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SINGULAR DIRECTIONS OF $WF(u)$ AND ITS APPLICATION

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Abstract

This paper introduces the notion of the singular direction of wave front sets for distributions and proves the invariance of the singular direction under the elliptic equivalent transformation. A kind of Fourier integral operators, which ensure such invariance, are also investigated. The results obtained are applied to the propagation of singularities for a class of differential operators with multiple characteristics.

As we know, [4] gave the important results about the propagation of singularities for operators of principal type. [2, 3] investigated the same problem near multiple characteristics for those of non-principal type. The present paper is to study a class of differential operators of non-principal type, too. The purpose of § 1 is to refine the wave front set of distributions and to introduce the notion of singular direction of $WF(u)$. § 2 is concerned with the invariance of singular direction under the elliptic equivalent transformation. Unfortunately, singular direction is not always invariant under the transformation of independent variables. In § 3, a necessary condition, in order that singular direction is invariant, is given and a class of Fourier integral operators, under the action of which singular direction is invariant, are studied. § 4 and § 5 are devoted to the propagation of singularities for a class of operators with multiple characteristics.

§ 1. Singular Directions of $WF(u)$

Let $u\in\mathscr{D}'(\Omega)$. As is well known, $(x_0, \xi^0)\overline{\in}WF(u)$ if and only if there exist a $\varphi(x)\in C_0^\infty(R^n)$ with $\varphi(x_0)\neq 0$ and a conical neighbourhood of ξ^0, $\Gamma_\delta(\xi^0)=\{\xi\,|\,|\xi/|\xi|-\xi^0/|\xi^0||<\delta\}$ such that $\widehat{\varphi u}(\xi)$ rapidly decreases in $\Gamma_\delta(\xi^0)$. Let $(x_0, \xi^0)\in WF(u)$. The question naturally asked is whether there exist some part of $\Gamma_\delta(\xi^0)$, $K_\delta(\xi^0, \eta^0)=\{\xi\,|\,\xi=\lambda\xi^0+\eta\in\Gamma_\delta(\xi^0),\ \eta\neq 0,\ \eta;\ \eta^0$ are perpendicular to ξ^0, and $|\eta/|\eta|-\eta^0/|\eta^0||<\delta\}$, such that $\widehat{\varphi u}$ rapidly decreases in $K_\delta(\xi^0, \eta^0)$ for some $\varphi(x)\in C_0^\infty(R^n)$ with $\varphi(x_0)\neq 0$. Indeed, we can introduce the following.

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Definition 1.1. *Let $u\in\mathscr{D}(\Omega)$ and let $(x_0, \xi^0)\in T^*(\Omega)$. We say that η^0 is not a singular direction of $WF(u)$ at (x_0, ξ^0) if there exist a conical neighbourhood of ξ^0, $\Gamma_\delta(\xi^0)$, and its some part $k_\delta(\xi^0, \eta^0)$ such that*

$$\int_{\xi=\lambda\xi^0+\eta\in k_\delta(\xi^0,\eta^0),\eta\perp\xi^0} |\widehat{\varphi u}(\xi)|^2(1+|\xi^2|)^{-l}(1+|\eta^2|)^N d\lambda\, d\eta\leqslant C_N \tag{1.1}$$

for some $\varphi(x)\in C_0^\infty(R^n)$ with $\varphi(x_0)\neq 0$ and some integer l and any integer N. Denote it by

$$\eta^0\overline{\in} SWF(u)(x_0, \xi^0). \tag{1.2}$$

Remark 1. (1. 1) is equivalent to

$$\int_{\xi+\lambda\xi^0+\eta\in k_\delta(\xi^0,\eta^0),\eta\perp\eta^0} |\widehat{\varphi u}(\xi)|(1+|\xi^2|)^{-l}(1+|\eta^2|)^N d\lambda\, d\eta\leqslant C_N \tag{1.1'}$$

for other l and C_N.

In fact, if (1. 1) holds, by choosing new $l>l+n/2$ and applying Cauchy inequality to (1. 1′) the boundness of (1. 1′) follows immediately. Assume, conversely, that (1. 1′) is valid. Because $\varphi v\in \varepsilon'(\Omega)$, the growth of $\widehat{\varphi u}(\xi)$ at infinity does not exceed that of some polynomial. It is easy to obtain (1. 1) after a choice of another big enough l.

Remark 2. Definition (1. 1) is also equivalent to that

$$|\widehat{\varphi u}(\xi)|(1+|\xi^2|)^{-l}(1+|\eta^2|)^N\leqslant C_N \tag{1.1''}$$
$$\text{when } \xi=\lambda\xi^0+\eta\in k_\delta(\xi^0, \eta^0),\ \eta\perp\xi^0,$$

for some $\Gamma_\delta(\xi^0)$, $K_\delta(\xi^0, \eta^0)$, some integer l and any integer N. Evidently, (1. 1″) implies (1. 1) and (1. 1′). But the converse proof is too long and is postponed until § 2. This is an immediate consequence of Lemma 2. 1.

Remark 3. (1. 1) is always valid for any l, N and any direction $\eta^0\perp\xi^0$ if $(x_0, \xi^0)\overline{\in} WF(u)$.

Let us consider several examples. If $u=H(x_n)H(x_{n-1})$, where $H(t)$ is the Heaviside function, obviously, $(x_0=0, \xi^0=(0, \cdots, 0, 1))\in WF(u)$. It is easily seen that $(\eta'', \eta_{n-1}, 0)\overline{\in} SWF(u)(x_0, \xi^0)$ with $\eta''\neq 0$ and $(0, 1, 0)\in SWF(u)(x_0, \xi^0)$. A direct computation yields $\eta^0\overline{\in} SWF(H(x_n))(x_0, \xi^0)$ for any $\eta^0\perp\xi^0$, whereas for any $\eta_0\perp\xi^0$, $\eta^0\in SWF(u)(x_0, \xi^0)$ if $u=\prod_{i=1}^n H(x_i)$. However, there exist some points (x_0, ξ^0) of wave front set without any singular direction, but in contrast there exist some points (x_0, ξ^0) of wave front set with singular directions full of all directions perpendicular to ξ^0.

Remark 4. If in Definition 1. 1 we demand that $\varphi(x)=1$ near x_0, no change of Definition 1. 1 will happen.

Remark 5. It is not difficult to see that Definition 1. 1 is invariant under the translation and rotation of coordinates.

Let $n_0\in T^*(R^n)$ and let P and Q be pseudodifferential operators defined in some neighbourhood of n_0, $\Gamma(n_0)$. According to the definition of [2], we call them microlocal equivalence and denote them by $P\sim Q$ if $\sigma(P)-\sigma(Q)\in S^{-\infty}(\Gamma(n_0))$. We say that P and Q are elliptic equivalence at n_0 if $A_1PA_2\sim Q$ for some pseudodifferential operators A_1, A_2 which are elliptic at n_0. Let P and Q be pseudodlfferential operators defined in $\Gamma_1(n_1)\subset T^*(\Omega_x)$ and $\Gamma_2(n_2)\subset T^*(\Omega_y)$, resp.. Assume that there exist a homogeneous canonical transformation χ from $\Gamma_1(n_1)$ onto $\Gamma_2(n_2)$ and an elliptic Fourier integral operator at (n_1, n_2) associated with χ such that $FPF^{-1}\sim Q$ at n_2. Then we say that P, Q are Fourier equivalence. In this paper, both of elliptic equivalent transformation and Fourier equivalent transformation are called the microlocal equivalent transformation. One wonder if the singular direction of $WF(u)$ is invariant under the microlocal equivalent transformation. Several propositions in § 2 will answer this question.

