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# THE CONVEX VECTOR PROGRAMMING ON AN ORDER COMPLETE ORDERED TOPOLOGICAL VECTOR SPACE

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**Abstract**

This paper first gives several basic theorems, including some equivalent forms of Hahn–Banach Theorem on an order complete ordered vector space and an order complete ordered topological vector space. Then, with these results, we can conveniently discuss the problems of general convex vector programming. In special cases, we give the corresponding results of Rockafellar Problem and Kuhn–Tucker's problem.

## § 1. Introduction

Now there was Kuhn–Tucker's theorem of convex programming, whether on the problems of practical projects, econometrics, or on management science, etc, the convex programming and its duality theory have widely been applied. But in application, one has met various kinds of more general problems of convex programming. Thus it is necessary to have it more perfect in theory. Recently, there have been many papers published in this field, such as [1—5]. Shi Shuzhong[1], as the pure algebraic case, has perfectly generalized Kuhn–Tucker's theorem of convex programming to an order complete vector lattice. This paper, based on [1] and consulting [2], discusses the general convex vector programming and its special cases on an order complete ordered topological vector space.

This paper first gives several basic theorems. Except for the theorem about the continuity of a convex map, they are the generalizations of Hahn–Banach Theorem on an order complete ordered vector space or ordered topological vector space. With these results, we can conveniently give a series of results about the general convex vector programming and its special cases. So, on ordered topological vector spaces, it makes the discussion on the convex vector programming more perfect.

[2], also in ordered topological vector spaces, discussed the problems of convex vector programming. But on the generalization of Kuhn–Tucker's theorem, which is principal result, the conditions required are strong, even causing the original

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Kuhn–Tucker's theorem not to be a special case. What is more, [2] only gives the resu lts without any proofs.

## § 2. Definitions and Symbols

This paper assumes that the vector spaces are on the real field $R$. We suppose that the reader has had the basic concepts of a vector space and a topological vector space.

Let $X$ and $Y$ be vector spaces. To a subset $A\subset X\times Y$, let

$$A_x=\{y\in Y\mid (x, y)\in A\}.$$

To a subset $B\subset X$, let $B^i$ be the set of all algebraic interior points of $B$. $B$ is absorb if $0\in B^i$. A convex subset $C\subset X$ is a convex cone if $\forall\lambda>0$, $\lambda C=C$.

If a vector space $Y$ is also defined as an order "$\leqslant$" and satisfies the following consistency conditions:

1) $y_1, y_2, y_3\in Y$, $y_1\geqslant y_2\Rightarrow y_1+y_3\geqslant y_2+y_3$;

2) $y\in Y$, $\lambda\geqslant 0$, $y\geqslant 0\Rightarrow \lambda y\geqslant 0$,

then $Y$ is called an ordered vector space.

Let $Y$ be an ordered vector space. $C_y=\{y\in Y\mid y\geqslant 0\}$ is called the the positive cone of $Y$. If $y\in C_y^i$, it is denoted by $y>0$. To a subset of $Y$, if there is a $y_1\in Y$ such that $\forall\ a\in A$, $a\leqslant y_1$, then $y_1$ is called a upper bound of $A$. If there exists a upper bound $y_0$ of $A$ such that $y_0\leqslant y_1$ for any upper bound $y_1$ of $A$, then $y_0$ is called the supremum of $A$; the notation is $y_0=\sup A$. Similarly, we can define the lower bound and the infimum $\inf A$. If $\sup A$ exists for any non–empty subset $A$ that is bounded above, then $Y$ is called order complete. Certainly, if $Y$ is order complete, any non–empty subset that is bounded below has its infimum.

If $Y$ is an ordered vector space and a topological vector space, and $C_y$ is closed, then $Y$ is called an ordered topological vector space. If $Y$, as an ordered vector space, is order complete, it is called an order complete ordered topological vector space.

Let $X$ and $Y$ be topological vector space. We know that if $A$ is convex and $\operatorname{int}A\neq\emptyset$, then $A^i=\operatorname{int} A$. The set of all continuous linear maps from $X$ to $Y$ is denoted by $B(X, Y)$. $X$ is called a locally convex space if it has a local basis consisting of convex sets.

Let $Y$ be an ordered topological vector space. For a subset $A\subset Y$, let $[A]=(A-C_y)\cap(A+C_y)$. $A$ is $C_y$–saturated if $[A]=A$. $C_y$ is normal if $Y$ has a local basis consisting of $C_y$–saturated sets.

Let $X$ be a vector space, $Y$ an ordered vector space. For a map $f: D(f)\subset X\to Y$, where $D(f)$ is the field of the definitions of $f$, if $D(f)$ is convex and $\forall x_1, x_2\in$

$D(f)$, $\lambda\in(0, 1)$, $f(\lambda x_1+(1-\lambda)x_2)\leqslant\lambda f(x_1)+(1-\lambda)f(x_2)$, then $f$ is called a convex map.

## § 3. Basic Theorems

**Lemma 1**[1]. *Let $X$ be a vector space, $Y$ an order complete ordered vector space, and $f$: $D(f)\subset X\to Y$ a convex map. If $f(0)\geqslant 0$ and $D(f)$ is absorb, then there exists a linear map $L$: $X\to Y$ such that $Lx\leqslant f(x)$, $\forall x\in D(f)$.*

**Theorem 1.** *Let $X$ be a vector space, $Y$ an order complete ordered vector space, $A\subset X\times Y$ a convex set, and $D=\{x\in X\mid A_x\neq\emptyset\}$. If*

*1) $D$ is absorb,*

*2) $\forall x\in D$, $A_x$ is bounded below,*

*3) $(0, y)\in A$, $y\geqslant 0$,*

*then there exists a linear map $\Lambda$: $X\to Y$ such that*

$$\Lambda x+y\geqslant 0,\ \forall(x, y)\in A.$$

*Proof* By defining a map $f$: $D\to Y$, $f(x)=\inf A_x$, we have $\forall x_1, x_2\in D$, $\lambda\in(0, 1)$, $\lambda A_{x_1}+(1-\lambda)A_{x_2}\subset A_{\lambda x_1+(1-\lambda)x_2}$. Hence

$$f(\lambda x_1+(1-\lambda)x_2)\leqslant\inf\{\lambda A_{x_1}+(1-\lambda)A_{x_2}\}=\inf\{\lambda A_{x_1}\}+\inf\{(1-\lambda)A_{x_2}\}$$
$$=\lambda f(x_1)+(1-\lambda)f(x_2),$$

then $f$ is convex. Moreover, it is clear that $f$ satisfies the conditions of Lemma 1, hence, there exists a linear map $L$: $X\to Y$ such that $Lx=f(x)$, $\forall x\in D$. That is $Lx\leqslant y$, $\forall y\in A_x$, $x\in D$. We take $\Lambda=-L$, and have $\Lambda x+y\geqslant 0$, $\forall(x, y)\in A$.

