Chin. Ann. of Math.
7B (2) 1986

ESTIMATE OF d_0/d^* FOR STARLIKE FUNCTIONS

HUANG XINZHONG (黄心中)*

Abstract

Let S^* be the class of functions $f(z)$ analytic, univalent in the unit disk $|z|<1$ and map $|z|<1$ onto a region which is starlike with respect to $w=0$ and is denoted as D_f. Let $r_0=r_0(f)$ be the radius of convexity of $f(2)$.

In this note, the author proves the following result:

$$\frac{d_0}{d^*}\geqslant 0.4101492,$$

where $d_0=\min\limits_{|z|=r_0} f(z)$, $d^*=\inf\limits_{\beta\in D_f}|\beta|$.

§ 1. Introduction

Let S^* be the class of functions $f(z)$ analytic, univalent in the unit disk $|z|<1$ and map $|z|<1$ onto a region which is starlike with respect to $w=0$ and is denoted as D_f. Let $r_0=r_0(f)$ be the radius of convexity of $f(z)$ (see Hayman[1] for the definition). Put $d_0=\min\limits_{|z|=r_0}|f(z)|$ and $d^*=\inf\limits_{\beta\in D_f}|\beta|$. Then in 1953, A. Schild[2] proved that $d_0/d^*\geqslant 2-\sqrt{3}=0.268\cdots$ and conjectured that $d_0/d^*\geqslant 2/3$ for all $f(z)\in S^*$ $\left(\text{where the } 2/3 \text{ is attained for the function } f(z)=\frac{z}{(1+z)^2}\right)$. He also proved the conjecture for p symmetric functions, $p\geqslant 7$. Lewandowski[3] proved in 1956 that the conjecture is true for certain subclasses of S^*. It was not until 1970 that Tepper[4] proved that $d_0/d^*\geqslant 0.343\cdots$, in quick succession McCarty and Tepper[5] proved that $d_0/d^*\geqslant 0.380\cdots$. However, Barnard and Lewis[6] showed that Schild's conjecture is false by giving two counterexamples in 1973. After commenting on the counterexamples of Barnard and Lewis in American Mathematics Review Vol. 48 in 1974, the reviewer remarked that the exact greatest lower bound for d_0/d^* is still an open question, At the conference on complex function theory held at the State University College at Brockport, New York in 1976, Schild put forward again the problem of finding the exact greatest lower bound for d_0/d^*. Recently Chen Yueqing proved that $d_0/d^*\geqslant 0.38177\cdots$. In this note we prove the following

Theorem 1. $d_0/d^*\geqslant 0.4101492$.

Manuscript received February 6, 1984.

* Deparment of Mathematics, Hua Chiao University, Chuanchow, Fukien, China.

§ 2. Preliminaries

Lemma 1. *If $a\geqslant 0$, $w=f_a(z)=z+az^2+\cdots\in S^*$. Let r_0 be the radius of convexity of $f_a(z)$, then*

$$r_0\geqslant r_0(a)=\frac{a+\sqrt{a^2+32}-\sqrt{[2a^2+2a\sqrt{(a^2+32)}+16]}}{4}, \tag{1}$$

where the equality holds when

$$f_a(z)=\frac{z}{1-az+z^2}.$$

Lemma 2. *The function*

$$r_0(a)=\frac{a+\sqrt{a^2+32}-\sqrt{[2a^2+2a\sqrt{(a^2+32)}+16]}}{4}$$

is monotone decreasing for $0\leqslant a\leqslant 2$ and $r_0(a)$ satisfies the following equations

$$1-ar_0(a)-6r_0^2(a)-ar_0^3(a)+r_0^4(a)=0, \tag{2}$$

$$r_0'(a)=\frac{r_0(a)[1+r_0^2(a)]}{4r_0^3(a)-3ar_0^2(a)-12r_0(a)-a}, \tag{3}$$

$$4r_0^3(a)-3ar_0^2(a)-12r_0(a)-a<0, \text{ for } 0\leqslant a\leqslant 2. \tag{4}$$

Lemmas 1 and 2 are due to Tepper.[4]

Lemma 3.[7] *Let $f(z)=z+az^2+\cdots$, $a\geqslant 0$, be regular and univalent in $|z|<1$. Then*

$$d^*\leqslant\left(\frac{4+\sqrt{2(2-a)}}{6+a}\right)^2, \tag{5}$$

and the inequality is sharp.

