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NONLINEAR INITIAL-BOUNDARY VALUE PROBLEM FOR QUASILINEAR HYPERBOLIC SYSTEM*

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Abstract

Consider the nonlinear initial-boundary value problem for quasilinear hyperbolic system:

$$(*)\quad \begin{cases} \partial_t u=\sum_{j=1}^{n} A_j(u)\partial_{x_j}u+E(u)u+F, \text{ in } (0,T)\times\Omega, \\ u(0)=0,\ P(u)u|_{\partial\Omega}=g. \end{cases}$$

Let $k\geqslant 2\left[\frac{n}{2}\right]+6$, $(F, g)\in H^k(R_+;\ \Omega)\times H^k(R_+;\ \partial\Omega)$, and their traces at $t=0$ are zero up to the order $k-1$.

If for $u=0$, the problem $(*)$ at $t=0$ is a Kreiss hyperbolic system, and the boundary conditions satisfy the uniformly Lopatinsky criteria, then there exists a $T>0$ such that $(*)$ has a unique H^k soluton in $(0, T)$.

In the Appendix, for symmetric hyperbolic systems, a comparison between the uniformly Lopatinsky condition and the stable admissible condition is given.

§ 1. Introduction

In his paper[5], H-O. Kreiss discussed the general initialboundary value problem for strictly hyperbolic system, employing the method of microlocal analysis. Since then, there have been a lot of developments in this direction. In [13], J. Ralston generalized the result of Kreiss to the problem with complex coefficients; in [9], J. Rauch proved the semigroup estimate of high order with non-zero initial data, and the existence of differentiable solution; afterwards, in [10], J. Rauch & F. Massey loosened the restriction that the traces of data at $t=0$ should be zero, and they only required the natural compatiability. In their paper [6], A. Majda & S. Osher discussed the problem with uniformly characteristical boundary. On the other hand, Chen Shuxing and Zheng Songmu discussed, in [2,

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3, 12], respectively, the linear boundary value problems for symmetric quasilinear hyperbolic system, within the framework of admissible boundary conditions for positive symmetric system.

In this paper, we want to generalize the above noncharacteristic uniform Lopatinsky boundary value problem for hyperbolic system to the situation with nonlinear boundary conditions and of quasilinear system. For such problems, we prove the local uniqueness and existence of differentiable solutions (Theorem 2). In our proof, the Newtonian iteration is not used for the boundary term, so the method is different from Majda's in [8] and can be used to simplify the proof in [8].

In fact, using the technique of Beals and Reed in their recent paper [1], the restriction on k_0 can be relaxed to

$$k_0 \geqslant \left[\frac{n+1}{2}\right]+3.$$

We'll not go into the details of the proof here. In the author's another paper on the boundary value problems of quasilinear hyperbolic-parabolic coupled system, we have the similar results and give the complete proof.

At first, we are going to give a simple description of the results for linear problems, obtained in the articles cited above.

Consider the following initial-boundary value problem for linear hyperbolic system:

$$\begin{cases} \partial_t u = A_1 \partial_{x_1} u + \sum_{j=2}^{n} A_j \partial_{x_j} u + Eu + F, \ t>0, \ x_1>0, \\ u(x,0) = f, \\ Pu|_{x_1=0} = g, \end{cases} \tag{1.1}$$

where A_j, E are $m\times m$ matrices, $A_1=\mathrm{diag}\,(A_1^1, A_1^2)$, $A_1^1<0$ is an $l\times l$ matrix, $A_1^2>0$ is an $(m-l)\times(m-l)$ matrix. Besides, we suppose that the elements of the above matrices are all smooth functions of (t,x), and are constants when $|t|+|x|\geqslant R_0$.

Define

$$M(t, x, s, \omega) = A_1^{-1}\Big(s - i\sum_{j=2}^{n} A_j\omega_j\Big),$$

where $\omega=(\omega_2,\cdots,\omega_n)\in R_{n-1}$, $s=\eta+i\xi$, $\eta\geqslant 0$.

Definition 1.1. *The system of equations in* (1.1) *will be called Kreiss' hyperbolic system if the following conditions are satisfied:*

For every fixed $(t_0, x_0, s_0=i\xi_0, \omega_0)\in\Omega\times\mathscr{S}$, $\mathscr{S}=\{(s, \omega);\ |s|^2+|\omega|^2=1,\ \mathrm{Re}\, s\geqslant 0\}$, *there's a neighborhood in which there exists a smooth invertible matrix* $T(t, x, s, \omega)$ *such that* $TMT^{-1}=\mathrm{diag}\,(M_1,\cdots,M_q)$, *where* $M_j(t, x, s, \omega)$ *at* $(t_0, x_0, i\xi_0, \omega_0)$ *are exactly the Jordan blocks of* $M\,(t_0, x_0, i\xi_0, \omega_0)$, *and* $M_j\,(t, x, i\xi, \omega)$ *is a pure imaginary matrix when the eigenvalue of* $M_j\,(t_0, x_0, i\xi_0, \omega_0)$ *is pure imaginary.*

Remark. Specifically, strictly hyperbolic systems are Kreiss' hyperbolic systems (cf. [5]); besides, the symmetric hyperbolic systems encountered in physics are Kreiss' hyperbolic systems (cf. [7]).