**Theorem 2.** *Let $X$ be a topological vector space; $Y$ an order complete ordered topologioal vector space, $C_y$ normal; $f$. $D(f)\subset X\to Y$ a convex map, continuous at a point of int$D(f)$; and $L_0$: $X_0\to Y$ a linear map, $X_0\subset X$ a vector subspace. If $D^i(f)\cap X_0\neq\emptyset$ and*

$$L_0x\leqslant f(x),\ \forall x\in X_0\cap D(f),$$

*then there exists a continuous linear extension $L$: $X\to Y$ of $L_0$ such that*

$$Lx\leqslant f(x),\ \forall x\in D(f).$$

*Proof* Let $K=\{(x, y)\mid f(x)\leqslant y,\ x\in D(f)\}$, $B=\{(x, L_0x)\mid x\in X_0\}$, $A=K-B$, and $D=\{x\in X\mid A_x\neq\emptyset\}$. It is clear that $A$ is convex and $D=D(f)-X_0$. Thus, we can easily find that $A$ satisfies the three conditions of Theorem 1, hence there exists a linear map $\Lambda$: $X\to Y$ such that

$$\Lambda x_1+y_1\geqslant\Lambda x_2+y_2,\ \forall(x_1, y_1)\in K,\ (x_2, y_2)\in B.$$

Since $(0, 0)\in B$ and $\forall x\in E(f)$, $(x, f(x))\in K$, $\Lambda x+f(x)\geqslant 0$.

Also, to the fixed point $(x_1, y_1)\in K$, we have $n(x, L_0x)\in B$, $\forall x\in X_0$, $n\in N$, hence $n(\Lambda x+L_0x)\leqslant\Lambda x_1+y_1$, $\forall n\in N$. But $Y$ is order complete, so $\Lambda x+L_0x\leqslant 0$. Also $\Lambda(-x)+L_0(-x)\leqslant 0$, therefore $\Lambda x+L_0x=0$, $\forall x\in X_0$.

So, $L=-A$ is the linear extension of $L_0$. Below, we will verify that $L$ is continuous.

We can assume that $f$ is continuous at $x=0$, and $f(0)=0$. For any 0-neighborhood $V$ in $Y$, since $C_y$ is normal, we can assume that $V$ is circled and $C_y$-saturated. But $f$ is continuous at 0, so there is a 0-neighborhood $U$ in $X$ such that $f(U\cap D(f))\subset V$. Since $D(f)$ is also a neighborhood of 0, $U\cap D(f)$ is a neighborhood of 0. We can take a circled 0-neighborhood $U_0\subset U\cap D(f)$. Certainly. $f(U)\subset V$. $\forall\in U_0$, we have $L(-x)\leqslant f(-x)$, hence $-f(-x)\leqslant Lx\leqslant f(x)$. As $-f(-x).\ f(x)\in V$, we have $Lx\in[V]=V$, namely $L(U_0)\subset V$. Therefore, we have proved that $L$ is continuous at 0, namely $L$ is continuous. This completes the proof.

**Theorem 3.** *Let $X$ be a topological vector space; $Y$ an ordered topological vector space, $C_y$ normal; and $f$: $X\to Y$ a convex map. If $f$ is bounded above on a non-empty open set, $f$ is continuous.*

*Proof* If $f$ has a upper bound $\bar{y}$ on a 0-neighborhood $U$, then $f$ is continuous at 0. In fact, we can assume that $U$ is circled and $f(0)=0$. $\forall V$, a neighborhood of 0 in $Y$, due to $C_y$'s normality, we can assume that $V$ is circled and $C_y$-saturated. There exists a number $\lambda$, $0<\lambda<1$, such that $\lambda\bar{y}\in V$. Hence, $\forall x\in\lambda U$, let $x=\lambda u$, we have $f(x)=f(\lambda u)=f(\lambda u+(1-\lambda)0)\leqslant\lambda f(u)\leqslant\lambda\bar{y}$. On the other hand,

$$0=f\left(\frac{1}{2}x-\frac{1}{2}x\right)\leqslant\frac{1}{2}(f(x)+f(-x))\leqslant\frac{1}{2}(f(x)+\bar{y}),$$

so $-\lambda\bar{y}\leqslant f(x)$. That is, $-\lambda\bar{y}\leqslant f(x)\leqslant\lambda\bar{y}$, $\forall x\in\lambda U$. Since $\lambda\bar{y}$, $-\lambda\bar{y}\in U$, $f(x)\in[V]=V$, that is, $f(\lambda U)\subset V$. Therefore, $f$ is continuous at 0.

If $f$ has a upper bound $\bar{y}$ on a neighborhood $x_0+u$ of a point $x_0$, where $U$ is a neighborhood of 0, then $f$ is continuous. In fact, to any $x_1\in X$, $\forall u\in V$, we have

$$f\left(x_1+\frac{1}{2}u\right)=f\left(\frac{1}{2}(x_0+u)+x_1-\frac{1}{2}x_0\right)\leqslant\frac{1}{2}f(x_0+u)+\frac{1}{2}f(2x_1-x_0)$$
$$\leqslant\frac{1}{2}\bar{y}+\frac{1}{2}f(2x_1-x_0),$$

that is, $f$ is bounded above on $x_1+\frac{1}{2}U$. Hence, to a convex map $g$, $g(x)=f(x_1+x)$, we know that $g$ is bounded above on the 0-neighborhood $\frac{1}{2}U$. From the result above, $g$ is continuous at 0. That is, $f$ is continuous at $x$. Therefore, $f$ is continuous.