Lemma 4. *Let $f(z)=z+az^2+\cdots$, $a\geqslant 0$ be regular and univalent in $|z|<1$. Then*

$$d_0/d^*\geqslant\left(\frac{1}{\sqrt{\frac{(1-r_0)^2}{4r_0}d^*+1}+\sqrt{\frac{(1-r_0)^2}{4r_0}d^*}}\right)^2. \tag{6}$$

Proof Let $z=g(w)$ denote the inverse function to $w=f(z)$, then the function

$$K(\zeta)=d^{*-1}\frac{g(d^*\zeta)}{(1-e^{i\phi}g(d^*\zeta))^2}=\zeta+\sum_{n=2}^{\infty}b_n\zeta^n$$

is regular and univalent for $|\zeta|<1$, $-\pi<\phi\leqslant\pi$. We have by the classical distortion theorem

$$d^{*-1}\cdot\frac{|g(d^*\zeta)|}{|1-e^{i\phi}g(d^*\zeta)|^2}\leqslant\frac{|\zeta|}{(1-|\zeta|)^2},\ |\zeta|<1. \tag{7}$$

Set $\zeta=\frac{w_0}{d^*}$, $\phi=-\arg z_0$ in (7), where $w_0=f(z_0)$ with $|z_0|=r_0$ and $|w_0|=d_0$, we obtain

$$d^*\frac{r_0}{(1-r_0)^2}\leqslant\frac{d_0/d^*}{(1-d_0/d^*)^2},$$

$$(1-d_0/d^*)\leqslant\sqrt{d_0}\frac{1-r_0}{\sqrt{r_0}},$$

$$d_0+\sqrt{d_0}\frac{1-r_0}{\sqrt{r_0}}d^*-d^*\geqslant 0,$$

$$\left(\sqrt{d_0}+\frac{1-r_0}{2\sqrt{r_0}}d^*\right)^2-d^*\left(1+\frac{(1-r_0)^2}{4r_0}d^*\right)\geqslant 0,$$

$$\left[\sqrt{d_0}+\frac{1-r_0}{2\sqrt{r_0}}d^*-\sqrt{d^*\left(\frac{(1-r_0)^2}{4r_0}d^*+1\right)}\right]\left[\sqrt{d_0}+\frac{1-r_0}{2\sqrt{r_0}}d^*\right.$$

$$\left.+\sqrt{d^*\left(\frac{(1-r_0)^2}{4r_0}d^*+1\right)}\right]\geqslant 0.$$

Hence we have

$$\sqrt{d_0}\geqslant\sqrt{d^*\left(\frac{(1-r_0)^2}{4r_0}d^*+1\right)}-\frac{1-r_0}{2\sqrt{r_0}}d^*,$$

$$\sqrt{d_0}\geqslant\sqrt{d^*}\left(\sqrt{\frac{(1-r_0)^2}{4r_0}d^*+1}-\sqrt{\frac{(1-r_0)^2}{4r_0}d^*}\right),$$

$$\sqrt{d_0/d^*}\geqslant\frac{1}{\sqrt{\frac{(1-r_0)^2}{4r_0}d^*+1}+\sqrt{\frac{(1-r_0)^2}{4r_0}d^*}},$$

which completes the proof of Lemma 4.