§ 2. Singular Directions and Elliptic Equivalent Transfomation

Because the properties discussed in this paper are all local, we need only to deal with the pseudodifferential operators or Fourier integral operators defined in some conical neighbourhood of a concerned point. Therefore, that we write $A\in L^m$ implies that $\sigma(A)\in S^m(\Gamma(n))$ where $\Gamma(n)$ is some conical neighbourhood of the point n under consideration. In the sequal, if no otherwise statement, all psendodifferential operators involved are properly supported.

Lemma 2.1. *Let $u\in\mathscr{D}'(\Omega)$ with $(x_0, \xi^0)\in WF(u)$. If $\eta^0\overline{\in}SWF(u)(x_0, \xi^0)$ for some η^0 perpendicular to ξ^0, then $\eta^0\overline{\in}SWF(Au)(x_0, \xi^0)$ for any $A\in L^m$.*

Proof In view of Remark 5, without loss of generality, we may suppose that $x_0=0$, $\xi^0=(0, \cdots, 0, 1)$. Let $\varphi(x)$, $h(x)\in C_0^\infty(R^n)$ equal 1 near x_0. $\varphi Au=\varphi Ahu+\varphi A(1-h)u$. It is evident that $\varphi A(1-h)u\in C^\infty$ near x_0. We need, therefore, only to investigate φAhu. By Definition 1. 1, we can assume that $\widehat{hu}(\xi)$ satisfies (1. 1′). Now we write

$$\widehat{\varphi A(hu)}(\xi)=\int e^{ix(\eta-\xi)}\varphi(x)\sigma(A)(x, \eta)\widehat{hu}(\eta)d\eta\, dx \tag{2.1}$$

$$=\int_{R^n\setminus\Gamma_\delta(\xi^0)}+\int_{\Gamma_\delta(\xi^0)\setminus K_\delta(\xi^0,\eta^0)}+\int_{K_\delta(\xi^0,\eta^0)}e^{i(\eta-\xi)x}F(x, \eta)d\eta\, dx$$

$$=I_1(\xi)+I_2(\xi)+I_3(\xi). \tag{2.2}$$

From the standard procedure of attacking the oscillatory integral, it follows that $I_1(\xi)$ decreases rapidly in $\Gamma_{\delta/2}(\xi^0)$.

Let us now study $I_2(\xi)$. Since hu$\in\mathscr{E}'(R^n)$, $|\widehat{hu}(\xi)|\leqslant C(1+|\xi^2|)^s$ for any $\xi\in R^n$ and some constant C. Take an integer $l_1>[s]+[m/2]+[n/2]+3$.

$$(1+|\xi^2|)^{-l_1}(1+|\xi'|^2)^N I_2(\xi)$$
$$=\int_{\Gamma_\delta(\xi^0)\backslash K_\delta(\xi^0,\eta^0)} e^{ix(\eta-\xi)} \frac{(1+|\xi|^2)^{-l_1}(1+|\xi'|^2)^N}{(1+|\xi-\eta|^2)^{N_1}(1+|\xi'-\eta'|^2)^{N_2}}$$
$$\cdot(I-\Delta_{x'})^{N_2}(I-\Delta_x)^{N_1} F\, dx\, d\eta,$$

where $\xi'=(\xi_1, \cdots \xi_{n-1})$. Note that $(1+|\xi|^2)^s\leqslant 2^{|s|}(1+|\eta|^2)^{|s|}$ and

$$|\xi'-\eta'|\geqslant C(|\xi'|+|\eta'|),\ \xi\in K_{\delta/2}(\xi^0, \eta^0),\ \eta\in\Gamma_\delta(\xi^0)\backslash K_\delta(\xi^0, \eta^0). \quad (2.3)$$

Direct computation yields

$$(1+|\xi|^2)^{-l_1}(1+|\xi'|^2)^N|I_2(\xi)|\leqslant C_N \quad (2.4)$$

when $l_1<N_1$, $N<N_2$ and $\xi\in K_{\delta/}(\xi^0, \eta)$.

It only remains to deal with $I_3(\xi)$. Choose $l'_1=\max(l_1, l+1+[m/2])$. Thus

$$|(1+|\xi|^2)^{-l'_1}(1+|\xi|^2)^N I_3(\xi)|$$
$$=\left|\int_{K_\delta(\xi^0,\eta^0)} e^{i(-x\xi+x_n\eta_n)} \frac{(1+|\xi|^2)^{-l'_1}(I-\Delta_{x'})^N[(e^{ix'\eta'}(I-\Delta_x)^{N_1} F}{(1+|\xi-\eta|^2)^{N_1}} dx\, d\eta\right|$$
$$\leqslant C_N \quad (2.5)$$

when $N_1>l'_1$. In getting (2. 5) we have used the fact that $\widehat{hu}(\eta)$ satisfies (1. 1′) in $K_\delta(\xi^0, \eta^0)$. Now the proof is complete, because (1. 1″) implies (1. 1).

Corollary 2.1. (1. 1) *implies that* (1. 1″) *is valid in* $K_{\delta/2}(\xi^0, \eta^0)$.

Proof Identity operator I is also a pseudodifferential operator. Let $\eta^0\overline{\in} SWF(u)(x_0, \xi^0)$. Substituting I for A in proving Lemma 2. 2, we can at once obtain (2. 4), (2. 5) and prove that $I_1(\xi)$ decreases rapidly in $\Gamma_{\delta/2}(\xi^0)$. So (1. 1″) is proved.

Corollary 2.2. *Let* $u\in\mathscr{D}'(\Omega)$ *and let* $A\in L^m$ *be elliptic at* (x_0, ξ^0). *Assume* η^0 *is perpendicular to* ξ^0. *Then the fact that* $\eta^0\in SWF(u)(x_0, \xi^0)$ *implies* $\eta^0\in SWF(Au)(x_0, \xi^0)$.

Proof Corollary 2. 2 is a direct consequence of Lamma 2. 1 and the existence of parametrix of elliptic pseudodifferential operator A.

§ 3. Singular Directions Under the Coordinate Transformation and the Fourier Equivalent Transformation

Before discussing the behavior of singular direction under the coordinate transformation, we have to introduce the following

Lemma 3.1. *Let* $u\in\mathscr{D}'(\Omega)$ *and let* $\Xi^0\overline{\in}SWF(u)(x_0, \xi^0)$ *where* Ξ^0 *is perpendicular to* ξ^0. *Then there exist positive constants* δ_x, δ_ξ *such that any* $(x, \xi)\overline{\in} WF(u)$ *if* $(x, \xi)\in O_{\delta_x}(x_0)\times K_{\delta_\xi}(\xi^0, \Xi^0)$ *where* $O_{\delta_x}(x_0)$ *stands for a ball of radius* δ_x *with* x_0 *as its centre.*

Proof Let $\Xi\overline{\in}SWF(u)(x_0, \xi^0)$. By the definition (1. 1) and Corollary 2. 1 one can find a cutoff function φ, which does not vanish in some neighbourhood $O_{\delta_x}(x_0)$, such that (1. 1″) holds and a cone $\Gamma_{\delta_1}(\xi^1)\subset K_\delta(\xi^0, \Xi^0)$. If $\xi=\lambda\xi^0+\eta\in\Gamma_{\delta_1}(\xi^1)$ with $\eta\perp\xi^0$, we have

$$|\lambda|\leqslant C|\eta|, \tag{3.1}$$

where C is a constant independent of λ, η. Thus from (1. 1″) it follows that when $\xi=\lambda\xi^0+\eta\in\Gamma_{\delta_1}(\xi^1)$,

$$|\widehat{\varphi u}(\xi)|(1+|\xi|^2)^N=|\widehat{\varphi u}(\xi)|(1+|\lambda\xi^0+\eta|^2)^N$$
$$\leqslant\text{Const.}\,|\widehat{\varphi u}(\xi)|(1+|\lambda\xi^0+\eta|^2)^{-l}(1+|\eta|)^{l+N}\leqslant C_N.$$

Lemma 3. 1 is proved.