**Theorem 4.** *Let $X$ be a vector space, $Y$ an order complete ordered vector space, and $A\subset X\times Y$ a convex set. If*

1) $\forall(0,y)\in A$, $y\geqslant0$,

2) *$\exists y\in Y$ such that $V_x=\{x\in X\mid(x,\hat{y})\in A\}$ is absorb, then there exists a linear map $A$: $X\to Y$ such that*

$$Ax+y\geqslant0,\quad(x,y)\in A.$$

*Proof* Take the convex cone $C=\bigcup_{\lambda>0}\lambda B$ into consideration. Of course, $C$ also satisfies the conditions of 1) and 2). According to this, we obtain $D\triangleq\{x|C_x\neq\emptyset\}=X$ and $C$ satisfies three conditions of Theorem 1. As a result, there exists a linear map $\Lambda$: $X\to Y$ such that $\Lambda_{x+y}\geqslant 0$, $\forall(x, y)\in C$. This completes the proof.

**Theorem 5.** *Let $X$ be a locally convex space; $Y$ an order complete ordered topological vector space, $C_y$ normal, and* $\operatorname{int} C_y\neq\emptyset$. *If $\Lambda_0$: $X_0\subset X\to Y$ is a continuous linear map and $X_0$ is a vector subspace, then there is a continuous linear extension $\Lambda$: $X\to Y$ of $\Lambda_0$.*

*Proof* Let $y_0\in\operatorname{int} C_y$, and we take a 0-neighborhood $V=y_0-C_y$. Since $\Lambda_0$ is continuous, $\Lambda_0^{-1}(V)$ is a neighborhood of 0 in $X_0$. Hence, there is a 0-neighborhood $U$ in $X$ such that $X_0\cap U=\Lambda_0^{-1}(V)$. But $X$ is a locally convex space, so there is a closed and circled convex 0-neighborhood $W\subset U$.

For $W$, we can obtain a continuous semi-norm $p$: $X\to R$, $p(x)=\inf\{t>0|x\in tW\}$, and have $W=\{x\in X|p(x)\leqslant 1\}$.

Hence $\forall x\in X_0$, we have $x/(p(x)+\varepsilon)\in W$, $\forall\varepsilon>0$. Therefore

$$\Lambda_0(x/(p(x)+\varepsilon))\in V,$$

that is, $\Lambda_0(x/(p(x)+\varepsilon))\leqslant y_0$. Then, $\Lambda_0 x\leqslant p(x)y_0+\varepsilon y_0$, $\forall\varepsilon>0$. Since $\varepsilon$ can be arbitrarily small and owing to the consistency condition of an ordered topological vector space, we have $\Lambda_0 x\leqslant p(x)y_0$.

By defining a continuous convex map $f$: $X\to Y$, $f(x)=p(x)y_0$, we have $\Lambda_0 x\leqslant f(x)$, $\forall x\in X_0$. Therefore, applying Theorem 2, we have a continuous linear extension $\Lambda$: $X\to Y$ of $\Lambda_0$.

In fact, all theorems and Lemma 1 except Theorem 3 in this section are equivalent to Hahn-Banach Theorem.

## § 4. General Convex Vector Programming

Let $X$ and $Z$ be vector spaces, and $Y_0$ an order complete ordered vector space. We associate two points $-\infty$ and $+\infty$ with $Y_0$, denoting it by $Y_0\cup\{\infty\}$. For any subset $A\subset Y_0$, we say $\sup A=+\infty$ if $A$ is non-empty and not bounded above; we say $\inf A=-\infty$ if $A$ is non-empty and not bounded below. Besides this, we say $\sup Y_0=+\infty$, $\inf Y_0=-\infty$, $\sup\emptyset=-\infty$, and $\inf\emptyset=+\infty$. As a result, to any subset in $Y_0\cup\{\infty\}$ there exists the supremum and the infimum. Here, we can assume $-\infty\neq+\infty$.

If there is a bilinear map $\langle\cdot, \cdot\rangle$: $X\times Z\to Y_0$, then, to any map $f$: $X\to Y_0\cup\{\infty\}$, we can define its conjugate $f^*$: $Z\to Y_0\cup\{\infty\}$, $f^*(z)=\sup\{\langle x, z\rangle-f(x)|x\in X\}$; and its secondary conjugate $f^{**}$: $X\to Y_0\cup\{\infty\}$, $f^{**}=(f^*)^*$. $f$ is a closed convex map if $f=f^{**}$.

**Proposition 1.** *$f^*$ and $f^{**}$ are closed convex.*

*Proof* $f^{**}(x)=\sup\{\langle x, z\rangle-f^*(z)\,|\,z\in Z\}=\sup\{\langle x, z\rangle-\alpha\,|\,\alpha\geqslant f^*(z);\ z\in Z\}=\sup\{\langle x, z\rangle-\alpha\,|\,\langle y, z\rangle-\alpha\leqslant f(z),\ \forall y\in X;\ z\in Z\}$, hence we know that if there exists a set $\{(z_\lambda, \alpha_\lambda)\,|\,z_\lambda\in Z,\ \alpha_\lambda\in Y_0,\ \lambda\in\Lambda\}$ such that $f(x)=\sup\ \{\langle x, z\rangle-\alpha_\lambda\,|\,\lambda\in\Lambda\}$, $\forall x\in X$, then $f$ is closed convex.

But $f^*(z)=\sup\{\langle x, z\rangle-f(x)\,|\,x\in X\}$, that is, the set $\{(x, f(x))\,|\,x\in X\}$ is that kind of set mentioned above, therefore $f^*$ is closed convex. For $f^{**}$, its position is symmetric with $f^*$ and it's also a conjugate ($f^*$'s). As is said above, any conjugate is closed convex, so $f^{**}$ is also closed convex.