§ 3. Estimates for d_0/d^*

Let $f(z)=z+az^2+\cdots\in S^*$, $a\geqslant 0$, by Lemma 4 we have

$$\sqrt{\frac{d_0}{d^*}}\geqslant\frac{1}{\sqrt{\frac{(1-r_0)^2}{4r_0}d^*+1}+\sqrt{\frac{(1-r_0)^2}{4r_0}d^*}}.$$

Since $\frac{(1-r_0)^2}{4r_0}$ is monotone decreasing for $0<r_0<1$, we obtain by (1) and (5)

$$\sqrt{d_0/d^*}\geqslant\frac{1}{\sqrt{\frac{(1-r_0(a))^2}{4r_0(a)}\left(\frac{2+\sqrt{1-a/2}}{3+a/2}\right)^2+1}+\sqrt{\frac{(1-r_0(a))^2}{4r_0(a)}\left(\frac{2+\sqrt{1-a/2}}{3+a/2}\right)^2}}. \tag{8}$$

We wish to estimate the right side of (8). Put

$$G(a)=\frac{1-r_0(a)}{2\sqrt{r_0(a)}}\left(\frac{2+\sqrt{1-a/2}}{3+a/2}\right).$$

Using the expression of $r_0'(a)$ in (3), we have

$$G'(a)=-\frac{1}{4\sqrt{r_0(a)}\,(3+a/2)}\left[\frac{2(1-r_0(a))}{3+a/2}+\frac{(5-a/2)(1-r_0(a))}{2(3+a/2)\sqrt{1-a/2}}\right.$$

$$\left.-\frac{(1+r_0(a))(1+r_0^2(a))(2+\sqrt{1-a/2})}{3ar_0^2(a)+12r_0(a)+a-4r_0^3(a)}\right].$$

Let $f_1(a)=\frac{5-a/2}{(3+a/2)\sqrt{1-a/2}}$, we want to show $f_1(a)$ is monotone decreasing for $a\in[0,\ 18-8\sqrt{5}]$ and is monotone increasing for $a\in[18-8\sqrt{5},\ 2]$.

In fact

$$f_1'(a)=-\frac{1-9a+a^2/4}{4(3+a/2)^2(1-a/2)^{3/2}},$$

$$=-\frac{(a-18-8\sqrt{5})(a-18+8\sqrt{5})}{16(3+a/2)^2(1-a/2)^{3/2}}.$$

Thus $f_1'(a)\leqslant 0$ for $a\in[0,\ 18-8\sqrt{5}]$ and $f_1'(a)\geqslant 0$ for $a\in[18-8\sqrt{5},\ 2]$. This is what we want to prove. Since $(1-r_0(a))$ is monotone increasing for $a\in[0,\ 2]$, $(1-r_0(a))f_1(a)$ is monotone increasing for $a\in[18-8\sqrt{5},\ 2]$.

Let $f_2(a)=3ar_0^2(a)+12r_0(a)+a-4r_0^3(a)$, then

$$f_2'(a)=6(2+ar_0(a)-2r_0^2(a))r_0'(a)+1+3r_0^2(a).$$

Applying (3) to this equation, we obtain

$$f_2'(a)=\frac{-8r_0^3(a)+3ar_0^2(a)-a}{4r_0^3(a)-3ar_0^2(a)-12r_0(a)-a}(1+r_0^2(a))+2r_0^2(a)$$

$$=\frac{-8r_0^3(a)-a(1-3r_0^2(a))}{4r_0^3(a)-3ar_0^2(a)-12r_0(a)-a}(1+r_0^2(a))+2r_0^2(a).$$

According to Lemma 2, we know $2-\sqrt{3}\leqslant r_0(a)\leqslant\sqrt{2}-1$ and obtain by (4) $f_2'(a)>0$. Hence $f_2(a)$ is monotone incresing for $a\in[0,\ 2]$.

Let $f_3(a)=(1+r_0(a))(1+r_0^2(a))(2+\sqrt{1-a/2})$, then

$$f_3'(a)=(1+2r_0^2(a)+3r_0^3(a))(2+\sqrt{1-a/2})r_0'(a)-\frac{(1+r_0(a))(1+r_0^2(a))}{4\sqrt{1-a/2}}.$$

We have by (3) $f_3'(a)<0$. Thus $f_3(a)$ is monotone decreasing for $a\in[0,2]$.