When Re $s>0$, the matrix $M(s, \omega)$ has l eigenvalues with negative real part, and $(m-l)$ eigenvalues with positive real part (cf [5]). Let $V(s, \omega)$ be the l-dimensional linear space spanned by the eigen-vectors corresponding to the eigenvalues with negative real part of $M(s, \omega)$.

Definition 1.2. *The initial-boundary value problem* (1.1) *will be called Kreiss' well-posed if* (1.1) *is Kreiss' hyperbolic system, and for every* $(t, 0, x')$, $\exists\delta>0$ *such that*

$$|P(t, x')v(s, \omega)| \geqslant \delta |v(s, \omega)|, \tag{1.2}$$

where the constant δ *is uniform for all* $v \in V(s, \omega)$, Re $s>0$, $|s|^2+|\omega|^2=1$.

For linear problem (1.1), we have the following

Theorem 1. *Let* (1.1) *be Kreiss well-posed. Then*

$$\forall (f, F, g) \in H^k(\Omega) \times H^k([0, T] \times \Omega) \times H^k([0, T] \times \partial\Omega)$$

satisfying the natural compatiable conditions up to the order $k-1$ *at the corner point,* $\exists \eta_0>0$ *such that when* $\eta \geqslant \eta_0$, (1.1) *has a unique strong solution*

$$(u, \bar{u}) \in \bigcap_{r=0}^{k} C^r([0, T]; H^{k-r}(\Omega)) \times H^k([0, T] \times \partial\Omega)$$

satisfying the following energy inequality

$$\sum_{r=0}^{k} \|\partial_t^r u(t)\|^2_{k-r, \Omega, \eta} + \eta \|u\|^2_{k, [0, t] \times \Omega, \eta} + \sum_{r=0}^{k} \|\partial_{x_1}^r u\|^2_{k-r, [0, t] \times \partial\Omega, \eta}$$

$$\leqslant C\left(\frac{1}{\eta}\|F\|^2_{k, [0, t] \times \Omega, \eta} + \|f\|^2_{k, \Omega, \eta} + \|g\|^2_{k, [0, t] \times \partial\Omega, \eta}\right), \tag{1.3}$$

where $t \leqslant T$. *When* $T=+\infty$, *the first term on the left part of* (1.3) *disappears. And the respective norms are defined by*

$$\|\varphi\|^2_{k, \sigma, \eta} = \sum_{|k_1|+|k_2| \leqslant k} \int_\sigma \eta^{2k_1} |D^{k_2}\varphi|^2 e^{-2\eta t}\, d\sigma,$$

where D *denotes the inner differentiation in* σ.

Remark 1. It is pointed out in [5] that for the problem (1.1) with constant coefficients, the condition (1.2) of the Kreiss well-posedness is necessary for the estimate (1.3) to hold.

Remark 2. By localization and changing the localized domain to half-space, we can consider the initial-boundary value problems in a bounded domain with sufficient smooth boundary, and we have the same result corresponding to Theorem 1.

§ 2. Nonlinear Boundary Value Problem for Quasilinear System, Main Result

Let Ω be a bounded domain in R^n, with sufficiently smooth boundary $\partial\Omega$. Consider the initial-boundary value problem of quasilinear hyperbolic system:

$$\begin{cases} \partial_t u=\sum_{j=1}^{n} A_j\partial_{x_j}u+Eu+F, \\ u(0)=0, \\ Pu|_{\partial\Omega}=g. \end{cases} \tag{2.1}$$

Here $A_j(t, x, u)$, $E(t, x, u)$ are $m\times m$ matrices, $P(t, x, u)$ is an $l\times m$ matrix, the elements of which are smooth functions with respect to all the arguments and depending only on x when $t\geqslant T_0$. Without loss of generality, we may assume that F, g are independant of u.

Let $k_0=2\left[\frac{n}{2}\right]+6$, $k\geqslant k_0$. In the following, we always assume $(F, g)\in H^k(R^+\times\Omega)\times H(R^+\times\partial\Omega)$, and the traces of F and g at $t=0$ are all zero up to the order $k-1$. For more general condition $u(0)=f\neq 0$, (f, F, g) satisfying the compatiable condition of order $k-1$, at the corner point, we can change it to the situation stated above. But by doing so, we may lose one order of differentiability.

