eta_{u2}p_2(u,\ v),\ u\neq v,\ i=j=2,\\ 0,\ \text{other}\ (u,\ i)\neq(v,\ j),\end{cases}$$

and for (1.14), we have

$$\beta((u,\ i),\ \eta_{ui})=\begin{cases}\lambda_1 a_u,\ i=1,\\ 0,\ i=2,\end{cases}$$

$$\delta((u,\ i),\ \eta_{ui})=\begin{cases}\lambda_4\eta_{u1},\ i=1,\\ 0,\ i=2,\end{cases}$$

$$\gamma((u,\ i),\ (v,\ j),\ \eta_{ui},\ \eta_{vj})=\begin{cases}\lambda_2 b_u\eta_{u1},\ u=v,\ i=1,\ j=2,\\ \lambda_3\binom{\eta_{u1}}{2}\eta_{u2},\ u=v,\ i=2,\ j=1,\\ \eta_{ui}p_i(u,\ v),\ u\neq v,\ i=j=1,\ 2,\\ 0,\ \text{other}\ (u,\ i)\neq(v,\ j).\end{cases}$$

§ 5. Recurrence

Lemma 1. *Let* $(q_{ij}: i, j\in E)$ *be a regular irreducible Q-matrix. Then, the* (q_{ij})-*process is recurrent iff*

$$x_i=\sum_{k\neq j_0}\bar{p}_{ik}x_k,\quad 0\leqslant x_i\leqslant 1,\ i\in E \tag{5.1}$$

has only zero solution for some $j_0\in E$ (*or equivalently, for any* $j_0\in E$), *where* (p_{ij}) *is the jump matrix of* (q_{ij}), i.e.,

$$\bar{p}_{ij}=\begin{cases}q_{ij}/q_i, & i\neq j,\\ 0, & i=j,\ i,\ j\in E.\end{cases} \tag{5.2}$$

Proof Clearly, we can fix a $j_0\in E$. By [7, Theorem 1], the (q_{ij})-process is recurrent iff the jump chain $(\bar{p}_{ij})$ is recurrent. Then, by [4, Theorem 6.6.1] it is equivalent to that the minimal nonnegative solution of

$$x_i=\sum_{k\neq j_0}\bar{p}_{ik}x_k+\bar{p}_{ij_0},\ i\in E, \tag{5.3}$$

is $x_i^*=1$ $(i\in E)$. Therefore, by [4, Theorem 5.6.3], the proof is reduced to showing that (5.3) has no nonnegative non-constant bounded solution iff (5.1) has only zero solution. To this end, let $(\tilde{x}_i:\ i\in E)$ be a non-zero solution of (5.1). Then $(y_i=1+\tilde{x}_i:\ i\in E)$ is a nonnegative non-constant bounded solution of (5.3). Conversely, if (5.3) has such a solution, then the minimal nonnegative solution of (5.3) satisfies $x_i^*\leqslant 1$ and there exists at least an $i_0\in E$ so that $x_{i_0}^*<1$, since $(y_i=1,\ i\in E)$ is a solution of (5.3). Hence, $(\tilde{x}_i\equiv 1-x_i^*:\ i\in E)$ is a non-zero solution of (5.1).

Proof of Theorem 6. By Lemma 1, to prove the first part of Theorem 6, it suffices to show that (5.1) has only zero solution implies that

$$u(\eta)=\sum_{\zeta\neq\theta}\bar{p}(\eta,\ \zeta)u(\zeta),\ 0\leqslant u(\eta)\leqslant 1,\ \eta\in X_0 \tag{5.4}$$

has only zero solution, where $(\bar{p}(\eta,\ \zeta))$ is the jump matrix of $(q(\eta,\ \zeta))$. Now, the proof is similar to the proof of Theorem 1. Suppose that $\{u(\eta):\ \eta\in X_0\}$ is a non-zero solution to (5.4). Then

$$q(\eta)u(\eta)=\sum_{\zeta\neq\theta,\eta}q(\eta,\ \zeta)u(\zeta),\quad 0\leqslant u(\eta)\leqslant 1,\ \eta\in X_0. \tag{5.5}$$