Let us consider a mapping Φ: $x_i=y_i(i=1, \cdots, n-1)$, $x_n=y_n-(y_1^2+\cdots+y_n^2)$ which is a diffeomorphism of R^n_y onto R^n_x. Let $u=H(y_n)$ with $(y_0=0, \eta^0=(0, \cdots, 0, 1))\in WF(u)$. It is evident that $\Theta^0\overline{\in} SWF(u)(y_0, \eta^0)$ for any Θ^0 perpendicular to η^0. Under the induced mapping of Φ, $(y_0, \eta^0)\to(x_0, \xi^0)=(0, 0, \cdots, 0, 1)$, $\Theta^0\to\Xi^0=(\partial y/\partial x)^t(x_0)\Theta^0$ and $u\to\Phi^*u=H\left(x_n+\sum\limits_{\lambda=1}^{n-1}x_i^2\right)$. But $\Xi^0\in SWF(\Phi^*u)(x_0, \xi^0)$. In fact, $WF(\Phi^*u)=\left\{(x,\xi)\,\middle|\, x_n+\sum\limits_{i=1}^{n-1}x_i^2=0,\ 0\neq\xi \text{ parallel to } (2x', 1)\right\}$. Hence $(O_{\delta_x}(x_0)\times K_{\delta_\xi}(\xi^0, \Xi^0))\cap WF(\Phi^*u)\neq\phi$ for any positive constant δ_x and any δ_ξ, which means that $\Xi^0\in SWF(\Phi^*u)(x_0, \xi^0)$. This example shows that singular directions are not always invariant under the coordinate transformation. Next we shall point out that singular directions may be invariant under some circumstance.

Definition 1.2. *Let χ be a diffeomorphism: $T^*(\Omega_x)\supset\Gamma_1(x_0, \xi^0)\to\Gamma_2(y_0, \eta^0)\subset T^*(\Omega_y)$ with $\chi(x_0, \xi^0)=(y_0, \eta^0)$ and let $y(x, \xi)$, $\eta(x, \xi)$ be homogeneous of degree zero and 1 in ξ, resp.. χ is said to be parallel at point x_0 with respect to ξ^0, η^0 if there exists a neighbourhood of x_0, $O(x_0)$, such that $\eta(x, \xi^0)=\eta^0$ for any $x\in O(x_0)$.*

Remark 1. If χ is parallel at x_0 with respect to ξ^0, η^0, it follows that $\partial\eta/\partial x(x, \xi^0)=0$ as $x\in O(x_0)$, which implies

$$\det\left(\frac{\partial\eta}{\partial\xi}(x_0, \xi^0)\right)\neq0,\quad \det\left(\frac{\partial y}{\partial x}(x_0, \xi^0)\right)\neq0 \tag{3.2}$$

since χ is a diffeomorphism.

Let $\zeta\in R^n$ and let Π_ζ stand for a plane with ζ as its normal. Denote by $\Pi_\zeta v$ the projection of the vector v on the plane Π_ζ i. e., $\Pi_\zeta v=v-\langle v, \zeta\rangle\zeta/|\zeta|^2$.

Lemma 3.2. *Let χ be a homogeneous diffeomorphism from $\Gamma(x_0, \xi^0)\subset T^*(\Omega_x)$ onto $\Gamma(y_0, \eta^0)\subset T^*(\Omega_y)$ with $\eta(x_0, \xi^0)=\eta^0$, $y(x_0, \xi^0)=y_0$ and let χ be parallel at x_0 with respect to ξ^0, η^0. Then for any $O_{\delta_x}(x_0)\times K_{\delta_\xi}(\xi^0, \Xi^0)\subset\Gamma(x_0, \xi^0)$ where $\Xi^0\perp\xi^0$ there exist constants δ_y, δ_η such that*

$$\chi(O_{\delta_x}(x_0)\times K_{\delta_\xi}(\xi^0, \Xi^0))\supset O_{\delta_y}(y_0)\times K_{\delta_\eta}(\eta^0, \Theta^0), \tag{3.3}$$

where $\Theta^0=\Pi_{\eta^0}(\partial\eta/\partial\xi(x_0, \xi^0))\Xi^0$.

Proof Since the conditions and conclusions in Lemma 3. 2 are invariant under the translation and rotation in base space, we may assume that $x_0=y_0=0$, $\xi^0=\eta^0=(0, \cdots, 0, 1)$. Without difficulty we can prove that the inverse mapping χ^{-1} of the homogeneous diffeomorphism χ is also parallel at y_0 with respect to η^0, ξ^0, too. Application of (3. 2) to χ^{-1} gives $\det(\partial\xi/\partial\eta(y_0, \eta^0))\neq 0$, $\det(\partial x/\partial y(x_0, \xi^0))\neq 0$. Of course, χ^{-1}: $\xi=\xi(y, \eta)$, $x=x(y, \eta)$ is also a homogeneous diffeomorphism from $\Gamma(y_0, \eta^0)$ onto $\Gamma(x_0, \xi^0)$. One can find a conical neighbourhood $O(y_0)\times\Gamma(\eta^0)\subset\Gamma(y_0, \eta^0)$, $\xi^0(y, \eta^0)$ as $y\in O(y_0)$. Under the present coordinates $(\xi^0)_j=(\eta^0)_j=0$ $(j=1, \cdots, n-1)$. By the homogeneity of $\xi(y, \eta)$, we have $O=(\xi^0)_j=\partial(\xi)_j/\partial\eta_l(y, \eta^0)(\eta^0)_l=\partial(\xi)_j/\partial\eta_n(y, \eta^0)$. It follows from $\det(\partial\xi/\partial\eta(y_0, \eta^0))\neq 0$ that

$$\partial(\xi)_n/\partial\eta_n(y_0, \eta^0)\neq 0. \tag{3.4}$$

The key to our proof is to find small enough constants δ_y and δ_η such that (3. 3) holds. Suppose that $(y, \eta)\in O_{\delta_y}(y_0)\times K_{\delta_\eta}(\eta^0, \Theta^0)$. Then

$$|\xi_j(y, \eta)|=|(\partial\xi_j/\partial\eta_l(y, \eta))(\eta)_l|\leqslant C_1\delta_\eta|\eta_n| \quad (j=1, \cdots, n-1), \tag{3.5}$$

where constant C_1 is independent of y, η. On the other hand, $|\xi_n(y, \eta)|\geqslant|\partial\xi_n/\partial\eta_n(y, \eta)\eta_n|-C_2\delta_\eta|\eta_n|$. Using (3. 4) and choosing sufficient small δ_η and δ_y we can get

$$|\xi_n(y, \eta)|\geqslant C_3|\eta_n| \text{ when } (y, \eta)\in O_{\delta_y}(y_0)\times K_{\delta_\eta}(\eta^0, \Theta^0) \tag{3.6}$$

for some positive constant C_3. From (3. 5), (3. 6) it follows immediately that

$$|\xi'(y, \eta)|\leqslant C_4\delta_\eta|\xi_n| \text{ when } (y, \eta)\in O_{\delta_y}(y_0)\times K_{\delta_\eta}(y_0, \Theta^0).$$