From now on, we always let $X$ be a vector space, $Y$ an ordered topological vector space, $Y_0$ an order complete ordered topological vector space; and choose $Z=B(X, Y_0)$ and $\langle\cdot, \cdot\rangle$ for $\langle T, x\rangle=Tx$, that is, to any map $f$: $X\to Y_0\cup\{\infty\}$, we define its conjugate and secondary conjugate respectively as

$$f^*: B(X, Y_0)\to Y_0\cup\{\infty\},\ f^*(T)=\sup\{Tx-f(x)\,|\,x\in X\};$$

$$f^{**}: X\to Y_0\cup\{\infty\},\ f^{**}(x)=\sup\{Tx-f^*(T)\,|\,T\in B(X, Y_0)\}.$$

Besides this, we define a set-valued map $\partial f$: $X\to B(X, Y_0)$ as

$$\partial f(x)=\{T\in B(X, Y_0)\,|\,T(y-x)\leqslant f(y)-f(x),\ \forall y\in X\},$$

and we call $\partial f$ the subdifferential of $f$, where we say $\partial f(x)=B(X, Y_0)$ if $f(x)=+\infty$; and $\partial f(x)=\emptyset$ if $f(x)=-\infty$.

A map $f$: $X\to Y_0\cup\{\infty\}$ is convex if its epigragh

$$\text{epi } f=\{(x, y)\in X\times Y_0\,|\,f(x)\leqslant y\}$$

is convex. Let dom $f=\{x\,|\,f(x)\neq+\infty\}$. Clearly, we have the following result:

$f$ is convex$\Leftrightarrow\forall x_1, x_2\in$ dom $f$, $\lambda\in(0, 1)$,

$$f(\lambda x_1+(1-\lambda)x_2)\leqslant\lambda f(x_1)+(1-\lambda)f(x_2).$$

**Proposition 2.** *Let $f$: $X\to Y_0\cup\{\infty\}$. We have the following results:*

a) *If $f(x)$ is finite, then $T\in\partial f(x)\Leftrightarrow f(x)+f^*(T)=Tx$.*

b) *If $f$ is closed convex and $f(x)$ or $f^*(T)$ is finite, then $T\in\partial f(x)\Leftrightarrow x\in\partial f^*(T)$.*

*Proof* For a), $T\in\partial f(x)\Leftrightarrow T(y-x)\leqslant f(y)-f(x)$, $\forall y\in X\Leftrightarrow Ty-f(y)\leqslant Tx-f(x)$, $\forall y\in X\Leftrightarrow Tx-f(x)=\sup\ \{Ty-f(y)\,|\,y\in X\}=f^*(T)\Leftrightarrow f(x)+f^*(T)=Tx$. Then, we have a).

Applying a), we can have b).

For some convex map $f$: $x\to Y_0\cup\{\infty\}$, suppose that we study the following problem:

$$(P)\qquad \inf\{f(x)\,|\,x\in X\},$$

which is called a basic problem. Corresponding to the problem $(P)$, we give some convex map $\Phi$: $X\times Y\to Y_0\cup\{\infty\}$ such that $\Phi(x, 0)=f(x)$, $\forall x\in X$. $\Phi$ is called a perturbation. Besides this, we give a corresponding problem

$$(P^*)\qquad \sup\{-\Phi^*(0, T)\,|\,T\in B(Y, Y_0)\},$$

which is called the dual problem of Problem $(P)$ ralated to the perturbation $\Phi$. The values of $(P)$ and $(P^*)$ are respectively denoted by $\inf P$ and $\sup P^*$. Again, we define a convex map $h: Y \to Y_0 \cup \{\infty\}$,

$$h(y) = \inf\{\Phi(x, y) \mid x \in X\}.$$

**Proposition 3.** a) $\sup P^* \leqslant \inf P$;

b) $h^*(T) = \Phi^*(0, T)$, $T \in B(Y, Y_0)$;

c) $\sup P^* = h^{**}(0)$, $\inf P = h(0)$;

d) *The set of the solutions of* $(P^*)$ *such that* $\sup P^*$ *is finite is* $\partial h^{**}(0)$.

*Proof* a), b) and c) are immediate from the definitions. We prove d) below. Since $h^*$ and $h^{**}$ are both closed convex, from Proposition 2, we have

$$\hat{T} \in \partial h^{**}(0) \Leftrightarrow 0 \in \partial h^{***}(\hat{T}) = \partial h^*(\hat{T}) \Leftrightarrow h^*(\hat{T}) \leqslant h^*(T),\ \forall T \in B(Y, Y_0)$$
$$\Leftrightarrow \Phi^*(0, \hat{T}) \leqslant \Phi^*(0, T),\ \forall T \in B(Y, Y_0) \Leftrightarrow \hat{T}$$

is the solution of $(P^*)$ such that $\sup P^*$ is finite. This completes the proof.

**Definition 1.** *Basic Problem* $(P)$ *is stable if* $h(0)$ *is finite and* $\partial h(0) \neq \emptyset$.

**Proposition 4.** *Problem* $(P)$ *is stable* $\Leftrightarrow$ *Problem* $(P^*)$ *is solvable,* $\inf P = \sup P^*$, *and this value is finite. Moreover, at this moment,* $\partial h(0) = \partial h^{**}(0)$.

*Proof* When we know that $\hat{T} \in \partial h(0)$, since $\hat{T}y \leqslant h(y) - h(0)$, $\forall y \in Y$, we have

$$h^{**}(0) = \sup_T\{-h^*(T)\} \geqslant -h^*(\hat{T}) = -\sup_y\{\hat{T}y - h(y)\} \geqslant h(0),$$

namely $\sup P^* \geqslant \inf P$. But we always have $\sup P^* \leqslant \inf P$, so $\sup P^* = \inf P$.

If $\inf P = \sup P^*$ and this value is finite, then

$$\hat{T} \in \partial h^{**}(0) \Leftrightarrow h^{**}(0) + h^{***}(\hat{T}) = 0 \Leftrightarrow h^{**}(0) + h^*(\hat{T}) = 0 \Leftrightarrow h(0) + h^*(\hat{T})$$
$$= 0 \Leftrightarrow \hat{T} \in \partial h(0),$$

and this means that $\partial h^{**}(0) = \partial h(0)$. From these, we can come to our conclusion.