Set $u_k=\sup\{u(\eta):\ \eta\in E_k\}$, $i\in Z_+$. Since $E_0=\{\theta\}$, E_k is a finite set $(k\geqslant 1)$, there exists an $\eta^{(k)}\in E_k$ $(k\geqslant 1)$ so that

$$u_0=u(\theta),\quad u_k=u(\eta^{(k)}), \tag{5.6}$$

$$q_0u_0=q(\theta)u(\theta)=\sum_{\zeta\neq\theta}q(\theta,\ \zeta)u(\zeta)\leqslant\sum_{\zeta\in E_1}q(\theta,\ \zeta)u_1=q_{01}u_1. \tag{5.7}$$

If $k\geqslant 1$, by using the method in the proof of (2.6), it follows that

$$q(\eta^{(k)},\ \theta)u_k+\sum_{j=1}^{k-1}\Big(\sum_{\zeta\in E_j}q(\eta^{(k)},\ \zeta)\Big)(u_k-u_j)\leqslant\sum_{\zeta\in E_{k+1}}q(\eta^{(k)},\ \zeta)(u_{k+1}-u_k),$$

so $u_k\uparrow$, and

$$q_{k0}u_k+\sum_{j=1}^{k-1}q_{kj}(u_k-u_j)\leqslant q_{k,k+1}(u_{k+1}-u_k),\quad k\geqslant 1. \tag{5.8}$$

Combining (5.7) and (5.8), we get

$$q_ku_k\leqslant\sum_{j\neq 0,k}q_{kj}u_j,\quad k\in Z_+.$$

It is now easy to show that $u_i=0$, $i\in Z_+$ (cf. the proof of § 2, Lemma 1) since (5.1) has only zero solution. But this is a contradction to $u(\eta)\neq 0$.

As for the second part of Thorem 6, by Lemma 1, we need only to prove that

$$x_i=\sum_{k\neq 0,i}\frac{q_{ik}}{q_i}x_k,\quad 0\leqslant x_i\leqslant 1, i\in Z_+ \tag{5.9}$$

has non-zero solution iff $\sum_{k=0}^{\infty}F_k^{(0)}<\infty$. Notice that (5.9) has non-zero solution iff

$$x_i=\sum_{k\neq 0,i}\frac{q_{ik}}{q_i}x_k,\ 0\leqslant x_i\leqslant 1, i\in Z_+ \tag{5.10}$$

has nonnegative bounded solution, and (5.10) has at most one solution which is

$$\tilde{x}_0 = 1 = \tilde{x}_1, \tag{5.11}$$

$$\tilde{x}_{i+1} - \tilde{x}_i = \Big(\sum_{j=1}^{i-1} q_{ij}(\tilde{x}_i - \tilde{x}_j) + q_{i0}\tilde{x}_i\Big)\Big/ q_{i,i+1}, \quad i \geq 1.$$

Clearly, $(\tilde{x}_i)$ is increasing. Thus, the proof is reduced to showing that $(\tilde{x}_i)$ is bounded iff

$$\sum_{k=0}^{\infty} F_k^{(0)} < \infty.$$

Observe that

$$\sum_{j=1}^{i-1} q_i^{(j)}(\tilde{x}_{j+1} - \tilde{x}_j) = \sum_{j=0}^{i-1} q_{ij}(\tilde{x}_i - \tilde{x}_j).$$

Hence

$$\tilde{x}_{i+1} - \tilde{x}_i = \sum_{j=1}^{i-1} \frac{q_i^{(j)}}{q_{i,i+1}}(\tilde{x}_{j+1} - \tilde{x}_j) + \frac{q_i^{(0)}}{q_{i,i+1}}, \quad i \geq 1.$$