Choosing δ_ε small enough we have $\xi(y, \eta)\in\Gamma_{\delta_\xi}(\xi^0)$. It only remains to show $\xi(y, \eta)\in K_{\delta_\xi}(\xi^0, \Xi^0)$. Evidently, $\xi'(y, \eta)\neq 0$ when $(y, \eta)\in O_{\delta_y}(y_0)\times K_{\delta_\eta}(\eta^0, \Theta^0)$, since the mapping χ^{-1} is a diffeomorphism parallel at y_0 with respect to η^0, ξ^0. Let us estimate $\xi'(y, \eta)/|\xi'(y, \eta)|$. By the definition of Θ^0 in the present lemma, $(\Theta^0)_j=\partial\eta_j/\partial\xi_l(x_0, \xi^0)(\Xi^0)_l$. From the fact that $\partial\xi'/\partial y(y, \eta^0)=0$, $\partial\xi_j/\partial\eta_n(y_0, \eta^0)=0$ $(j=1, \cdots, n-1)$ we deduce

$$(\Xi^0)_j=\sum_{l=1}^{n-1}\frac{\partial\xi_j}{\partial\eta_l}(y_0, \eta^0)(\Theta^0)_l. \tag{3.7}$$

Let $\eta=\lambda\eta^0+\eta'\in K_{\delta_\eta}(\eta^0, \Theta^0)$. Then

$$\begin{aligned}\xi'(y, \lambda\eta^0+\eta')&=\sum_{l=1}^{n-1}\int_0^1\frac{\partial\xi'}{\partial\eta_l}(y, \lambda\eta^0+\theta\eta')\eta_l\,d\theta\\&=\sum_{l=1}^{n-1}\int_0^1\frac{\partial\xi'}{\partial\eta_l}\left(y, \eta^0+\frac{\theta}{\lambda}\eta'\right)\eta_l\,d\theta.\end{aligned} \tag{3.8}$$

Hence when $\lambda\eta^0+\eta'\in K_{\delta_\eta}$,

$$\begin{aligned}&\lim_{y\to y_0,\,\delta_\eta\to 0}\xi'(y, \eta)/|\xi'(y, \eta)|\\&=\sum_{l=1}^{n-1}\partial\xi'/\partial\eta_l(y_0, \eta^0)(\Theta^0)_l\Big/\Big|\sum_{l=1}^{n-1}\partial\xi'/\partial\eta_l(y_0, \eta^0)(\Theta^0)_l\Big|\\&=\Xi^0/|\Xi^0|.\end{aligned} \tag{3.9}$$

In getting (3. 9), we have used (3. 7). Now we come to the conclusion that one can choose δ_y, δ_η small enough such that

$$|\xi'(y,\eta)/|\xi'(y,\eta)|-\Xi^0/|\Xi^0||<\delta_\xi$$

when $(y,\eta)\in O_{\delta_y}(y_0)\times K_{\delta_\eta}(\eta^0,\Theta^0)$, which means $\xi(y,\eta)\in K_{\delta_\xi}(\xi^0,\Xi^0)$.

The remainder of the proof is to deal with $x(y,\eta)$. Because

$$|x(y,\eta)|=\left|x\left(y,\frac{\eta}{|\eta|}\right)-x\left(y_0,\frac{\eta^0}{|\eta^1|}\right)\right|\leqslant C_5(\delta_y+\delta_\eta),$$

it follows that $x(y,\eta)\in O_{\delta_x}(x_0)$ when δ_y and δ_η are small enough. The proof is complete.

Proposition 3.1. *Let $u\in\mathscr{D}'(\Omega_y)$ with $\Theta^0\overline{\in}SWF(u)(y_0,\eta^0)$. If the diffeomorphism φ: $\Omega_x\to\Omega_y$ with $\varphi(x_0)=y_0$ satisfies*

$$\left(\frac{\partial y}{\partial x}\right)^t(x)\eta^0=\left(\frac{\partial y}{\partial x}\right)^t(x_0)\eta^0=\xi^0,$$

$$\text{when } x \text{ is in some neighbourhood of } x_0,\ O(x_0), \tag{3.10}$$

then $\Xi^0\overline{\in}SWF(\varphi^*u)(x_0,\xi^0)$, where $\Xi^0=\Pi_{\xi^0}((\partial y/\partial x)^t(x_0)\Theta^0)$.

Proof The induced mapping in cotangent bundle of φ, φ^*: $x=\varphi^{-1}(y)$, $\xi=(\partial y/\partial x)^t\eta$ is parallel at y_0 with respect to η^0, ξ^0 because of (3.10). Remark 5 provides that we can assume $x_0=y_0=0$, $\xi^0=\eta^0=(0,\cdots,0,1)$. Under the present circumstance (3.10) may be written as

$$\partial y_n/\partial x_j(x)=0,\ x\in O(x_0) \text{ and } \partial x_n/\partial y_j(y)=0,\ y\in\varphi(O(x_0))\quad (j=1,\cdots,n-1). \tag{3.10'}$$

Let us turn to the study of the behavior of φ^*u near (x_0,ξ^0). Take functions $h_i\in C_C^\infty(\Omega_x)$ $(i=1,2)$ in such a way that $h_i(x_0)\neq 0$ and $\widehat{h_1(\varphi^{-1}(y))u}(\eta)$ satisfies (1.1') for some $\Gamma_\delta(\eta^0)$ and $K_\delta(\eta^0,\Theta^0)$. Other restrictions on h_2 will be described later. This is possible since $\Theta^0\overline{\in}SWF(u)(y_0,\eta^0)$.

$$\begin{aligned}\widehat{h_1h_2\varphi^*u}(\xi)&=\langle h_1h_2\varphi^*u,e^{-ix\xi}\rangle\\&=\int_{R^n\setminus\Gamma_\delta(\eta^0)}+\int_{\Gamma_\delta(\eta^0)\setminus K_\delta(\eta^0,\Theta^0)}+\int_{K_\delta(\eta^0,\Theta^0)}\widehat{h_1u}(\eta)I(\xi,\eta)d\eta\\&=I_1(\xi)+I_2(\xi)+I_3(\xi),\end{aligned}\tag{3.11}$$

where $h_1u=h_1(\varphi^{-1}(y))u(y)$ and

$$I(\xi,\eta)=\int e^{-i(x_l(y)\xi_l-y_j\eta_j)}h_2(\varphi^{-1}(y))|\partial x/\partial y(y)|dy.$$

By the standard procedure of dealing with oscillatory integral, we can find a positive constant δ_1 such that $I_1(\xi)$ decreases rapidly in $\Gamma_{\delta_1}(\xi^0)$. Consider, now, $I_3(\xi)$. From Lemma 3. 2 we see that for some $O_{\delta_x}(x_0)\times K_{\delta_1}(\xi^0,\Xi^0)$, if necessary, taking smaller δ_1,

$$\varphi^*(\varphi(O(x_0))\times K_\delta(\eta^0,\Theta^0))\supset O_{\delta_x}(x_0)\times K_{\delta_1}(\xi^0,\Xi^0) \tag{3.12}$$

since φ^* is parallel at y_0 with respect to η^0, ξ^0. Choose $h_2\in C_C^\infty(O_{\delta_x}(x_0))$ with support so small that $\partial x_n/\partial y_j=0$ when $y\in\text{Supp }h_2(\varphi^{-1}(y))$ $(j=1,\cdots,n-1)$. Then for any $N\in Z$,

$$|I_3(\xi)|(1+|\xi|^2)^{-l}(1+|\xi'|^2)^N$$
$$=\int_{K_\delta(\eta^0,\Theta^0)} e^{-i(x_l(y)\xi_l-y_j\xi_j)}\widehat{h_1u}(\eta)(1+|\xi|^2)^{-l}(1+|\xi'|^2)^N$$
$$\cdot\frac{(I-\Delta_y)^{l+N}(h_2\det(\partial x/\partial y))}{(1+|(\partial x/\partial y)^t\xi-\eta|^2)^{l+N}}\,dx\,d\eta. \tag{3.13}$$