Because $f_2(a)$ and $f_3(a)$ are positive functions, we see that

$$f_4(a)=\frac{f_3(a)}{f_2(a)}=\frac{(1+r_0(a))(1+r_0^2(a))(2+\sqrt{1-a/2})}{3ar_0^2(a)+12r_0(a)+a-4r_0^3(a)}$$

is monotone decreasing for $a\in[0,\ 2]$.

Let $f_5(a)=\frac{1-r_0(a)}{3+a/2}$, then

$$f_5'(a)=\frac{-r_0'(a)\left(3+\frac{a}{2}\right)-\frac{1}{2}(1-r_0(a))}{(3+a/2)^2}.$$

Putting $f_6(a)=-r_0'(a)(3+a/2)-\frac{1}{2}(1-r_0(a))$, we obtain by (3) and (2)

$$f_6(a)=\frac{r_0(a)(1+r_0^2(a))(3+a/2)}{3ar_0^2(a)+12r_0(a)+a-4r_0^3(a)}-\frac{1}{2}(1-r_0(a))$$

$$=\frac{-4r_0^4(a)+10r_0^3(a)+4ar_0^3(a)-3ar_0^2(a)+12r_0^2(a)-6r_0(a)+2ar_0(a)-a}{2[3ar_0^2(a)+12r_0(a)+a-4r_0^3(a)]}$$

$$=\frac{10r_0^3(a)-3ar_0^2(a)-12r_0^2(a)-6r_0(a)-2ar_0(a)-a+4}{2[3ar_0^2(a)+12r_0(a)+a-4r_0^3(a)]}.$$

If we write $f_7(a)=10r_0^3(a)-3ar_0^2(a)-12r_0^2(a)-6r_0(a)-2ar_0(a)-a+4$, from

$$f_7'(a) = (30r_0^2(a) - 6ar_0(a) - 24r_0(a) - 2a - 6)r_0'(a) - (1+2r_0(a)+3r_0^2(a))$$

$$= \frac{r_0(a)[1+r_0^2(a)][30r_0^2(a) - 6ar_0(a) - 24r_0(a) - 2a - 6]}{4r_0^3(a) - 3ar_0^2(a) - 12r_0(a) - a}$$

$$- [1+r_0^2(a)][1+2r_0(a)] - 2r_0^2(a)[1-r_0(a)]$$

$$= \frac{[1+r_0^2(a)][-8r_0^4(a) + 26r_0^3(a) + 6ar_0^3(a) - 3ar_0^2(a) + 6r_0(a) + a]}{4r_0^3(a) - 3ar_0^2(a) - 12r_0(a) - a}$$

$$- 2r_0^2(a)[1-r_0(a)]$$

$$= \frac{[1+r_0^2(a)][2r_0(a)(3+13r_0^2(a) - 4r_0^3(a)) + a(1-3r_0^2(a)+6r_0^3(a))]}{4r_0^3(a) - 3ar_0^2(a) - 12r_0(a) - a}$$

$$- 2r_0^2(a)[1-r_0(a)],$$

we obtain $f_7'(a) < 0$ for $a \in [0, 2]$. Hence $f_7(a)$ is monotone decreasing for $a \in [0, 2]$. Direct computation gives $r_0(0.16) = 0.3981891$, we have

$$f_7(a) \leqslant f_7(0.16) \leqslant 4.6313469 - 4.655316 < 0, \text{ for } a \in [0.16, 2],$$

which shows that $f_5'(a) < 0$ for $a \in [0.16, 2]$. Therefore, $f_5(a)$ is monotone decreasing for $a \in [0.16, 2]$. Since $r_0(18 - 8\sqrt{5}) = 0.402939$, we obtain by computation

$$f_7(a) \geqslant f_7(18 - 8\sqrt{5}) = 4.6542111 - 4.6215161 > 0, \text{ for } a \in [0, 18 - 8\sqrt{5}],$$

which means that $f_5(a)$ is monotone increasing for $a \in [0, 18 - 8\sqrt{5}]$.