Theorem 2 (**Main Theorem**). *If for $u=0$, problem (2.1) is a Kreiss well-posed initial-boundary value problem, then $\exists h>0$ such that in $[0, h]$, (2.1) has a unique differentiable solution $u\in H^k([0, h]\times\Omega)$, which is a classical solution.*

Theorem 3. *If for $u=0$, (2.1) is Kreiss well-posed, then $\forall\, h>0$, $\exists\, \varepsilon_1>0$ such that for any (F,g) satisfying the compatiable condition and*

$$\|F\|_{k,[0,h]\times\Omega,\eta}+\|g\|_{k,[0,h]\times\partial\Omega,\eta}<\varepsilon_1,$$

problem (2.1) has a unique solution $u\in H([0, h]\times\Omega)$.

According to Theorem 3, we may also consider the uniqueness and existence of solution for large h, near any given smooth solution.

§ 3. The Estimate with H^k Coefficients

In order to prove the above theorems, we'll practise iteration. By this, it is necessary to consider the Kreiss initial-boundary value problem with coefficients in H^k and the dependency on its coefficients of the energy inequality of order k.

Since the boundary is assumed to be smooth, and the composition of H^k function with C^1-homeomorphism (which belongs to H^k at the sametime) is again an H^k function, we need only analyze the energy estimate (1.3) for localized problem with its boundary defined by $x_1=0$. When $t\geqslant T_0$, the localized problem is a linear problem and has nothing to do with our iteration. So it remains to consider the following problem:

$$\begin{cases} \partial_t u=\sum_{j=1}^{n}(A_j+a_j)\partial_{x_j}u+(E+e)u+F,\ t>0,\ x_1>0, \\ u(0)=0, \\ (P+p)u|_{x_1=0}=g. \end{cases} \tag{3.1}$$

Here $u\in H^k$; A_j, E, P are constant matrices when $|t|+|x|\geqslant 2R_0$; a_j, e, and p are perturbations in H^k; (F, g) are the same as in (2).

Proposition 3.1. *Let $k\geqslant 2\left[\frac{n}{2}\right]+6$, and (3.1) be Kreiss well-posed when a_j, e, p vanish. Then $\exists\varepsilon>0$ such that if $\|a_j\|_{k,(0,\infty)\times\Omega,\eta}+\|e\|_{k,(0,\infty)\times\Omega,\eta}+\|p\|_{k,(0,\infty)\times\partial\Omega,\eta}\leqslant\varepsilon$, (3.1) is again a Kreiss well-posed problem, and the constants C and η_0 in the energy inequality (1.3) depend only of the H^k norm of A_j, E, P, and are independent of a_j, e, p.*

Proof Let $R(t, x, s, \omega)$ be the Kreis's symmetrilizer for the problem (3.1). As in [7], R is of H^k_{ul} with respect to (t, x). Here H^k_{ul} is the uniformly H^k space, defined in [4]. Without confusion, we'll denote by R the pseudo-differential operator of order zero with η-weighted symbol $R(s, \omega)$.

Set $w=e^{-\eta t}u$. Applying the operator R to the equation (3.1) and taking L^2 inner product with w in $(0, \infty)\times\Omega$, we have

$$\operatorname{Re}\,(w, R\partial_{x_1}w)=\operatorname{Re}\left(w, RA^{-1}\left(\partial_t w+\eta w-\sum_{j=2}^{n}A_j\partial_{x_j}w\right)\right)-\operatorname{Re}(w, RA^{-1}Ew)$$
$$-\operatorname{Re}(w, Re^{-\eta t}F).$$

Since R is a Hermite matrix, $R-R^*$ is an opertor of order -1. It is easily seen that

$$2\operatorname{Re}\,(w, R\partial_{x_1}w)\leqslant-\operatorname{Re}(w, Rw)|_{x_1=0}+\frac{C}{\eta}\|w\|^2_{0,(0,\infty)\times\Omega}.$$

While R is the operator of order 0, it is a bounded operator in L^2. So

$$|(w, RA^{-1}Ew)|\leqslant C\|w\|^2_{0,(0,\infty)\times\Omega},$$
$$|(w, RAe^{-\eta t}F)|\leqslant C\|w\|_{0,(0,\infty)\times\Omega}\|e^{-\eta t}F\|_{0,(0,\infty)\times\Omega}$$
$$\leqslant\delta_0\eta\|w\|^2_{0,(0,\infty)\times\Omega}+\frac{C}{\eta}\|e^{-\eta t}F\|^2_{0,(0,\infty)\times\Omega}.$$

It is easily seen that the constants C and δ_0 depend only on the local H^k norms of the coefficients. So we have

$$(w, Rw)|_{x_1=0}+\operatorname{Re}\left(w, RA^{-1}\left(\eta+\partial_t-\sum_{j=l}^{n}A_j\partial_{x_j}\right)w\right)$$
$$\leqslant\delta_0\eta\|w\|^2_{0,(0,\infty)\times\Omega}+\frac{C}{\eta}\|e^{-\eta t}F\|^2_{0,(0,\infty)\times\Omega}+\frac{C}{\eta}\|w\|^2_{0,(0,\infty)\times\partial\Omega}. \tag{3.2}$$