This proves that $(y_i \equiv \tilde{x}_{i+1} - \tilde{x}_i;\ i \geq 1)$ is a nonnegative finite solution to

$$z_i = \sum_{j=1}^{i-1} \frac{q_i^{(j)}}{q_{i,i+1}} z_j + \frac{q_i^{(\,)}}{q_{i,i+1}}, \quad i \geq 1. \tag{5.12}$$

On the other hand, the solution to (5.12) is unique, i.e.,

$$z_1^* = q_1^{(0)}/q_{12},$$

$$z_i^* = \sum_{j=1}^{i-1} (q_i^{(j)}/q_{j,j+1}) z_j^* + q_i^{(0)}/q_{i,i+1}, \quad i \geq 1,$$

and it is easy to show

$$z_i^* = F_i^{(0)}, \quad i \geq 1$$

by induction. Using the above remarks, we get

$$\tilde{x}_0 = \tilde{x}_1 = 1 = F_0^{(0)}, \ \tilde{x}_{i+1} - \tilde{x}_i = F_i^{(0)}, \quad i \geq 1.$$

This of course implies what we need.

Proposition 4. *Let S be finite. Then the $(q(\eta, \zeta))$-process defined by (1.12) is recurrent.*

Proof. Take $E_k = \{\eta \in X: |\eta| = k\}$. By (3.4) and (3.6), we have $s_{k+1}/r_{k+1} \to \infty$ $(k \to \infty)$. Hence

$$\sum_{k=0}^{\infty} F_k^{(0)} = 1 + \sum_{k=1}^{\infty} \frac{s_k \cdots s_1}{r_k \cdots r_1} = \infty.$$

Now Theorem 6 is available.

§ 6. Positive Recurrence and Ergodicity

Proof of Theorem 7. If the (q_{ij})-process is positive recurrent, then for some (each) $i_0 \in Z_+$ there exists a nonnegative solution to

$$x_i \geq \sum_{j \neq i_0, i} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \quad i \neq i_0, \tag{6.1}$$

$$\sum_{j\neq i_0}\frac{q_{i_0j}}{q_{i_0}}x_j<\infty$$

by [4, Theorem 9.4.1]. But (6.1) is just the condition (1.22) if we set $x_{i_0}=0$, in addition. Therefore (1.22) is necessary.

Conversely, if $\{u_i: i\in Z_+\}$ is a nonnegative solution of (1.22) for an $i_0\in Z_+$, we define

$$c_i=1,\ i\neq i_0;\ =-\sum_j q_{i_0j}u_j,\ i=i_0. \tag{6.2}$$

Then $c_i+\sum_j q_{ij}u_j\leqslant 0$, $i\in Z_+$, and so

$$u_i-c\geqslant\sum_{j\neq i}\frac{q_{ij}}{\lambda+q_i}(u_j-c)+\frac{c_i-\lambda c}{\lambda+q_i},\ i\in Z_+, \tag{6.3}$$

where $c\equiv(\inf\{c_j/\lambda:\ j\in Z_+\})\wedge 0>-\infty$. Denote the Laplace transform of the minimal (q_{ij})-process by $(P_{ij}^{\min}(\lambda))$. It follows from [4, Theorem 3.3.3 and Theorem 3.3.1] that

$$u_i-c\geqslant\sum_j P_{ij}^{\min}(\lambda)(c_j-\lambda c). \tag{6.4}$$

Since (q_{ij}) is regular, $\lambda\sum_j P_{ij}^{\min}(\lambda)=1$, $i\in Z_+$, $\lambda>0$, it follows from (6.2) and (6.4) that

$$u_i\geqslant\sum_j P_{ij}^{\min}(\lambda)c_j=\sum_{i\neq i_0}P_{ij}^{\min}(\lambda)+P_{ii_0}^{\min}(\lambda)c_{i_0}=\lambda^{-1}+P_{ii_0}^{\min}(\lambda)(c_{i_0}-1),$$

i.e.,

$$\lambda u_i\geqslant 1+\lambda P_{ii_0}^{\min}(\lambda)(c_{i_0}-1),\ \mathrm{i}\in Z_+.$$