Now we want to prove $G(a)$ is monotone decreasing for $a \in [0.455, 2]$.

i) Since $r_0(0.7) = 0.3510004$, by computation, we obtain

$$2f_5(a) + \frac{1}{2}(1 - r_0(a))f_1(a) - f_4(a)$$

$$\geqslant 2f_5(2) + \frac{1}{2}(1 - r_0(0.7))f_1(0.7) - f_4(0.7)$$

$$= 0.3660254 + 0.5586835 - 0.8520418 = 0.0726671$$

for $a \in [0.7, 2]$, therefore, we have $G'(a) < 0$.

ii) Since $r_0(0.5) = 0.3673648$, by computation we have

$$2f_5(a) + \frac{1}{2}(1 - r_0(a))f_1(a) - f_4(a)$$

$$\geqslant 2f_5(0.7) + \frac{1}{2}(1 - r_0(0.5))f_1(0.5) - f_4(0.5)$$

$$= 0.3874624 + 0.5338299 - 0.9054016$$

$$= 0.0158907$$

for $a \in [0.5, 0.7]$. So we have $G'(a) < 0$.

iii) Since $r_0(0.455) = 0.3712165$, we obtain again by computation

$$2f_5(a) + \frac{1}{2}(1 - r_0(a))f_1(a) - f_4(a)$$

$$\geqslant 2f_5(0.5) + \frac{1}{2}(1 - r_0(0.455))f_1(0.455) - f_4(0.455)$$

$$= 0.3893139 + 0.5289340 - 0.9179516 = 0.000296369$$

for $a \in [0.455, 0.5]$, hence we have $G'(a) < 0$. From this we conclude that $G(a)$ is monotone decreasing for $a \in [0.455, 2]$.

On the other hand, we wish to prove $G(a)$ is monotone increasing for

$$a\in[0, 0.453].$$

From the expression of $G(a)$ we have

$$G'(a)=\frac{1}{4\sqrt{r_0(a)}\,(3+a/2)}\left[\frac{(1+r_0(a))(1+r_0^2(a))(2+\sqrt{1-a/2})}{3ar_0^2(a)+12r_0(a)+a-4r_0^3(a)}-\frac{2(1-r_0(a))}{3+a/2}\right.$$

$$\left.-\frac{\left(5-\frac{a}{2}\right)(1-r_0(a))}{2\left(3+\frac{a}{2}\right)\sqrt{1-a/2}}\right].$$

I) Since $r_0(0.114561)=0.4026323$,

$$\frac{1}{2}(1-r_0(a))f_1(a)\leqslant\begin{cases}\frac{5}{6}(1-r_0(0.1114561))=0.4975508,\\ \quad\text{for } a\in[0, 0.1114561],\\ \frac{1}{2}f_1(0.114561)\,[1-r_0(0.114561)]=0.4828835,\\ \quad\text{for } a\in[0.1114561, 0.114561],\end{cases}$$

$$2f_5(a)\leqslant\begin{cases}\dfrac{2[1-r_0(0.1114561)]}{3+0.1114561/2}=0.3907815, \text{ for } a\in[0, 0.1114561],\\ \dfrac{2[1-r_0(0.114561)]}{3+0.114561/2}=0.3909822, \text{ for } a\in[0.1114561, 0.114561],\end{cases}$$

we have

$$f_4(a)-\frac{1}{2}(1-r_0(a))f_1(a)-2f_5(a)\geqslant f_4(0.114561)-0.4975508-0.3909822$$
$$=1.0214946-0.4975508-0.3909822=0.13929616$$

for $a\in[0, 0.114561]$, hence $G'(a)>0$, which means that $G(a)$ is monotone increasing for $a\in[0, 0.114561]$.