In what follows, we want to make use of the Lemma 4.2 in [7], which states that if $k\geqslant 2\left[\frac{n}{2}\right]+6$, the operator $RA^{-1}\left(\eta+\partial_+-\sum_{j=2}^{n}A_j\partial_{x_j}\right)$ differs from the oparetor with the symbol $R(s,\ \omega)\ A^{-1}\left(s-i\sum_{j=2}^{n}A_j\omega_j\right)$ by an operator R_η^0, which is a bounded operator in L^2 and has its norm bounded unilormly with respect to η. From the definition of R and the Sharp Garding inequality, we have

$$\operatorname{Re}\left(R(s,\ \omega)A^{-1}\left(s-i\sum_{j=2}^{n}A_j\omega_j\right)\right)\geqslant\delta_1\eta I,$$

and when $\eta\gg 1$,

$$\operatorname{Re}\left(w,\ RA^{-1}\left(\eta+\partial_t-\sum_{j=2}^{n}A_j\partial_{x_j}\right)w\right)\geqslant\frac{\delta_1}{2}\eta\|w\|^2_{0,(0,\infty)\times\Omega},$$

where the positive constant δ_1 depeneds only on the local H^k norms of the coefficients.

For fixed $(t_0,\ x_0)$, let $\tilde{R}=R\ (t_0,\ x_0,\ s,\ \omega)$. From the definition of $R(s,\ \omega)$ in [5], we have

$$\begin{aligned}\frac{1}{2}(w,\ \tilde{R}w)\,|_{x_1=0}&=\frac{1}{2}\int\hat{w}(s,\ \omega)\tilde{R}(s,\ \omega)\hat{w}(s,\ \omega)\ d\xi\, d\omega\\&\geqslant\frac{C_0}{2}\int(\varkappa\,|\hat{w}_+(s,\ \omega)\,|^2-|\hat{w}_-(s,\ \omega)\,|^2)d\xi\, d\omega,\end{aligned}\tag{3.3}$$

where $\hat{w}(s,\ \omega)=\hat{w}_+(s,\ \omega)+\hat{w}_-(s,\ \omega)$ is the decomposition of $\hat{w}(s,\ \omega)$ with respect to the eigen-spaces of $M\ (t_0,\ x_0,\ s,\ \omega)$ corresponding to the eigenvalues with positive and negtive real part respectively. Since R is continuous in $(t,\ x)$, taking the localized neighborhood small enough, we have

$$(w,\ \tilde{R}w)\leqslant(w,\ Rw)+\varepsilon\|w\|^2_{0,(0,\infty)\times\partial\Omega},\tag{3.4}$$

where ε is sufficiently small.

Similarly, without loss of generality, we may assume the matrix P to be constant.

Making Fourier transform of the boundary condition, we get

$$P\hat{w}=\widehat{e^{-\eta t}g}(s,\ \omega),\ \text{i.e.}$$

$$P_1\hat{w}_++P_2\hat{w}_-=\widehat{e^{-\eta t}g}.$$

From the Kreiss' well-posedness, $\det P_2\neq 0$. So one has

$$\hat{w}_-(s,\ \omega)=P_2^{-1}\widehat{e^{-\eta t}g}(s,\ \omega)-P_2^{-1}P_1\hat{w}_+.\tag{3.5}$$

Taking the parameter $\varkappa$ in (3.3) sufficiently large, we have

$$\begin{aligned}\frac{1}{2}(w,\tilde{R}w)_{x_1=0}&\geqslant\frac{C_0}{2}\int(\varkappa\,|\hat{w}_+|^2+|\hat{w}_-|^2-2\,|P_2^{-1}\widehat{e^{-\eta t}g}-P_2^{-1}P_1\hat{w}_+|^2)d\xi\, d\omega\\&\geqslant C_1\|w\|^2_{0,(0,\infty)\times\partial\Omega}-C_2\|e^{-\eta t}g\|^2_{0,(0,\infty)\times\partial\Omega}.\end{aligned}$$

From this, one gets the energy inequality of order zero straightly, with the constants C and η_0 depending only on the local H^k norms of the coefficients.

Now we turn to the energy inequality of order k.

Differentiating the problem (3.1) with respect to (t, x) k times, one gets the new problem with unknowns $D^k u$. The principal part of the new system is a block-wise diagonal matrix, every block being equal to the principal part of the original system. So we can construct a new symmetrilizer which is block-wise diagonal, every block being equal to the symmetrilizer $R(s, \omega)$. Then proceeding as with the estimate of order zero, we can get the energy inequality of order k.

Among the terms of the lower order in the new system, the one containing normal differentiation $\partial_{x_1} u$ can be expressed by the one containing only the tangential differentiation by making use of the original system. So, generally, the terms of the lower order can be written as

$$(D^{k_1}Q)(D^{k_2}u), \ |k_1| \leqslant k, \ |k_2| \leqslant k, \ |k_1+k_2| \leqslant k.$$

Evidently, one of $|k_1|$ and $|k_2|$ must be $\leqslant \frac{k}{2}$. Consequently, one of the terms $D^{k_1}Q$ and $D^{k_2}u$ is uniformly continuous and bounded, which can be estimated by its continuous maximal norm, and so by its local H^k norm.

