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NON-EXISTENCE OF LIMIT CYCLE AROUND A WEAK FOCUS OF ORDER THREE FOR ANY QUADRATIC SYSTEM*

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Abstract

In 1980 Professor Ye Yangian proposed a conjecture that around a weak focus of order 3 of any real quadratic differential system there can exist no limit cycle. It is the purpose of this paper to give a proof of the conjecture by using the method of continuous variation of a coefficient.

It is an interesting problem to determine whether there exists a limit cycle around a weak focus of order 3 of a quadratic system (E_2), because this problem is closely related to whether there exist four limit cycles around one singular point of (E_2). In 1976, L. Cerkas[1] proved that there can exist no limit cycle around a weak focus of order 3 of (E_2) when $|n|$ (the absolute value of a coefficient) becomes large enough. In 1980, Professor Ye said[2]: "...... Wang Mingshu and Cai Suilin proved indepently that there can exist no limit cycle around a weak focus of order 3 of (E_2) for the case of $n=0$. But up to now none can prove the same conclusion for the case of $n\neq 0$, although it seems right." The above-mentioned two papers were published in 1981 (Cai[3]) and in 1982 (Wang[4]). Afterwards Du Xingfu[5] generalized this result to the case of $n=\pm\frac{1}{2}$, ± 1. In this paper, we will prove that this conclusion is true for any case of $-\infty<n<+\infty$. Therefore, a proof of prof. Ye's conjecture will be given. The proof will be divided into ten sections.

§ 1. Transformation of Equations and Problem: $(E_2)\Rightarrow$ Lienard's; Limit Cycle $\Rightarrow$ Intersection of Curves

Without loss of generality, we can assume that a quadratic system with a weak focus of order 3 at the origin is of the form (see [6], Corollary 3 and Remark 2 of

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Theorem 1)

$$\begin{cases} \dfrac{dx}{dt} = -y + ax^2 + bxy, \\ \dfrac{dy}{dt} = x + \dfrac{b-3n}{5} x^2 + 3axy + ny^2, \end{cases} \tag{1}$$

moreover

$$W_3 = a^3(n-2b)(3n-b)[5(n+3b)a^2 + (n-b)(2n+b)^2] \neq 0. \tag{2}$$

Obviously, from the first equation of (1), the straight line $1-bx=0$ is a line without contact. So we only need to consider a half plane including the origin: $1-bx>0$.

By Liu Jun's transformation[7] the system (1) can be transformed into an equation of Lienard's type

$$\begin{cases} \dfrac{dx}{d\tau} = V - F(x), \\ \dfrac{dV}{d\tau} = -g(x), \end{cases} \tag{3}$$

Where

$$\begin{cases} F(x) = a\displaystyle\int_0^x \frac{E(\xi)}{1-b\xi}[2(2b-n)\xi - 5]\xi\, d\xi, \\ g(x) = \dfrac{xE^2(x)}{1-bx}\left\{\left[\dfrac{(b-3n)b^2}{5} - 3a^2b + na^2\right]x^3 + \left[b^2 + 3a^2 - \dfrac{2b(b-3n)}{5}\right]x^2 \right. \\ \qquad\left. + \left(\dfrac{b-3n}{5} - 2b\right)x + 1\right\}, \\ E(x) = \exp\displaystyle\int_0^x \frac{b+n}{1-b\xi}\, d\xi. \end{cases} \tag{4}$$

The origin remains his position when system (1) is transformed into (3). If there exists $\bar{x} \neq 0$ such that $g(\bar{x}) = 0$, then there exists a singular point $P(\bar{x}, F(\bar{x}))$ of (3) on the straight line l: $x = \bar{x}$. Suppose that system (3) has a limit cycle around the origin. Then it can not intersect with the line l. In fact, on the line l above the point P: $\frac{dx}{d\tau} > 0$, below the point P: $\frac{dx}{d\tau} < 0$. On the other hand, a limit cycle of (E_2) can not include two singular points. Therefore, we only need to consider the interval $x_0 < x < X$, in which $xg(x) > 0$ when $x \neq 0$. Let $G(x) = \int_0^x g(\xi)d\xi$. We will prove that in $x_1 - x_2$ plane two curves defined by

$$F(x_1) = F(x_2), \quad G(x_1) = G(x_2) \quad (x_0 < x_1 < 0 < x_2 < X), \tag{5}$$

do not have any intersection point for any a, b and n, so long as the condition (2) holds. Therefore, system (3) can not have any limit cycle around the origin (see [8]).

Notice that F and G are continuous fuctions with regard to b. So, if we prove that two curves defined by (5) do not intersect for any $b \neq 0$, then they can not have any s-point of intersection for $b=0$ (the defination of s-point is given in [9]). It implies that system (5) can not have any limit cycle around the origin. Therefore we can suppose $b\neq 0$, and further suppose $b=1$ (otherwise, let $x=\frac{1}{b}\tilde{x}$, $y=\frac{1}{b}\tilde{y}$ in (3), then $\tilde{b}=1$). In this case the system (1), condition (2) and formulas (4) become respectively

$$\begin{cases} \dfrac{dx}{dt}=-y+ax^2+xy, \\ \dfrac{dy}{dt}=x+\dfrac{1-3n}{5}x^2+3axy+ny^2, \end{cases} \tag{6}$$

$$W_3=a^3(n-2)(3n-1)[5(n+3)a^2+(n-1)(2n+1)^2]\neq 0, \tag{7}$$

and

$$\begin{cases} F_1(x)=a\displaystyle\int_0^x \frac{\xi[2(2-n)\xi-5]}{(1-\xi)^{n+2}}d\xi, \\ g_1(x)=\dfrac{x}{5}\dfrac{[(1-3n)+5(n-3)a^2]x^3+3[(2n+1)+5a^2]x^2-3(n+3)x+5}{(1-x)^{2n+3}}. \end{cases}$$