Because of $\lim\limits_{\lambda\searrow 0}\lambda p_{ij}^{\min}(\lambda)=\lim\limits_{t\to\infty}p_{ij}(t)=\pi_{ij}$, $i,\ j\in Z_+$, we have

$$0\geqslant 1+\pi_{ii_0}(c_{i_0}-1),\ \ i\in Z_+,$$

and so $\pi_{ii_0}\neq 0$ for each $i\in Z_+$. This implies that $\pi_i=\pi_{ij}>0$, $i,\ j\in Z_+$ and $\sum_i\pi_i=1$. So the (q_{ij})-process is ergodic.

Proof of Theorem 8. By Theorem 7, it suffices to prove that

$$\sum_\zeta q(\eta,\zeta)u(\zeta)+1\leqslant 0,\ \eta\neq\theta,$$
$$\Big|\sum_\zeta q(\theta,\zeta)u(\zeta)\Big|<\infty \tag{6.5}$$

has nonnegative solution. To this end, put

$$u(\eta)=u_k,\ \eta\in E_k,\ k\in Z_+.$$

Given $\eta\notin E_0$, there exists only one $k\in Z_+\setminus\{0\}$ so that $\eta\in E_k$. Hence

$$\sum_\zeta q(\eta,\zeta)u(\zeta)+1=-\sum_{j=0}^{k-1}\sum_{\zeta\in E_j}q(\eta,\zeta)(u_k-u_j)+\sum_{\zeta\in E_{k+1}}q(\eta,\zeta)(u_{k+1}-u_k)+1$$
$$\leqslant-\sum_{j=0}^{k-1}q_{kj}(u_k-u_j)+q_{k,k+1}(u_{k+1}-u_k)+1=\sum_j q_{kj}u_j+1\leqslant 0$$

by (1.1), (1.2), $u_k\nearrow$ and the assumption of this theorem. But

$$\sum_\zeta q(\theta,\zeta)u(\zeta)=\sum_{\zeta\in E_1}q(\theta,\zeta)u_1=q_{01}u_1$$

is finite, so (6.5) holds.

Proof of Theorem 9 Let $\{u_i: i\in Z_+\}$ be a nonnegative increasing solution to (1.22) with $i_0=0$. Then

$$q_{01}(u_1-u_0)=|\sum_j q_{0j}u_j|<\infty, \tag{6.6}$$

$$\sum_j q_{kj}u_j+1\leqslant 0,\ k>0. \tag{6.7}$$

By (1.7), (6.7) and Abelian transform, we get

$$q_{k,k+1}(u_{k+1}-u_k)+1\leqslant\sum_{j=0}^{k-1}q_{kj}(u_k\cdots u_j)=\sum_{j=0}^{k-1}q_k^{(j)}(u_{j+1}-u_j). \tag{6.8}$$

Put $v_k=u_{k+1}-u_k(k\geqslant 0)$. Then v_0 is finite by (6.6). By (1.9), (1.25), (6.8) and induction it follows that

$$v_k\leqslant F_k^{(0)}v_0-d_k,\ k\in Z_+. \tag{6.9}$$

Now, (1.24) follows from (6.9) since $u_k\uparrow$, $v_k\geqslant 0$, $k\in Z_+$.

Conversely, if (1.24) holds, then (6.6) holds for $(u_k: k\in Z_+)$ defined by (1.26). Hence, by (1.9) and (1.25), for each $k>0$, we have

$$1+q_{k,k+1}(u_{k+1}-u_k)=q_{k,k+1}F_k^{(0)}u_1-q_{k,k+1}d_k+1=\sum_{s=0}^{k-1}q_k^{(s)}F_s^{(0)}u_1-\sum_{s=0}^{k-1}q_k^{(s)}d_s$$

$$=\sum_{s=0}^{k-1}q_k^{(s)}(F_s^{(0)}u_1-d_s)=\sum_{s=0}^{k-1}q_k^{(s)}(u_{s+1}-u_s)=\sum_{j=0}^{k-1}q_{kj}(u_k-u_j),$$

i. e. $\sum_j q_{kj}u_j+1=0$, and so (1.22) has a nonnegative increasing solution when $i_0=0$.