Thus we have the H^k estimate of the tangential differentiations of u. By the non-characteristics of the boundary $\partial\Omega$, we can get the estimate of the normal differentiation of u. This concludes the proof.

§ 4. The Solution of the Quasilinear Problem

We are now to prove the existence and uniqueness of the local differentiable solution of the nonlinear initial-boundary value problem (2.1) for the quasilinear hyperbolic system. The basic method is linearized iteration. Noting that the right side of the problem (2.1), i. e. (F, g), is independant of u, by the energy inequality obtained in § 3, we can use the usual method of iteration to handle the boundary condition, without applying the Newtonian iteration used by A. Majda in [8], and thus simplify the proof of the Theorem. In [10], it has been mentioned that the nonlinear boundary value conditions can be handled by the simple iteration. But to the author's knowledge, the proof has never been given. In fact, if we retain u in (F, g), i.e. in the deduction of the energy inequality we consider only the principal part of the system, it'll be necessary to apply the Newtonian iteration used by A. Majda in [8].

For the simplicity of notation, in this paragraph we'll always write

$$\|u\|_{k,[0,h]\times\Omega,\eta} = \|u\|_{k,h,\eta}, \ \|\bar{u}\|_{k,[0,h]\times\partial\Omega,\eta} = |\bar{u}|_{k,h,\eta},$$

$$|\|(u,\bar{u})|\|^2_{k,h,\eta} = \|u\|^2_{k,h,\eta} + |\bar{u}|^2_{k,h,\eta}.$$

Now consider the linearized problem of (2.1):

$$\begin{cases}\partial_t+U=\sum_{j=1}^{n}A_j(u)\partial_{x_j}U+E(u)U\equiv L(u)U+F,\\ U(0)=0,\\ P(u)U|_{\partial\Omega}=g.\end{cases}\tag{4.1}$$

Denote by $\mathscr{B}(\varepsilon, \mathrm{h})$ the set

$$\mathscr{B}(\varepsilon, \mathrm{h})=\{(u, \bar{u});\ |\|(u, \bar{u})\||^2_{k,h,\eta}\leqslant\varepsilon, \text{ the traces at } t=0 \text{ are zero up to the order } k-1\}.$$

For fixed η, the η-weighted norms of $(u, \bar{u})$ are equivalent to the usual Sobolev norms. According to the assumption in Theorem 2, the problem (2.1) is Kreiss well-posed when $u=0$. So with ε sufficiently small, the $(u, \bar{u})$ in $\mathscr{B}(\varepsilon, h)$ guarantees the well-posedness of problem (4.1). By the result of linear problem and Proposition 3.1, (4.1) has a unique solution

$$(U, \bar{U})\in H^k([0, \mathrm{h}]\times\Omega)\times H^k([0, h]\times\partial\Omega)$$

satisfying the energy inequality

$$\sum_{j=0}^{k}\|\partial_t^j U(t)\|^2_{k-j,\Omega,\eta}+\eta\|U\|^2_{k,t,\eta}+|\bar{U}|^2_{k,t,\eta}\leqslant C_k\left(\frac{1}{\eta}\|F\|^2_{k,t,\eta}+|g|^2_{k,t,\eta}\right).\tag{4.2}$$

Here $t\leqslant h$, and the constant C_k may depend on η.

Fixing η and taking h sufficiently small, we have

$$C_k\left(\frac{1}{\eta}\|F\|^2_{k,h,\eta}+|g|^2_{k,h,\eta}\right)\leqslant\frac{\varepsilon}{2}.$$

From it

$$\|U\|^2_{k,h,\eta}+|\bar{U}|^2_{k,h,\eta}\leqslant\varepsilon.$$

Thus the resolvent operator $\mathscr{T}$ of (4.1), $\mathscr{T}$: $(u, \bar{u})\mapsto(U, \bar{U})$, is a map from $\mathscr{B}(\varepsilon, h)$ into $\mathscr{B}(\varepsilon, h)$. So we can simply take $(u_0, \bar{u}_0)=(0, 0)$ to guarantee that all the iteration could be practised within a common interval of $t\in[0, h]$.