II) Since $r_0(0.16)=0.3981891$,

$$f_4(a)-\frac{1}{2}(1-r_0(a))f_1(a)-2f_5(a)$$
$$\geqslant f_4(0.16)-\frac{1}{2}(1-r_0(0.16))f_1(0.16)-\frac{3(1-r_0(0.16))}{3+0.114561/2}$$
$$=1.0066471-0.5011301-0.3936903=0.1118266$$

for $a\in[0.114561, 0.16]$, we obtain $G'(a)>0$.

III) Since $r_0(0.45)=0.3716485$,

$$f_4(a)-\frac{1}{2}(1-r_0(a))f_1(a)-2f_5(a)$$
$$\geqslant f_4(0.45)-\frac{1}{2}(1-r_0(0.45))f_1(0.45)-2f_5(0.16)$$
$$=0.91936-0.5284032-0.3907862=0.000170501$$

for $a\in[0.16, 0.45]$, we obtain $G'(a)>0$.

IV) Since $r_0(0.453)=0.3713892$,

$$f_4(a)-\frac{1}{2}(1-r_0(a))f_1(a)-2f_5(a)$$
$$\geqslant f_4(0.453)-\frac{1}{2}(1-r_0(0.453))f_1(0.453)-2f_5(0.45)$$
$$=0.9185147-0.5287214-0.3896753=0.0001180$$

for $a\in[0.45, 0.453]$, we have $G'(a)>0$. And we conclude that $G(a)$ is monotone increaing for $a\in[0, 0.453]$.

In the light of above-mentioned results, we derive

$$G(a)\leqslant G(0.455)\leqslant 0.4602786, \text{ for } a\in[0.455, 2],$$
$$G(a)\leqslant G(0.453)\leqslant 0.4602787, \text{ for } a\in[0, 0.453].$$

Hence we obtain

$$\frac{d_0}{d^*}\geqslant 0.4103233, \text{ for } a\in[0.455, 2],$$
$$\frac{d_0}{d^*}\geqslant 0.4103234, \text{ for } a\in[0, 0.453].$$

And we see easily

$$G(a)\leqslant\frac{1-r_0(0.455)}{2\sqrt{r_0(0.455)}}\left(\frac{2+\sqrt{1-0.453/2}}{3+0.453/2}\right)=0.4605122, \text{ for } a\in[0.453, 0.455],$$

therefore we have

$$\frac{d_0}{d^*}\geqslant 0.4101492, \text{ for } a\in[0.453, 0.455].$$

Thus we obtain

$$\frac{d_0}{d^*}\geqslant 0.4101492,$$

which completes the proof of Theorem 1.

References

[1] Hayman, W. K., Multivalent functions, *Cambridge Tracts in Math. and Math. Phys.* No. 48, Cambridge Univ. Press, Cambridge, 1958.

[2] Schild, A., On a problem in conformal mapping of schlicht functions, *Proc. Amer. Math. Soc.*, **4** (1953), 43—51.

[3] Lewandowski, Z., Nouvelles remarques sur les théoremès de Schild relatifs d'une class de fonctions univalentes (Démonstration d'une hypothèse de Schild), *Ann. Univ. Mariae Curie-Sklodowska Sect.* **A** **10** (1956), 81—94.

[4] Tepper, D. E., On the radius of convexity and boundary distortion of schlicht functions, *Trans. Amer. Math, Soc.*, **150** (1970), 519—528.

[5] McCarty, C. P. & Tepper, D. E., A note on the 2/3 conjecture for starlike functions, *Proc. Amer. Math. Soc.*, **34** (1972), 417—421.

[6] Barnard, R. W. & Lewis, J. L., A counterexample to the two-thirds conjecture, *Proc. Amer. Math. Soc.*, **41** (1973), 525—529.

[7] Netanyahu, E., On univalent functions in the unit disk whose image contains a given disk, *J. Analyse Math.*, **23** (1970), 305—322.