The special choice of h_2 provides

$$\left(\left(\frac{\partial x}{\partial y}\right)^t\xi\right)'=\left(\frac{\partial x'}{\partial y'}\right)^t\xi' \text{ when } y\in\operatorname{supp} h_2(\varphi^{-1}(y)). \tag{3.14}$$

Thus

$$(1+|\xi'|^2)\leqslant\text{const.}\left(1+\left|\left(\frac{\partial x'}{\partial y'}\right)^t\xi'\right|^2\right)$$
$$\leqslant\text{const.}\left(1+\left|\left[\left(\frac{\partial x}{\partial y}\right)^t\xi\right]'-\eta'\right|^2\right)(1+|\eta'|^2).$$

By an argument similar to Lemma 2. 1, we have

$$|I_3(\xi)|(1+|\xi|^2)^{-l}(1+|\xi'|^2)^N\leqslant C_N. \tag{3.15}$$

The remainder of the proof is to study $I_2(\xi)$. From (3. 12), (3. 14) we can deduce that when $\xi\in K_{\delta_1}(\xi^0, \Xi^0)$, $\eta\in\Gamma_\delta(n_2)\backslash K_\delta(\eta^0, \Theta^0)$ and $y\in O(y_0)$,

$$0\neq\left[\left(\frac{\partial x}{\partial y}\right)^t\xi\right]'-\eta'=\left(\frac{\partial x'}{\partial y'}\right)^t\xi'-\eta',$$

which implies that for some constant C,

$$\left|\left(\frac{\partial x'}{\partial y'}\right)^t\xi'-\eta'\right|\geqslant C(|\xi'|+|\eta'|) \text{ when } \xi\in K_{\delta_1}(\xi^0, \Xi^0)$$
$$\text{and } (y, \eta)\in\varphi(O(x_0))\times(\Gamma_\delta(\eta^0)\backslash K_\delta(\eta^0, \Theta^0)). \tag{3.16}$$

On the other hand

$$(I-\Delta_{y'})\exp(-i(x_l(y)\xi_l-y_j\eta_j))$$
$$=\left(1+\left|\left[\left(\frac{\partial x}{\partial y}\right)^t\xi\right]'-y'\right|^2\right)\exp(-i(x_l(y)\xi_l-y_j\eta_j))$$
$$=\left(1+\left|\left(\frac{\partial x'}{\partial y'}\right)^t\xi'-\eta'\right|^2\right)\exp(-i(x_l(y)\xi_l-y_j\eta_j)). \tag{3.17}$$

Combining (3. 16) and (3. 17) and repeating the same argument as in dealing with $I_2(\xi)$ in Lemma 2. 1 we can get

$$|I_2(\xi)|(1+|\xi|^2)^{-l}(1+|\xi'|^2)^N\leqslant C_N \text{ when } \xi\in K_{\delta_1}(\xi^0, \Xi^0).$$

Proposition 3. 1 is proved.

Remark 1. Proposition gives the invariance of singular directions under the affine transformation of independent variables.

Remark 2. Consider a local diffeomorphism: $y_n=x_n$, $y_j=\varphi_j(x)$ $(j=1, \cdots, n-1)$ with det $(\partial\varphi/\partial x'\ (0, 0))\neq 0$. Then its induced mapping in cotangent bundle: $x_n=y_n$, $x_j=\psi_j(y)$, $\xi=(\partial y/\partial x)^t\eta$ is parallel at $y_0=0$ with respect to $\eta^0=\xi^0=(0, \cdots, 0, 1)$, since $(\partial y/\partial x)^t(x)\eta^0=\xi^0$ as x is in some neighbourhood of $x_0=0$.

Proposition 3.2. *Let χ be a homogeneous canonical transformation from $\Gamma(y_0, \eta^0)\subset T^*(\Omega_y)$ onto $\Gamma(x_0, \xi^0)\subset T^*(\Omega_x)$ and let χ be parallel at y_0 with respect to*

η^0, ξ^0. *If* $\Theta^0 \in SWF(u)(y_0, \eta^0)$, *then* $\Pi_{\xi^0}(\partial\xi/\partial\eta)(y_0, \eta^0)\Theta^0 = \Xi^0 \in SWF(Fu)(y_0, \eta^0)$ *for any Fourier integral operator associated with* χ.

Proof It suffices to prove the case of $\eta^0=\xi^0=(0, \cdots, 0, 1)$. The parallel property of χ yields $\det\ (\partial\xi/\partial\eta(y_0, \eta^0)) \neq 0$. By the theory of Fourier integral operators ([5]) one can find a generating function $S(x, \eta)$ such that the phase function of Fourier integral operator $\varphi(x, y, \eta) = S(x, \eta) - y\eta$ and the homogeneous canonical transformation may be written as

$$\xi = \partial S/\partial x(x, \eta), \quad y = \partial S/\partial\eta(x, \eta). \tag{3.18}$$

Now the parallel property of χ gives

$$S_{x'}(x, \eta^0) = 0, \quad S_{x_n}(x, \eta^0) = 0 \tag{3.19}$$

when x is in some neighbourhood of x_0, $O(x_0)$.
The assumption $\Theta^0 \in SWF(u)(y_0, \eta^0)$ provides that there is a function $h \in C_C^\infty(\Omega_y)$ with $h=1$ as y is in some neighbourhood of y_0, $O(y_0)$, such that $\widehat{hu}(\eta)$ satisfies (1.1′) for some $\Gamma_\delta(\eta^0)$ and $K_\delta(\eta^0, \Theta^0)$. Furthermore, from (3.19) we have

$$\det\ (S_{x\eta}(x, \eta)) \neq 0, \ \forall (x, \eta) \in O(x_0) \times \Gamma_\delta(\eta^0), \tag{3.19′}$$

if necessary, shrinking $\Gamma_\delta(\eta^0)$. Because $F(u) = F(hu) + F((1-h)u)$ and $(x_0, \xi^0) \overline{\in} WF(F(1-h)u)$, it suffices to study $F(hu)$. From Lemma 3.2 we can get

$$\chi(O(y_0) \times K_\delta(\eta^0, \Theta^0)) \supset O(x_0) \times K_{\delta_1}(\xi^0, \Xi^0) \tag{3.20}$$

for some $O(x_0)$ and $K_{\delta_1}(\xi^0, \Xi^0)$. Without loss of generality, we can assume that $\chi(O(y_0) \times \Gamma_\delta(\eta^0)) \supset O(x_0) \times \Gamma_{\delta_1}(\xi^0)$. Let $\psi \in C_C^\infty(O(x_0))$. Then

$$\begin{aligned}\widehat{\psi F(hu)}(\xi) &= \left[\int_{R^n\setminus\Gamma_\delta(\eta^0)} + \int_{\Gamma_\delta(\eta^0)\setminus K_\delta(\eta^0,\Theta^0)} + \int_{K_\delta(\eta^0,\Theta^0)}\right] \widehat{hu}(\eta) I(\xi, \eta)\, d\eta \\ &= I_1(\xi) + I_2(\xi) + I_3(\xi),\end{aligned} \tag{3.21}$$

where

$$I(\xi, \eta) = \int \psi(x) a(x, \eta) e^{i(S(x,\eta) - x(\xi))}\, dx$$

and $a(x, \eta)$ is the amplitude of the Fourier integral operator under consideration. Along the familiar line of dealing with oscillatory integral we can also find another smaller δ_1 such that $I_1(\xi)$ decreases rapidly in $\Gamma_{\delta_1}(\xi^0)$. Note that

$$I(\xi, \eta) = \int e^{-i(x\xi - S(x,\eta))} \frac{(I - \Delta_x)^{N+l}(\psi(x)a(x, \eta))}{(1 + |S_x(x, \eta) - \xi|^2)^{N+l}}\, dx. \tag{3.22}$$