To prove the convergence of the iteration, let

$$\mathscr{T}(u_{\nu-1}, \bar{u}_{\nu-1})=(u_\nu, \bar{u}_\nu),\ v_\nu=u_\nu-u_{\nu-1},\ (\nu=1, 2, \cdots).$$

So v_ν satisfy

$$\begin{cases}\partial_t v_{\nu+1}-L(u_\nu)v_{\nu+1}=(L(u_\nu)-L(u_{\nu-1}))u_\nu,\\ v_{\nu+1}(0)=0,\\ P(\bar{u}_\nu)v_{\nu+1}|_{\partial\Omega}=(P(\bar{u}_\nu)-P(\bar{u}_{\nu-1}))u_\nu.\end{cases}\tag{4.3}$$

Noticing that $P(\bar{u}_\nu)-P(\bar{u}_{\nu-1})=0(|\bar{u}_\nu-\bar{u}_{\nu-1}|)$, for $v_{\nu+1}$ we have the estimate

$$|\|(v_{\nu+1}, \bar{v}_{\nu+1})|\|^2_{k-1,h,\eta}\leqslant O(\|v_\nu\|^2_{k-1,h,\eta}\|u_\nu\|^2_{k,h,\eta}+|\bar{v}_\nu|^2_{k-1,h,\eta}|\bar{u}_\nu|^2_{k-1,h,\eta}).$$

In deriving the above energy estimate, we have used the fact that when

$$k\geqslant\left[\frac{n+1}{2}\right]+1,$$

H^k is a Banach algebra, so that

$$\|uv\|_{k,h,\eta}\leqslant C'\|u\|_{k,h,\eta}\|v\|_{k,h,\eta}.$$

Because the traces of u and v at $t=0$ are zero of order $k-1$, the constant C' is independant of small h.

For ε sufficiently small, $\varepsilon C \leqslant \frac{1}{2}$, we get from the estimate of $v_{\nu+1}$:

$$|||(v_{\nu+1}, \bar{v}_{\nu+1})|||_{k-1,h,\eta} \leqslant \frac{1}{2} |||(v_\nu, \bar{v}_\nu)|||_{k-1,h,\eta}.$$

From this we know that the consequence $(u_\nu, \bar{u}_\nu)$ converges to a $(u, \bar{u})$ in

$$H^{k-1}([0, h] \times \Omega) \times H^{k-1}([0, h] \times \partial\Omega).$$

Since $\{(u_\nu, \bar{u}_\nu)\} \subset \mathscr{B}(\varepsilon, h)$, we have $(u, \bar{u}) \in \mathscr{B}(\varepsilon, h) \subset H^k([0, h] \times \Omega) \times H^k([0, h] \times \partial\Omega)$. $(u, \bar{u})$ is the fixed point of the resolvent operator $\mathscr{T}$, i. e. the solution of (2.1).

The proof of uniqueness is similar. Suppose $(u_1, \bar{u}_1)$ and $(u_2, \bar{u}_2)$ are two solutions of (2.1). Let $v = u_1 - u_2$. Then v satisfies

$$\begin{cases} \partial_+ v - L(u_1)v = (L(u_1) - L(u_2))u_2, \\ v(0) = 0, \\ P(\bar{u}_1)\bar{v}|_{\partial\Omega} = (P(\bar{u}_2) - P(\bar{u}_1))\bar{u}_2. \end{cases}$$

And for v the following estimate is valid:

$$|||(v, \bar{v})|||^2_{k-1,h,\eta} \leqslant C(\|v\|^2_{k-1,h,\eta}\|u_2\|^2_{k,h,\eta} + |v|^2_{k-1,h,\eta}|\bar{u}_2|^2_{k-1,h,\eta}).$$

So for ε sufficiently small, we have $(v, \bar{v}) = (0, 0)$. This finishes the proof of Theorem 2.

Theorem 3 may be proved similarly. Taking notice of (4.2), when $\varepsilon_1 \ll 1$, one easily sees that the map $\mathscr{T}$ is an injective one from $\mathscr{B}(\varepsilon, h)$ to $\mathscr{B}(\varepsilon, h)$. To prove the convergence and uniqueness, one needs only to take ε sufficiently small.

§ 5. Application

As the application of Theorem 2, we consider the Euler equation in hydrodynamics which depicts the unstationary flow of ideal gas:

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \frac{1}{(\rho c)^2} & \\ & & & & 1 \end{bmatrix} \partial_t w + \begin{bmatrix} u_1 & & & \frac{1}{\rho} & \\ & u_1 & & & \\ & & u_1 & & \\ \frac{1}{\rho} & & & \frac{u_1}{(\rho c)^2} & \\ & & & & u_1 \end{bmatrix} \partial_{x_1} w$$

$$+ \begin{bmatrix} u_2 & & & & \\ & u_2 & & \frac{1}{\rho} & \\ & & u_2 & & \\ & \frac{1}{\rho} & & \frac{u_2}{(\rho c)^2} & \\ & & & & u_2 \end{bmatrix} \partial_{x_2} w + \begin{bmatrix} u_3 & & & & \\ & u_3 & & \frac{1}{\rho} & \\ & & u_3 & & \\ & & \frac{1}{\rho} & \frac{u_3}{(\rho c)^2} & \\ & & & & u_3 \end{bmatrix} \partial_{x_3} w$$

$$= F.$$

Here $w=(u_1, u_2, u_3, p, s)^t$, (u_1, u_2, u_3) denotes the velosity of the gas, p is the pressure, s is the entropy, ρ is the density, and c is the speed of sound.