**Theorem 6** (Criterion of Stability). *If* $C_{y_0}$ *is normal,* $\inf P$ *is finite, and there exist* $x'_0, x_0 \in X$ *(perhaps* $x'_0 = x_0$*) such that* $\Phi(x_0, \cdot)$ *doesn't take* $+\infty$ *on an absorb set and* $\Phi(x'_0, \cdot)$ *is continuons at some point, then* $(P)$ *is stable.*

*Proof* $\forall y \in Y$, since $\Phi(x_0, \cdot)$ doesn't take $+\infty$ on an absorb set, we have

$$\Phi(x_0, ty) \neq +\infty$$

when $t$ is small enough. But

$$h(ty) \leqslant \Phi(x_0, ty),$$

so $ty \in \operatorname{dom} h$, that is, $\operatorname{dom} h$ is absorb.

On the other hand, $\forall y \in Y$, we can get $\lambda_0 \in (0,1)$ such that $\{(\lambda_0 - 1)y/\lambda_0 \in \operatorname{dom} h$. And we can assume $y \in \operatorname{dom} h$, otherwise we have had $h(y) \neq -\infty$. From

$$0 = \lambda_0((\lambda_0 - 1)y/\lambda_0) + (1 - \lambda_0)y,$$

we have

$$\inf P = h(0) \leqslant \lambda_0 h(\lambda_0 - 1)y/\lambda_0) + (1 - \lambda_0)h(y).$$

Then there must be $h(y) \neq -\infty$. That is, $h$ does not take the value of $-\infty$.

Let $A=\{(y, y_0)\in Y\times Y_0 | h(y)-h(0)\leqslant y_0\}$. We have $D\triangleq\{y\in Y | A_y\neq\emptyset\}=\mathrm{dom}\, h$. Since $h$ doesn't take $-\infty$, we can see that $A$ satisfies three conditions of Theorem 1. Then there exists a linear map $\Lambda$: $Y\to Y_0$ such that

$$\forall (y, h(y)-h(0))\in A, \Lambda y+h(y)-h(0)\geqslant 0.$$

That is

$$-\Lambda y\leqslant h(y)-h(0), \ \forall y\in Y.$$

But $h(y)\leqslant \Phi(x_0', y)$, so $-\Lambda y\leqslant \Phi(x_0', y)-h(0)$. Since $\Phi(x', \cdot)$ is continuous at some point, similar to the proof of $L$'s continuity in Theorem 2, we know that $\Lambda$ is continuous. Then $-\Lambda\in\partial h(0)$. So $(P)$ is stable.

**Proposition 5.** *$\hat{x}\in X$, $\hat{T}\in B(Y, Y_0)$ are respectively the solutions of $(P)$ and $(P^*)$, $\inf P=\sup P^*$, and this value is finite $\Leftrightarrow \Phi(\hat{x}, 0)+\Phi^*(0, \hat{T})=0$.*

*Proof* $\Rightarrow$: From Proposition 4, $(P)$ is stable. Then $\hat{T}\in\partial h(0)$. So $h(0)+h^*(\hat{T})=0$. But, from Proposition 3, $h^*(\hat{T})=\Phi^*(0, \hat{T})$; and $\hat{x}$ is a solution, then $\Phi(\hat{x}, 0)=h(0)$. Therefore $\Phi(\hat{x}, 0)+\Phi^*(0, \hat{T})=0$. $\Leftarrow$: Since $\inf P\leqslant\Phi(\hat{x}, 0)=-\Phi^*(0, \hat{T})\leqslant \sup P^*\leqslant\inf P$, we have $\inf P=\Phi(\hat{x}, 0)=-\Phi^*(0, \hat{T})=\sup P^*$. Therefore, we have the conclusion.

## § 5. Saddle Point and Lagrangian

**Definition 2.** *Let $A$ and $B$ are two sets. For some map $L$: $A\times B\to Y_0\cup\{\infty\}$, we call $(a, b)$ the saddle point of $L$ if $L(a, b)$ is finite and*

$$L(a, y)\leqslant L(a, b)\leqslant L(x, b), \ \forall x\in A, y\in B.$$

We clearly have

**Proposition 6** (Saddle Point Theorem). *The following two conditions are equivalent:*

1) *$(a, b)$ is the saddle point of $L$: $A\times B\to Y_0\cup\{\infty\}$;*

2) *$a$ is the solution of $\inf\limits_A \sup\limits_B L(x, y)$; $b$ is the solution of $\sup\limits_B \inf\limits_A L(x, y)$; and the values of the two problems are finite and equal (to be $L(a, b)$).*

**Definition 3.** *The Lagrangian of Problem $(P)$ related to the perturbation $\Phi$ is defined as $L$: $X\times B(Y, Y_0)\to Y_0\cup\{\infty\}$ such that*

$$L(x, T)=-\sup\{Ty-\Phi(x, y) | y\in Y\}.$$

Since

$$\inf_x L(x, T)=\inf_x\{-\sup_y[Ty-\Phi(x, y)]\}=-\sup_{x,y}\{Ty-\Phi(x, y)\}=-\Phi^*(0, T),$$

the Dual Problem $(P^*)$ can be written in the following form:

$$(P^*) \quad \sup_{B(Y,Y_0)}\inf_X L(x, T).$$

If we let $\Phi_x$ represent the map $\Phi(x, \cdot)$, then $L(x, T)=-\Phi_x^*(T)$. Hence

$$\sup_T\{L(x, T)\}=\sup_T\{-\Phi_x^*(T)\}=\Phi_x^{**}(0).$$

Therefore, if $\forall x\in X$, $\Phi_x(0)=\Phi_x^{**}(0)$ or $\Phi_x$ is closed convex, then the Basic Problem $(P)$ can be written in the following form

$$(P) \quad \inf_X \sup_{B(Y,Y_0)} L(x, T).$$

From Saddle Point Theorem, we have:

**Proposition 7.** *If* $\forall x\in X$, $\Phi_x(0)=\Phi_x^{**}(0)$ *or* $\Phi_x$ *is closed convex, then the following two conditions are equivalent:*

a) $(\hat{x}, \hat{T})$ *is the saddle point of the Lagrangian* $L$ *of Problem* $(P)$;

b) $\hat{x}$ *is the solution of* $(P)$, $\hat{T}$ *is the solution of* $(P^*)$, $\inf P=\sup P^*$, *and this value is finite.*

## § 6. Rockafellar Problem

For a given convex map $J$: $X\times Y\to Y_0\cup\{\infty\}$ and a linear map $\Lambda$: $X\to Y$, we consider the basic problem

$$(\tilde{P}) \quad \inf\{J(x, \Lambda x)\,|\,x\in X\}.$$

We define $\Phi$: $X\times Y\to Y_0\cup\{\infty\}$ such that $\Phi(x, y)=J(x, \Lambda x-y)$ as its perturbation. It is easy to find that the dual problem of $(\tilde{P})$ is

$$(\tilde{P}^*) \quad \sup\{-J^*(T\Lambda, -T)\,|\,T\in B(Y, Y_0)\}.$$

When $J(x, y)=F(x)+G(y)$, where $F$ and $G$ are convex, it is the original Rockafellar Problem, i. e. Fenchel Problem.

From Theorem 6, we have

**Proposition 8.** *If* $C_{y_0}$ *is normal,* $\inf \tilde{P}$ *is finite, and there exists* $x_0\in X$ *such that* $J(x_0, \cdot)$ *is continuous at* $y=\Lambda x_0$, *then Problem* $(\tilde{P})$ *is stable.*

From Proposition 5, we have

**Proposition 9.** $\hat{x}\in X$, $\hat{T}\in B(Y, Y_0)$ *are respectively the solutions of* $(\tilde{P})$ *and* $(\tilde{P}^*)$, $\inf \tilde{P}=\sup \tilde{P}^*$, *and this value is finite* $\Leftrightarrow$ $J(\hat{x}, \Lambda\hat{x})+J^*(\hat{T}\Lambda, -\hat{T})=0$.

## § 7. Kuhn-Tucker's Theorem

Let $Y_1$ be an ordered topological vector space, $Y_2$ a topological vector space, and $X$ a topological vector space. Let $f$: $D(f)\subset X\to Y_0$ and $g$: $D(g)\subset X\to Y_0$ are convex maps, and $h$: $X\to Y_2$ an affine map. Applying the results above, we discuss the following problem of convex vector programming

$$(\hat{P}) \quad \begin{cases} f(x)\to\min \\ g(x)\leqslant 0 \\ h(x)=0. \end{cases}$$

When $X=R^n$, $Y_0=R$, $Y_1=R^p$, and $Y_2=R^q$, $p$, $q$, $n\in N$, it's the classical Kuhn-

Tucker Problem.

Let $X_0=\{x\mid h(x)=0\}$. For convenience, we mark $H$ the hypothetic conditions:

$H$: $C_{y_0}$ is normal, $\operatorname{int} C_{y_0}\neq\emptyset$, $\operatorname{int} C_{y_1}\neq\emptyset$; $Y_2$ is a locally convex space; $D^i(f)\cap D^i(g)\cap X_0\neq\emptyset$; $f$ is continuous at a point of $\operatorname{int} D(f)$, $h$ is an open map from $X$ to $h(X)$; and $\exists\, x_0\in D(f)\cap D(g)$, $g(x_0)<0$, $h(x_0)=0$.

**Lemma 2.** *To problem*

$$(\hat{P}_1)\ \begin{cases} f(x)\to\min\\ g(x)\leqslant 0,\end{cases}$$

*we take perturbation* $\Phi_1$: $X\times Y_1\to Y_0\cup\{\infty\}$ *such that* $\Phi_1(x, y)=f(x)$, *if* $g(x)\leqslant y$; $\Phi_1(x, y)=+\infty$, *otherwise. If* $C_{y_0}$ *is normal,* $\inf\hat{P}_1$ *is finite, int* $C_{y_1}\neq\emptyset$, *and* $\exists x_0\in D(f)\cap D(g)$, $g(x_0)<0$, *then* $(\hat{P}_1)$ *is stable.*

*Proof* Since $\Phi_1(x_0, \cdot)$ constantly take $f(x_0)$ on $\{y\in Y_1\mid y\geqslant g(x_0)\}$, from Theorem 6, $(\hat{P}_1)$ is stable.

**Lemma 3.** *If* $C_{y_0}$ *is normal,* $\operatorname{int} C_{y_0}\neq\emptyset$, $f$ *is continuous at a point of* $\operatorname{int} D(f)$, $h$ *is an open map from* $X$ *to* $h(X)$, *and* $Y_2$ *is locally convex, then for problem* $(\hat{P}_2)$

$$(\hat{P}_2)\ \begin{cases} f(x)\to\min\\ h(x)=0,\end{cases}$$

*if* $\inf\hat{P}_2$ *is finite and* $D^i(f)\cap X_0\neq\emptyset$, *there exists a continuous linear map* $\hat{M}$: $Y_2\to Y_0$, *such that*

$$f(x)+\hat{M}\circ h(x)\geqslant\inf\hat{P}_2,\ \forall x\in D(f).$$

*Proof* We can assume that $\inf\hat{P}_2=0$ and $h$ is linear. Take

$$A=\{(h(x), y)\mid y\geqslant f(x)\}.$$

We know that $A$ is convex. We discuss $A$ in $h(X)\times Y_0$.

$\forall(0, y)\in A$, since $h(x)=0$, hence $f(x)\geqslant\inf\hat{P}_2=0$, so $y\geqslant f(x)\geqslant 0$, namely $y\geqslant 0$.