(3.19′) provides that

$$|\xi| = |S_x(x, \eta)| \geq \text{Const.}\ |\eta| \text{ when } (x, \eta) \in O(x_0) \times \Gamma_\delta(\eta^0).$$

Because $S_{x'\eta_n}(x, 0, 1) = 0$ and $S_{x'\eta_n}(x, \eta) = S_{x'\eta_n}(x, \eta'/\eta_n, 1)$ we have

$$\begin{aligned}|S_{x'}(x, \eta)| &= \left|\sum_{j=1}^{n-1} S_{x'\eta_j}(x, \eta)\eta_j + S_{x'\eta_n}(x, \eta)\eta_n\right| \\ &= \left|\sum_{j=1}^{n-1} S_{x'\eta_j}(x, \eta)\eta_j + \int_0^1 \partial S_{x'\eta_n}/\partial\eta'(x, \theta\, \eta'/\eta_n, 1)\eta'\, d\theta\right| \qquad (3.23) \\ &\leq \text{Const.}\ |\eta'|. \qquad (3.24)\end{aligned}$$

In view of the inequality

$$(1+|\xi|^2)^{-l}\leqslant 2^{|l|}(1+|\xi-S_x(x,\eta)|^2)^{|l|}(1+|S_x(x,\eta)|^2)^{-l}$$
$$\leqslant \text{Const.}(1+|\xi-S_x(x,\eta)|^2)^l(1+|\eta|^2)^{-l} \tag{3.25}$$

and by the same argument as in Lemma 2. 1, we can obtain

$$|I_3(\xi)|(1+|\xi|^2)^{-l}(1+|\xi'|^2)^N\leqslant C_N \tag{3.26}$$

for some l and any N.

Now we proceed to study $I_2(\xi)$. (3. 20) gives

$$S_{x'}(x,\eta)-\xi'\neq 0 \tag{3.27}$$

when $x\in\overline{O(x_0)}$, $\xi\in K_{\delta_1}(\xi^0, \Xi^0)$ and $\eta\in\Gamma_\delta(\eta^0)\backslash K_\delta(\eta^0, \Theta^0)$, which implies

$$|S_{x'}(x,\eta)-\xi'|\geqslant C(|\xi'|+|\eta'|) \tag{3.28}$$

for some positive constant C when x, ξ, η are in the region in question. In fact, if there exist sequences of $x_k\in\overline{O(x_0)}$ and $(\eta^k, \xi^k)\in(\Gamma_\delta(\eta^0)\backslash K_\delta(\eta^0, \Theta^0))\times K_{\delta_1}(\xi^0, \Xi^0)$ with $|(\eta^k)'|+|(\xi^k)'|=1$ such that

$$x_k\to x^*,\ |S_{x'}(x_k,\eta^k)-(\xi^k)'|\to 0(k\to+\infty), \tag{3.29}$$

obviously, $(\xi^k)'\to 0(k\to+\infty)$. If $\{(\eta^k)_n\}$ are bounded and $\eta^k\to\eta^*\in\Gamma_\delta(\eta^0)\backslash K_\delta(\eta^0, \Theta^0)$, then $S_{x'}(x^*,\eta^*)=\lim S_{x'}(x_k,\eta^k)=\lim(\xi^k)'=0$. More over, the canonical transformation χ maps $(y^*=S_\eta(x^*,\eta^*), \eta^*/S_{x_n}(x^*,\eta^*))$ to (x^*, ξ^0). By the parallel property of the diffeomorphism χ one can come to the conclusion: $\eta^*=S_{x_n}(x^*,\eta^*)\eta^0=(0, \cdots, 0, S_{x_n}(x^*,\eta^*))$, which contradicts $|(\eta^k)'|+|(\xi^k)'|=1$. Let us consider the case that $\{(\eta^k)_n\}$ are unbounded. we can assume that $(\eta^k)_n\to+\infty$ and $(\eta^k)'\to\theta\in R^{n-1}\backslash\{0\}$.

$$S_{x'}(x_k,\eta^k)=S_{x'\eta'}(x_k,(\eta^k)'/(\eta^k)_n,1)(\eta^k)'+S_{x'\eta_n}(x_k,(\eta^k)'/(\eta^k)_n,1)(\eta^k)_n. \tag{3.30}$$

Noting (3. 23) and $\partial/\partial\eta'(S_{x'\eta_n})(x^*, 0, 1)=\partial/\partial\eta_n\left(\dfrac{\partial\xi'}{\partial\eta'}\right)(x^*, 0, 1)=0$, and letting $k\to=\infty$ in both sides of (3. 30), we have at once

$$0=S_{x'\eta'}(x^*, 0, 1)\theta,$$

which means $\theta=0$ since $\det(S_{x\eta}(x^*,\eta^0))=\det(S_{x'\eta'}(x^*,\eta^0))$. This also contradicts the fact that $|(\eta^k)'|+|(\xi^k)'|=1$. The assertion of (3. 28) is proved. By means of (3. 28) and the same way as in Lemma 2. 1 it is not difficult to prove that

$$|I_3(\xi)|(1+|\xi|^2)^{-l}(1+|\xi'|^2)^N\leqslant C_n,$$

when $\xi\in K_{\delta_1}(\xi^0, \Xi^0)$ for some l and any N. The proof of Proposition 3. 2 is complete.

Corollary 3.1. *Let the assumption in Proposition 3. 2 be fulfilled. If Fourier integral operator F is elliptic at $(x_0, \xi^0, y_0, \eta^0)$ and $\Theta^0\in SWF(u)(y_0,\eta^0)$, then $\Xi^0\in SWF(Fu)(x_0,\xi^0)$.*

The proof of Corollary 3. 1 is the same as that in Corollary 2. 2 and need not be repeated.