This is a quasilinear hyperbolic symmetric system, and as pointed out by A. Majda in [7], its linearized equation is a Kreiss' hyperbolic system (cf. Definition 1.1). In [2], Chen Shuxing has considered the homogeneous initial-boundary value problem in the domain $[0, T]\times\Omega$ for this system. He has proved the existence and uniqueness of local differentiable solution for the stable admissible boundary value problems. Now, applying the Theorem 2 of the present paper, we know that when the boundary conditions are nonlinear and satisfy only a more general Kreiss well-posed condition, the local differentiable solution is also existent and unique. From the view point of physics, the usual linear boundary conditions are rather the approximations of nonlinear relations. Our Theorem 2 ascertains that such approximations don't make any difference upon the existence and the uniqueness of the local differentiable solution.

On the other hand, it is worth pointing out that the Kreiss well-posedness is the necessary condition for the energy estimate (1.3) to be valid in the case of the constant coefficient. Since generally the stable admissible boundary conditions and the Kreiss' condition are not equivalent (cf. Appendix), the initial-boundary value problems which could be treated by Theorem 2 are wider than within the framework of stable admissibliitiy.

§ 6. Appendix: Comparison Between the Kreis's Well-Posedness and the Stable Admissibility

Here, we want to discuss the relation between the Kreis's well-posedness and the stable admissibility of the noncharacteristic initialboundary value problems for the Kreiss symmetric hyperbolic system.

Proposition 5.1. *For the noncharacteristic initial-boundary value problem of the symmetric hyperbolic system with two independent variables (it is necessarily a Kreiss hyperbolic system), the stable admissibility is equivalent to the Kreiss well-posedness.*

Proof In this case, the principal part of the system can be written as $\partial_t u+A\partial_x u$, while matrix A can be assumed to be diagonal, $A=\mathrm{diag}\,(a_1,\cdots,a_l, a_{l+1},\cdots, a_m)$, where $a_1,\cdots,a_l>0$, $a_{l+1},\cdots,a_m<0$.

Let the boundary conditions on $x=0$ be written as $Mu|_{x=0}=g$. Suppose V to be the subspace spanned by the former l components of vector u. It is easily seen that the stable admissible conditions and the Kreiss well-posed ones are all the same, i. e.

$$|Mv|\geqslant\delta|v|,\ \forall v\in V.$$

This concludes the proof.

Proposition 5.2. *For the noncharacteristic initial-boundary value problem of the symmetric hyperbolic system with two unknowns, the stable admissibility is equivalent to the Kreiss well-posedness.*

Proof Without loss of generality, we can confine our consideration to the following problem:

$$\begin{cases} \partial_t u=A\partial_x u+\sum_{j=1}^{n}B_j\partial_{x_j}u+Eu+F,\ x>0,\ t>0, \\ u(0)=0,\ Pu|_{x=0}=g, \end{cases} \tag{5.1}$$

where $A=\operatorname{diag}(-1, \varkappa)$, $\varkappa>0$, $Pu=u_1-\alpha u_2$, and

$$B_j=\begin{bmatrix} l_j & b_j \\ b_j & a_j \end{bmatrix}.$$

It is readily affirmed that the boundary conditions are stable admissible iff $|\alpha|<\sqrt{\varkappa}$.

We'll examine the Kreiss well-posedness. Denote

$$B\cdot\omega=\sum_{j=1}^{n}B_j\omega_j,$$

similarly are defined $l\cdot\omega$, $b\cdot\omega$ and $a\cdot\omega$. Let $\mu=\frac{1}{\varkappa}$. Then one can compute:

$$M(s,\omega)=A^{-1}(sI-iB\cdot\omega)=\begin{bmatrix} -s+il\cdot\omega & ib\cdot\omega \\ -ib\cdot\omega\mu & (s-ia\cdot\omega)\mu \end{bmatrix}.$$

The eigenvector of $M(s,\omega)$, corresponding the eigenvalue λ_- with negative real part, is

$$v_-=(ib\cdot\omega,\ \lambda_-+s-il\cdot\omega),$$

with

$$\lambda_-=-\frac{(1-\mu)s+i\omega\cdot(a\mu-l)}{2}-\Big[\Big(\frac{(1-\mu)s+i\omega\cdot(a\mu-l)}{2}\Big)^2 +(\omega\cdot b)^2\mu+(s-il\cdot\omega)(s-ia\cdot\omega)\mu\Big]^{\frac{1}{2}}.$$

If $b\cdot\omega=0$, then $M(s,\omega)$ is a diagonal matrix, its eigenvector corresponding to the eigenvalue with negative real part will be $(1.0)^t$, and we'll always have $|Pv|\neq 0$.