Take $\hat{y}=y_0+f(x_0)$, $x_0\in D^i(f)\cap X_0$ and $y_0\in\operatorname{int} C_{y_0}$. Let

$$V_x=\{u\mid(u, \hat{y})\in A\}.$$

$\forall u\in h(X)$, we suppose $u=h(x)$. Since $x_0\in D^i(f)$, hence $\exists\delta>0$, $\forall\lambda\in(0, \delta)$, $x_0+\lambda x\in D(f)$. But

$$f(\lambda x+x_0)=f\left(\frac{\lambda}{\delta}(\delta x+x_0)+\left(1-\frac{\lambda}{\delta}\right)x_0\right)\leqslant\frac{\lambda}{\delta}f(\delta x+x_0)+\left(1-\frac{\lambda}{\delta}\right)f(x_0)$$
$$=f(x_0)+\frac{\lambda}{\delta}(f(\delta x+x_0)-f(x_0)),$$

and $y_0>0$, so when $\lambda\geqslant 0$ is much smaller, there is $\frac{\lambda}{\delta}(f(\delta x+x_0)-f(x_0))\leqslant y_0$. At this moment, $f(\lambda x+x_0)\leqslant f(x_0)+y_0=\hat{y}$. On the other hand, $\lambda u=h(\lambda x)=h(\lambda x+x_0)$. Then $(\lambda u, \hat{y})\in A$, that is, $\lambda u\in V_x$. Therefore, $V_x$ is an absorb set in $h(X)$.

So far, we have proved that $A$ satisfies the conditions of Theorem 4. Therefore there exists a linear map $\hat{M}_0$: $h(X)\to Y_0$ such that $\hat{M}_0\circ h(x)+f(x)\geqslant 0$, $\forall x\in D(f)$. Since $f$ is continuous at a point of $\operatorname{int} D(f)$, similar to the proof of Theorem 2, $\hat{M}_0\circ h$ is continuous. For any 0-neighborhood $V$ in $Y_0$, there is a 0-neighborhood $U$

in $X$ such that $\hat{M}_0 \circ h(U) \subset V$. But $h$ is open, so $h(U)$ is a neighborhood of 0 in $h(X)$, that is, there is $W = h(U)$ such that $\hat{M}_0(W) \subset V$. Therefore $\hat{M}_0$ is continuous in $h(X)$.

Moreover, from Theorem 5, there is $\hat{M}: X \to Y_0$, a continuous linear extension of $\hat{M}_0$. This completes the proof.

**Remark.** The conditions "$f$ is continuous at a point of int $D(f)$" and "$h$ is an open map from $X$ to $h(X)$" are requisite. In fact, we know that the first condition is necessary if we take $h = I$ and let $f(0) = 0$. If we take $f = -I$ and let $h$ be an isomorphism, then $\hat{M} \circ h = I$, that is, the inverse map $\hat{M}$ of $h$ is continuous, so $h$ is open. This means that the second condition is necessary.

From now on, we begin to use $T$ to represent the element of $B(Y_1, Y_0)$, and $M$ the element of $B(Y_2, Y_0)$. Let $B^+(Y_1, Y_0) = \{T \geqslant 0\}$, and

$$K = \{x \in X \mid g(x) \leqslant 0,\ h(x) = 0,\ x \in D(f) \cap D(g)\}.$$

We take the perturbation $\Phi$: $X \times Y_1 \times Y_2 \to Y_0 \cup \{\infty\}$ of $(\hat{P})$ such that

$$\Phi(x, y_1, y_2) = \begin{cases} f(x), \text{ if } g(x) \leqslant y_0,\ h(x) = y_2 \\ +\infty, \text{ otherwise.} \end{cases}$$

Since

$$\begin{aligned} -\Phi^*(0, -T, -M) &= -\sup_{\lambda, y_1, y_2} [-Ty_1 - My_2 - \Phi(x, y_1, y_2)] \\ &= \inf [\Phi(x, y_1, y_2) + Ty_1 + My_2] \\ &= \inf [f(x) + Ty_1 + M \circ h(x) \mid g(x) \leqslant y_1] \\ &= \begin{cases} -\infty, \text{ if } T \not\geqslant 0 \\ \inf [f(x) + Ty_1 + M \circ h(x) \mid g(x) \leqslant y_1], \text{ if } T \geqslant 0 \end{cases} \\ &= \begin{cases} -\infty, \text{ if } T \not\geqslant 0, \\ \inf_x [f(x) + T \circ g(x) + M \circ h(x)], \text{ if } T \geqslant 0, \end{cases} \end{aligned}$$

where, if $-\Phi^*(0, -T, -M)$ is finite, $\forall y_1' \geqslant 0$, we have $nTy_1' \geqslant -Ty_1 - M \circ h(x) - f(x) - \Phi^*(0, -T, -M)$, $\forall n \in N$, and since $Y_0$ is order complete, we have $Ty_1' \geqslant 0$, that is, at this moment, $T \geqslant 0$; but if $D(f) \cap D(g) \neq \emptyset$, there is $-\Phi^*(0, -T, -M) \neq +\infty$, therefore the dual problem of $(\hat{P})$ is

$$(\hat{P}^*) \quad \sup_{T \geqslant 0, \forall M} \inf_{x \in D(f) \cap D(g)} [f(x) + T \circ g(x) + M \circ h(x)].$$

**Theorem 7.** *If* inf $\hat{P}$ *is finite and Condition H holds, then Problem* $(\hat{P})$ *is stable.*

*Proof* At first, we consider the problem

$$(\hat{P}_3) \begin{cases} f(x) \to \min \\ g(x) \leqslant 0, \end{cases}$$

where $f$ is defined on $D(f) \cap X_0$. We know that inf $\hat{P}_3 =$ inf $\hat{P}$ is finite. $(\hat{P}_3)$ satisfies the conditions of Lemma 2, so $(\hat{P}_3)$ is stable. Taking $h = 0$ in $(\hat{P}^*)$, we obtain the dual problem of $(\hat{P}_3)$

$$(\hat{P}_3^*) \sup_{T \geqslant 0} \inf_{x \in D(f) \cap D(g) \cap X_0} [f(x) + T \circ g(x)].$$