§ 4. Propagation of Singularities for a Special Differential Operator

The purpose of the present section is to apply the results obtained previously to the propagation of singularities for a special differential operator

$$Lu=tu_t+b(x)u=f,\ \text{in}\ \Omega=\Omega_x\times\Omega_t\subset R^n_{xt}. \tag{4.1}$$

Here $b(x)\in C^\infty_C(\overline{\Omega}_x)$ and $O\in\Omega_t$. We have

Lemma 4.1. *Let u, $Lu\in L^2(\Omega)$. If* Re $b(x)>1/2$ *as $x\in\overline{\Omega}_x$, then*

$$u=\int_0^1 \lambda^{b(x)-1}f(x,\lambda t)\,d\lambda. \tag{4.2}$$

For the proof of Lemma 4. 1, refer to [6, Lemma 3. 1]. By analogy with [6, Lemma 3. 2], we have

Lemma 4.2. *Let $f\in L^2(\Omega)$. Set $n_0=(x_0, t_0, \xi^0, \tau^0)=(0, 0, 0, 1)$. If $\Xi^0\in SWF(f)(n_0)$, it follows that there exists a function $h\in C^\infty_C(\Omega)$ with $h(x_0, t_0)\neq 0$ satisfying*

$$|\widehat{hf}_\lambda(\xi,\tau)|(1+|\xi|^2+\tau^2)^{-l}(1+|\xi|^2)^N\leqslant C_N \tag{4.3}$$

for some l and any N. Here $hf_\lambda=h(x,t)f(x,\lambda t)$ and C_N is independent of λ. Throughout this section, if no otherwise statement, C is always independent of λ.

Proof It suffices to prove the case that for, $f\in\varepsilon'(\Omega)\cap L^2(\Omega)$ there exist $\Gamma_\delta(\xi^0,\tau^0)$ and $K_\delta(\xi^0,\tau^0,\Xi^0)$ such that

$$\int_{K_\delta(\xi^0,\tau^0,\Xi^0)}|\hat{f}(\xi,\tau)|(1+|\xi|^2+\tau^2)^{-l}(1+|\xi|^2)^N dt\,d\xi\leqslant C_N \tag{4.3'}$$

for some l and any N. Otherwise, we investigate φf instead of f, where $\varphi\in C^\infty_C(\Omega)$ with $\varphi=1$ near (x_0, t^0). Obviously, when $h\in C^\infty_C(\Omega)$,

$$\begin{aligned}\widehat{hf}_\lambda(\xi,\tau)&=\int\hat{h}(\xi-\eta,\tau-\lambda\sigma)\hat{f}(\eta,\sigma)\,d\eta\,d\sigma\\&=\int_{R^n\setminus\Gamma_\delta(\xi^0,\tau^0)}+\int_{\Gamma_\delta(\xi^0,\tau^0)\setminus K_\delta(\xi^0,\tau^0,\Xi^0)}+\int_{K_\delta(\xi^0,\tau^0,\Xi^0)}\hat{h}(\xi-\eta,\tau-\lambda\sigma)\hat{f}(\eta,\sigma)\,d\eta\,d\sigma\\&=I_1(\xi,\tau)+I_2(\xi,\tau)+I_3(\xi,\tau).\end{aligned} \tag{4.4}$$

We have

$$|\hat{h}(\xi-\eta,\tau-\lambda\sigma)|\leqslant C_N(1+|\xi-\eta|^2+|\tau-\lambda\sigma|^2)^{-N} \tag{4.5}$$

since $h\in C^\infty_C(\Omega)$. Without difficulty, we can derive

$$|\xi-\eta|^2+|\tau-\lambda\sigma|^2\geqslant C_1(|\xi|^2+|\eta|^2+|\tau|^2+|\sigma|^2) \tag{4.6}$$

when $(\xi,\tau)\in\Gamma_{\delta/2}(\xi^0,\tau^0)$, $(\eta,\sigma)R^n\setminus\Gamma_\delta(\xi_0,\tau^0)$ and $\lambda\in[0, 1]$. Along the familiar line it follows at once that $I_1(\xi,\tau)$ decreases rapidly in $\Gamma_{\delta/2}(\xi^0,\tau^0)$. Inserting (4.5) into (4. 4), we have

$$|I_3(\xi,\tau)|\leqslant C_{Nl}\int_{K(\xi^0,\tau^0,\Xi^0)}\frac{|\hat{f}(\eta,\sigma)|}{(1+|\xi-\eta|^2+|\tau-\sigma\lambda|^2)^{N+l}}\,d\eta\,d\sigma.$$

By means of (4. 3′) and the same argument as in Lemma 2. 1, the inequality

$$|I_3(\xi, \tau)|(1+|\xi|^2+\tau^2)^{-l}(1+\tau^2)^N \leqslant C_N \tag{4.7}$$

is obtained immediately. In view of the fact that when $(\xi, \tau) \in K_{\delta/2}(\xi^0, \tau^0, \Xi^0)$, $(\eta, \sigma) \in \Gamma_\delta(\xi^0, \tau^0) \setminus K_\delta(\xi^0, \tau^0, \Xi^0)$, the inequality

$$|\xi-\eta| \geqslant C_2(|\xi|+|\eta|) \tag{4.8}$$

holds. By a standard procedure we get the estimate of $I_2(\xi, \tau)$ similar to (4. 7), if necessary, choosing larger l.

Lemma 4.3. *Let $u, Lu \in L^2(\Omega)$. Assume that* Re $b(x) > 1/2$ *as $x \in \bar{\Omega}_\varepsilon$ and $\Xi^0 \bar{\in} SWF(f)(n_0)$. Then $\Xi^0 \bar{\in} SWF(u)(n_0)$.*

Proof From the assumption of this lemma and Lemma 4. 1, it follows that

$$u(x, t) = \int_0^1 \lambda^{b(x)-1} f(x, \lambda t) d\lambda.$$

Choose $h, h_1 \in C_C^\infty(\Omega)$ in such a way that $h_1 = 1$ as $(x, t) \in \operatorname{Supp} h$ and $\widehat{hf}_\lambda(\eta, \sigma)$ satisfies (4. 3). We then write

$$\widehat{hu}(\xi, \tau) = \widehat{hh_1u}(\xi, \tau) = \int_0^1 \int \hat{h}_1(\xi-\eta, \tau-\sigma, \lambda) hf_\lambda(\eta, \sigma)\, d\lambda\, d\sigma\, d\eta, \tag{4.9}$$

where

$$\hat{h}_1(\xi, \tau, \lambda) = \int \lambda^{b(x)-1} h_1(x, t) e^{-i(x\xi+t\tau)} dx\, dt,$$

the absolute value of which is bounded by

$$C_N \lambda^{b^*-1}(1+|\xi|^2+\tau^2)^{-N}. \tag{4.10}$$

Here $1/2 < b^* < \min$ Re $b(x)$ over $\bar{\Omega}_x$. This condition ensures $\int_0^1 \lambda \chi^{b^*-1} d\lambda < +\infty$. The rest of the proof is the same as that in Lemma 4. 2 and need not be repeated.

Proposition 4.1. *Let u, $Lu \in L^2(\Omega)$ and let Re $b(x) > 1/2$. Assume that $\Xi^0 \bar{\in} SWF(Lu)(n_0)$ and $(x_0, t_0, \Xi^0, \lambda T^0) \bar{\in} WF(Lu)$ for some T^0 and any $\lambda > 0$. Then $(x_0, t_0, \Xi^0, \lambda T^0) \bar{\in} WF(u)$ for any $\lambda > 0$.*

Proof From Lemma 4. 3 and the assumption that $\Xi^0 \bar{\in} SWF(Lu)(n_0)$, it follows that $\Xi^0 \bar{\in} SWF(u)(n_0)$. Lemma 3. 1 shows that there exists an $O_\delta(x_0, t_0) \times K_\delta(\xi^0, \tau^0, \Xi^0) \subset T^*(R^n_{xt})$ not meeting with $WF(u)$. Evidently, one can find a constant $\lambda^0 > 0$ such that $(x_0, t_0, \Xi^0, \lambda^0 T^0) \in O_\delta(x_0, t_0) \times K_\delta(\xi^0, \tau^0, \Xi^0)$. It is easily seen that Hamilton vector field, H_L, is unradial everywhere at the bicharacteristic strip γ: $(x_0, t_0, \Xi^0, \lambda T^0)$ $\lambda > 0$. Therefore γ does not meet $WF(u)$ since γ does not meet $WF(Lu)$.