If $b\cdot\omega\neq 0$, substituting the above v_- into $u_1-\alpha u_2$, we have $Pv=0$ iff

$$ib\cdot\omega-\alpha(\lambda_-+s-il\cdot\omega)=0,$$

$$\text{iff } ib\cdot\omega+i\alpha\gamma(\eta)-\alpha\sqrt{(b\cdot\omega)^2\mu-\gamma^2(\eta)}=0, \tag{5.3}$$

where

$$\gamma(\eta)=\frac{(a\mu+l)\cdot\omega-(1+\mu)(-i\eta+\xi)}{2}.$$

Evidently, (5.3) can hold only when $\eta=0$ and $(b\cdot\omega)^2\mu-\gamma^2(0)\leqslant 0$. When

$(b\cdot\omega^2)\mu-\gamma^2(0)=0$, we have $|\alpha|=\frac{2}{\sqrt{\mu}}=\sqrt{\varkappa}$. When $(b\cdot\omega)^2\mu-\gamma^2(0)<0$, according to the convention that we'll take the quadratic root with positive real part for $\eta>0$, we know that $\sqrt{(b\cdot\omega)^2\mu-\gamma^2(0)}$ is a positive imaginary if $\gamma^2(0)>0$, and $\sqrt{(b\cdot\omega)^2\mu-\gamma^2(0)}$ is a negative imaginary if $\gamma^2(0)<0$. We are going to consider the situation with $\gamma^2(0)>0$ (the situation with $\gamma^2(0)<0$ can be discussed similarly). Then (5.3) is equivalent to: $b\cdot\omega+\alpha\gamma(0)-\alpha\sqrt{\gamma^2(0)-(b\cdot\omega)^2\mu}=0$. So we have

$$\alpha=\frac{-b\cdot\omega}{\gamma(0)-\sqrt{\gamma^2(0)-(b\cdot\omega)^2\mu}}.$$

Let $X=\frac{\gamma(0)}{|b\cdot\omega|}$. Noticing the definition of γ, we can see that X may take any real value when (ξ, ω) varies on the unit sphere. Now that $|\alpha|=\frac{1}{X-\sqrt{X^2-\mu}}$, we should have $X\geqslant\sqrt{\mu}$. Since $X-\sqrt{X^2-\mu}$ is a monotonously decreasing function of X, and $X=\sqrt{\mu}$ is the minimal point of the function $|\alpha|$, we have $|\alpha|\geqslant\frac{1}{\sqrt{\mu}}$ i. e. $|\alpha|\geqslant\sqrt{\varkappa}$. So the condition for Kreiss well-posedness is also $|\alpha|<\sqrt{\varkappa}$. This ends the proof.

Generally speaking, the stable admissibility is not equivalent to the Kreiss well-posedness. It could be shown by the following example:

$$\begin{cases}\partial_t u=\begin{bmatrix}-1 & & \\ & -1 & \\ & & 1\end{bmatrix}\partial_x u+\begin{bmatrix} & & 1\\ & & 1\\ 1 & 1 & \end{bmatrix}\frac{\partial u}{\partial y}+Eu=F,\ t>0,\ x>0,\\ u(0)=0,\ u_1-\alpha_1u_3|_{x=0}=g_1,\ u_2-\alpha_2u_3|_{x=0}=g_2.\end{cases}\tag{5.4}$$

Now we have

$$M(s, \omega)=\begin{bmatrix}-s & 0 & i\omega\\ 0 & -s & i\omega\\ -i\omega & -i\omega & s\end{bmatrix}$$

with its eigenvalue $-s$, $\pm\sqrt{s^2+2\omega^2}$. When $\omega\neq0$, the three eigenvalues are different. When $\omega=0$, $M(s, \omega)$ is already a diagonal matrix. So (5.4) is a Kreiss hyperbolic symmetric system.

The eigenvectors of the $M(s, \omega)$, corresponding to the eigenvalues with negative real part, are $v_1=(1, -1, 0)^t$, $v_2=(i\omega, i\omega, s-\sqrt{s^2+2\omega^2})^t$. Substituting them into the boundary condition, we can calculate that the Kreiss' well-posedness will be determined by

$$(\alpha_1+\alpha_2)(s-\sqrt{s^2+2\omega^2})\neq2i\omega,\ |s|^2+|\omega|^2=1,\ \mathrm{Re}\ s\geqslant0.$$

If $\eta=0$, $\xi^2<2\omega^2$, the above condition is evidently satisfied. If $\xi^2>2\omega^2$, the invalidity of the above condition means

$$\text{for } \xi>0,\ (\alpha_1+\alpha_2)=2\omega(\xi-\sqrt{\xi^2-2\omega^2})^{-1};$$
$$\text{for } \xi<0,\ (\alpha_1+\alpha_2)=2\omega(\xi+\sqrt{\xi^2-2\omega^2})^{-1}.$$

From this, one gets $|\alpha_1+\alpha_2|\geqslant\sqrt{2}$. So for (5.4), the condition for the Kreiss' well-posedness is $|\alpha_1+\alpha_2|<\sqrt{2}$. The satable admissible condition is easily proved to be $\alpha_1^2+\alpha_2^2<1$, which is evidently stronger than the Kreiss condition.

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