From Proposition 4, $(\hat{P}_3^*)$ is solvable and $\sup \hat{P}_3^* = \inf \hat{P}_3$, that is, there exists a positive continuous linear map $\hat{T}: Y_1 \to Y_0$ such that

$$f(x) + \hat{T} \circ g(x) \geqslant \sup \hat{P}_3^* = \inf \hat{P}, \ \forall x \in D(f) \cap D(g) \cap X_0.$$

Hence, to problem

$$(\hat{P}_4) \begin{cases} f(x) + \hat{T} \circ g(x) \to \min \\ h(x) = 0, \end{cases}$$

we know that $\inf \hat{P}_4 = \inf \hat{P}$ is finite, where $f(\cdot) + \hat{T} \circ g(\cdot)$ is defined on $D(f) \cap D(g)$. $(\hat{P}_4)$ satisfies the conditions of Lemma 3. Therefore there exists a continuous linear map $\hat{M}: Y_2 \to Y_0$ such that

$$f(x) + \hat{T} \circ g(x) + \hat{M} \circ h(x) \geqslant \inf \hat{P}_4 = \inf \hat{P}, \ \forall x \in D(f) \cap D(g).$$

Moreover, $\sup \hat{P}^* \geqslant \inf [f(x) + \hat{T} \circ g(x) + \hat{M} \circ h(x) \mid x \in D(f) \cap D(g)\} \geqslant \inf \hat{P}$, that is, $\inf \hat{P} = \sup \hat{P}^*$. Again from Proposition 4, $(\hat{P})$ is stable. This completes the proof.

From Theorem 7 and Proposition 4, we clearly have

**Theorem 8** (Kuhn-Tucker's Theorem). *If Condition $H$ holds, then Problem $(\hat{P})$ has the solution $x = \hat{x} \Leftrightarrow \hat{x} \in K$, and there exists a positive continuous linear map $\hat{T}: Y_1 \to Y_0$, and a continuous linear map $\hat{M}: Y_2 \to Y_0$ such that*

$$f(x) + \hat{T} \circ g(x) + \hat{M} \circ h(x) \geqslant f(\hat{x}), \ \forall x \in D(f) \cap D(g).$$

*Moreover,* $\hat{T} \circ g(\hat{x}) = 0$.

If we take

$$L: D(f) \cap D(g) \times B^+(Y_1, Y_0) \times B(Y_2, Y_0) \to Y_0,$$
$$L(x, T, M) = f(x) + V \circ g(x) + M \circ h(x),$$

then, we can have Theorem 8 in another way.

**Theorem 9.** *If Condition $H$ holds, then: Problem $(\hat{P})$ has the solution $x = \hat{x} \Leftrightarrow \hat{x} \in K$, and there exists a positive continuous linear map $\hat{T}: Y_1 \to Y_0$ and a continuous linear map $\hat{M}: Y_2 \to Y_0$ such that $(\hat{x}, (\hat{T}, \hat{M}))$ is the saddle point of $L$. Moreover,*

$$\hat{T} \circ g(\hat{x}) = 0.$$

**Theorem 10.** *If Conditiin $H$ holds, then: $(\hat{x}, (\hat{T}, \hat{M}))$ is the saddle point of $L$ $\Leftrightarrow$ $\hat{x}$ is a solution of $(\hat{P})$, $(\hat{T}, \hat{M})$ is a solution of $(\hat{P}^*)$ and* $\inf \hat{P} = \sup \hat{P}^*$.

*Proof* $\Rightarrow$: Applying Theorem 9's "$\Leftarrow$", we know that $\hat{x}$ is a solution of $(\hat{P})$ and $\hat{T} g(\hat{x}) = 0$. From the Saddle Point Theorem, we know that $(\hat{T}, \hat{M})$ is a solution of $(\hat{P}^*)$ and $\inf \sup L = \sup \inf L = L(\hat{x}, \hat{T}, \hat{M}) = f(\hat{x})$, that is, $\inf \hat{P} = f(\hat{x}) = \sup \inf L = \sup \hat{P}^*$.

$\Leftarrow$: Since $(\hat{T}, \hat{M})$ is a solution of $(\hat{P})$, $\inf L(x, \hat{T}, \hat{M}) = \sup \hat{P}^* = \inf \hat{P} = f(\hat{x})$, that is, $L(x, \hat{T}, \hat{M}) \geqslant f(\hat{x})$, $\forall x \in D(f) \cap D(g)$. If we take $x = \hat{x}$, then $\hat{T} \circ g(\hat{x}) \geqslant 0$. But $g(\hat{x}) \leqslant 0$ and $\hat{T} \geqslant 0$, so $\hat{T} \circ g(\hat{x}) \leqslant 0$. Therefore $\hat{T} \circ g(\hat{x}) = 0$. On the other hand, we always have $L(\hat{x}, T, M) \leqslant f(\hat{x}) = L(\hat{x}, \hat{T}, \hat{M})$. Therefore

$$L(\hat{x}, T, M) \leqslant L(\hat{x}, \hat{T}, \hat{M}) \leqslant L(x, \hat{T}, \hat{M}), \ \forall x \in D(f) \cap D(g),$$
$$T \in B^+(Y_1, Y_0), \ M \in B(Y_2, Y_0).$$

That is, $(\hat{x}, (\hat{T}, \hat{M}))$ is the saddle point of $L$.

Usually one obtains the last two results by changing $(\hat{P})$ into the following form

$$\inf_{x\in D(f)\cap D(g)} \sup_{T\geqslant 0,\forall M} L(x, T, M),$$

and then applying the Saddle Point Theorem. But this requires "$\forall x\in X$, $\Phi_x^{**}(0)=\Phi_x(0)$". Hence, this requires that $Y_1$ has the property "$Ty_1\leqslant 0$, $\forall T\geqslant 0 \Rightarrow y_1\leqslant 0$". This condition is much stronger. Generally, we only have "$Ty_1\leqslant 0$, $\forall T\geqslant 0 \Rightarrow y_1 \not> 0$". Therefore, we try to make use of supinf $L$, a form of $(\hat{P}^*)$, and to apply Theorem 7, and deduce the last two result. Thus, on one hand, the condition required are weaker; but on the other hand, this makes the conditions consistent in discussing the Kuhn-Tucker's problems.

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