§ 5. Applications

Consider a pseudodifferential operator P with

$$\sigma(P) \sim p_2 + p_1 + p_0 + \cdots$$

with real principal symbol

$$p_2=t\tau^2+e(x,\ t,\ \xi,\ \tau),\tag{5.1}$$

where p_k is homogeneous of degree k in ξ, τ. The fundamental hypothesis in this section is

$$e(x,\ t,\ 0,\ \tau)=0,\ e_{\xi_j}(x,\ 0,\ 0,\ \tau)=0\quad (j=1,\ \cdots,\ n-1).\tag{5.2}$$

Evidently, on $\Sigma=\{t=0,\ \xi=(\xi_1,\ \cdots,\ \xi_{n-1})=0\}$, H_{p_2} is parallel to $\xi\dfrac{\partial}{\partial\xi}+\tau\dfrac{\partial}{\partial\tau}=\tau\dfrac{\partial}{\partial\tau}$.

[6] studied the microlocal equivalent transformation about (5. 1) satisfying (5. 2). Now without proof we list the relevant results.

Lemma 5.1. *Let (5. 1) satisfy (5. 2). Then for any $(x_0,\ 0,\ 0,\ \tau^0)$ there exist conical neighbourhoods $\Gamma_1(x_0,\ 0,\ 0,\ \tau^0)\subset T^*(R^n_{xt})$ and $\Gamma_2(0,\ 0,\ 0,\ \tau^0)\subset T^*(R^n_y)$, a homogeneous canonical transformation χ: $\Gamma_1\in(x,\ t,\ \xi,\ \tau)\to(y_1,\ \cdots,\ y_n,\ \eta_1,\ \cdots,\ \eta_n)\in\Gamma_2$ and a Fourier integral operator associated with χ such that*

1. *$\eta_n=\tau$, $y_n=a(t+e(x,\ t,\ \xi/\tau,\ 1))$ for some $a\in S^0(\Gamma_1)$ with $a|_\Sigma\neq0$;*
2. *with $\Sigma'=\{y_n=0,\ \eta'=0\}$, we have $\chi(\Sigma\cap\Gamma_1)=\Sigma'\cap\Gamma_2$;*
3. *$FPF^{-1}\sim A_1(y_nD_n+q(y'))A_0$ for certain pseudodifferential operators $A_1\in L^1$, $A_0\in L^0$ elliptic at $(0,\ 0,\ 0,\ \tau^0)$, where*

$$q(y')=(p_1/\tau)|_{\Sigma^0}\chi^{-1}+\sqrt{-1},\quad \textit{mod real part}.\tag{5.4}$$

We shall show that the canonical transformation χ is parallel at $(x_0,\ 0)$ with respect to $(0,\ \tau^0)$. Indeed, from 1 it follows at once that $\partial\eta_j/\partial t=\{\eta_n,\ \eta_j\}=0$, $\partial y_j/\partial t=\{\eta_n,\ y_j\}=0(j=1,\ \cdots,\ n-1)$. Since $\chi(\Sigma\cap\Gamma_1)=\Sigma'\cap\Gamma_2$, $\eta'(x,\ t,\ 0,\ \tau^0)=\eta'(x,\ 0,\ 0,\ \tau^0)=0$, which implies the assertion expected. In view of proposition 3. 2, the singular direction is invariant under the microlocal equivalent transformation defined by Lemma 5. 1.

Lemma 5.2. *Let the principal symbol of P,*

$$p=t\tau^2+t\tau e_1(x,\ t,\ \xi)+e_2(x,\ t,\ \xi)\tag{5.5}$$

where e_1 and e_2 are real and homogeneous of degree 1 and 2 in ξ. If $e_2(x_0,\ 0,\ \xi^*)\neq0$, there exist the null-bicharacteristics of P, $\gamma_+(s)$ and $\gamma_-(s)$ defined in $(0,\ s_1]$ and $[-s_1,\ 0)$ for some positive constant s_1 and $\gamma_\pm(s)=(x_\pm(s)\ t_\pm(s),\ \xi_\pm(s),\ \tau_\pm(s))\to(x_0,\ 0,\ \xi^*,\ \pm\infty)$ as $s\to\pm0$.

Proof Hamilton vector field of P is governed by

$$\frac{dx}{ds}=e_{1,\xi}t\tau+e_{2,\xi},\quad \frac{d\xi}{ds}=-t\tau e_{1,x}-e_{2,x},$$
$$\frac{dt}{ds}=2t\tau+te_1,\quad \frac{d\tau}{ds}=-(\tau^2+(te_1)_t\tau+e_{2,t}).\tag{5.6}$$

A change of variables: $x'=x$, $\xi'=\xi$, $\mu=1/\tau$, $T=t\tau^2$, gives at once the reduction of (5. 6) to

$$\frac{dx'}{ds}=T\mu A_1+A_2,\quad \frac{d\xi'}{ds}=-T\mu A_3-A_4,$$

$$\frac{dT}{ds}=T(A_5+A_6), \quad \frac{d\mu}{ds}=1+A_7\mu+A_8\mu^2, \tag{5.7}$$

where $A_i(i=1, \cdots, 8)$ are C^∞ functions of x', ξ', T, μ in the domain under consideration. Take

$$x'=x_0,\ \xi'=\xi^*,\ \mu=0,\ T=-e_2(x_0,\ 0,\ \xi^*) \tag{5.8}$$

as initial value. As is well known, (5. 8) (5. 7) admits a unique C^∞ solution $x'=x'(s)$, $\xi'=\xi'(s)$, $\mu=\mu(s)$, $T=T(s)$, $s\in[-s_1, s_1]$. Hence $\gamma_\pm(s)=(x'(s), T(s)\ \mu^2(s), \xi'(s), 1/\mu(s))$ defined in $s\in(0, s_1]$ and $s\in[-s_1, 0)$, resp., are just the null-bicharacteristics of P required. The proof is complete. Next we shall call $\gamma_\pm(s)$ the ξ^*—family of bicharacteristics of P, denoted by $\gamma_\pm(\xi^*)$. It seems that the ξ^*—family of bicharacteristics is analogous to the ξ^*—reflective family of bicharacteristics in [1].

Theorem 5.1. *Let (5, 1) satisfy (5. 5). Assume that* Re $\sqrt{-1}\,p_1/\tau>3/2$ *on* Σ *and* $e_2(x_0, 0, \xi^*)\neq 0$ *for some* $\xi^*\in R^{n-1}\backslash\{0\}$. *If* u, $Pu\in L^2(\Omega)$, $\xi^*\in SWF(Pu)(x_0, 0, 0, \pm 1)$ *and* $\gamma_\pm(\xi^*)\cap WF(Pu)=\varnothing$, *then* $\gamma_\pm(\xi^*)\cap WF(u)=\varnothing$.

Proof. Evidently, Theorem 5. 1 is an immediate consequence of Lemmas 5. 1, 5. 2 and Proposition 4. 1. The details of the proof are omitted.

Remark. The condition: Re $\sqrt{-1}\,p_1/\tau>3/2$ on Σ may be expressed by a coordinate-free form. Indeed, it is equivalent to that the real part of the subprincipal symbol of $\sqrt{-1}\,P$ on $\Sigma>1$.

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