Corollary. *Let (q_{ij}) be a birth-death Q-matrix:*

$$r_k=q_{k,k+1}>0,\ k\geqslant 0,$$

$$s_k=q_{k,k-1}>0,\ k\geqslant 1.$$

Then, the condition (1.24) *is equivalent to*

$$\sum_{k=0}^{\infty}\frac{r_0\cdots r_k}{s_1\cdots s_{k+1}}<\infty. \tag{6.11}$$

Proof By (1.9) and (1.25), we have

$$\frac{d_k}{F_k^{(0)}}=\frac{r_1\cdots r_k}{s_1\cdots s_k}\cdot\frac{1}{r_k}[1+s_kd_{k-1}]=\cdots=\frac{1}{r_0}\sum_{l=0}^{k-1}\frac{r_0r_1\cdots r_l}{s_1s_2\cdots s_{l+1}}.$$

Proposition 5. *Let S be finite. Then the Q-process corresponding to Schlögl model is ergodic.*

Proof By (3.4) and (3.6), $\frac{r_k}{s_k}\to 0$ $(k\to\infty)$. Hence (1.24) holds.

Proof of Theorem 10 Let $(u_i: i\in Z_+)$ be a nonnegative solution to (1.22) with $i_0=0$, and put $v_k=u_{k+1}-u_k$. Then (6.9) holds, and so

$$0\leqslant u_{k+1}=\sum_{l=0}^{k}v_k+u_0\leqslant\Big(\sum_{l=0}^{k}F_l^{(0)}\Big)v_0-\sum_{l=0}^{k}d_l+u_0\leqslant\Big(\sum_{l=0}^{k}F_l^{(0)}\Big)u_1-\sum_{l=0}^{k}d_l.$$

Therefore, we have $\hat{d}\leqslant u_1<\infty$. Conversely, if (1.27) holds, we choose $u_0=0$, $u_1\geqslant\hat{d}$, and define u_k $(k\geqslant 2)$ according to (1.26). Then, by using the proof of Theorem 9, it follows that

$$\sum_j q_{ij}u_j+1=0,\ k\neq 0,$$

$$\sum_j q_0 u_j=q_{01}u_1 \text{ is finite},$$

and

$$u_k=\sum_{l=0}^{k-1}(F_l^{(0)}u_1-d_l)\geqslant\hat{d}\sum_{l=0}^{k-1}F_l^{(0)}-\sum_{l=0}^{k-1}d_l\geqslant 0,\ k\geqslant 2.$$

We get a nonnegative solution to (1.22) with $i_0=0$.

The last assertion of Theorem 10 is obvious.

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References

[1] Nicolis, G. and Prigogine, I., Self-orgamzation in Non-equilibrium Systems, *John Wiley & Sons*, 1977.
[2] Haken, H., Synergetics, Sringer-Verlag. 1977.
[3] 严士健、李占柄，非平衡系统的概率模型及 Master 方程的建立。物理学报，29 (1980)，139—152
[4] 侯振挺、郭青峰，齐次可列马尔可夫过程，科学出版社，北京，1978.
[5] Reuter, G. E. H., Competition processes, Proc. of the 4th Berkeley Symposium on Math. Statis. and Prob. 1961. Vol. 2. 421—430.
[6] 张建康，广生灭过程的性质，数学物理学报，4:2(1984)，241—259.
[7] 吴立德，可数马尔科夫过程的状态的分类，数学学报，15 (1965)，32—41
[8] 王梓坤，生灭过程与马尔科夫链，科学出版社，北京，1980.
[9] 侯振挺，Q-过程唯一性准则，湖南科学技术出版社，长沙 1982.
[10] 陈木法，马链的基本耦合，北京师范大学学报，4(1984)，3—10.