Let

$$\tilde{F}(x)=\frac{F_1(x)}{a}, \quad \tilde{G}(x)=\int_0^x g_1(\xi)d\xi.$$

Consider

$$\tilde{F}(x_1)=\tilde{F}(x_2), \quad \tilde{G}(x_1)=\tilde{G}(x_2) \quad (x_0<x_1<0<x_2<X). \tag{8}$$

By calculation we have

$$\tilde{F}(x)=\frac{1}{n(n^2-1)}\left.\frac{\sum_{k=0}^{2}\alpha_k z^k}{z^{n+1}}\right|_{z=1}^{z=1-x}, \quad \text{when } n\neq 0, \pm 1,$$

$$\tilde{G}(x)=\frac{1}{10n(n^2-1)(4n^2-1)}\left.\frac{\sum_{k=0}^{4}\tilde{\beta}_k z^k}{z^{2(n+1)}}\right|_{z=1}^{z=1-x}, \quad \text{when } n\neq 0, \pm\frac{1}{2}, \pm 1,$$

where

$$\begin{cases} \alpha_0=-n(n-1)(2n+1), \\ \alpha_1=(n^2-1)(4n-3), \\ \alpha_2=-2n(n+1)(n-2), \end{cases} \tag{9}$$

$$\begin{cases} \tilde{\beta}_0=5n^2(n-1)(4n^2-1)a^2, \\ \tilde{\beta}_1=-10n(n^2-1)(2n-1)(4n-3)a^2, \\ \tilde{\beta}_2=3(n^2-1)(4n^2-1)[(2-n)+5(2n-3)a^2], \\ \tilde{\beta}_3=-2n(n^2-1)(2n+1)[(7-6n)+5(4n-9)a^2], \\ \tilde{\beta}_4=n(n+1)(4n^2-1)[(1-3n)+5(n-3)a^2]. \end{cases} \tag{10}$$

By transformation $z=1-x$, the origin of system (3) becomes point (1, 1). We can consider the following equations instead of (8) when $n\neq 0, \pm\frac{1}{2}, \pm 1$:

$$\bar{F}(z_1)=\bar{F}(z_2), \quad \bar{G}(z_1)=\bar{G}(z_2) \quad (0\leqslant z_0<z_1<1<z_2<Z), \tag{11}$$

where

$$\bar{F}(z)=\frac{\sum_{k=0}^{2}\alpha_k z^k}{z^{n+1}},\quad \bar{G}(z)=\frac{\sum_{k=0}^{4}\tilde{\beta}_k z^k}{z^{2(n+1)}}. \tag{12}$$

Notice that two curves defined by (11) do not change when we add a constant to $\bar{F}(z)$ or $\bar{G}(z)$ in (12). So we can suppose that $\bar{F}(z_1)\cdot\bar{F}(z_2)\neq 0$ when we consider whether the point (z_1, z_2) is or not the intersection point of the two curves. From (11) and (12), we have

$$\frac{\sum_{k=0}^{4}\tilde{\beta}_k z_1^k}{\sum_{k=0}^{4}\tilde{\beta}_k z_2^k}=\left(\frac{z_1}{z_2}\right)^{2(n+1)}=\left(\frac{\sum_{k=0}^{2}\alpha_k z_1^k}{\sum_{k=0}^{2}\alpha_k z_2^k}\right)^2.$$

Calculating it by using (9) and (10), and reducing the non-zero common factor $n(n^2-1)[5(n+3)a^2+(n-1)(2n+1)^2](z_1-z_2)$ (see (7)), we obtain

$$\begin{aligned}H(z_1, z_2)\equiv&\lambda_0(z_1+z_2)+\lambda_1(z_1^2-z_1z_2+z_2^2)+\lambda_2z_1z_2+\lambda_3(z_1+z_2)(z_1^2+z_2^2)\\&+\lambda_4(z_1+z_2)z_1z_2+\lambda_5(z_1z_2)^2+\lambda_6z_1z_2(z_1^2+z_1z_2+z_2^2)\\&+\lambda_7(z_1z_2)^2(z_1+z_2)+\lambda_8(z_1z_2)^3=0,\end{aligned} \tag{13}$$

where

$$\begin{cases}\lambda_0=3n(n-2)(n-1)(4n^2-1),\\ \lambda_1=-2n^2(n-1)(2n+1)(6n-7),\\ \lambda_2=-6(n^2-1)(n-2)(2n-1)(4n-3),\\ \lambda_3=n^2(4n^2-1)(3n-1),\\ \lambda_4=4n(n^2-1)(4n-3)(6n-7),\\ \lambda_5=-6(n^2-1)(2n+1)(8n^2-15n+3),\\ \lambda_6=-2n(n+1)(2n-1)(3n-1)(4n-3),\\ \lambda_7=3(n+1)(4n^2-1)(5n^2-6n-3),\\ \lambda_8=-12n(n-2)(n+1)^2(2n+1).\end{cases} \tag{14}$$

Below (in section 4) we will use (13) instead of the second equation of (11). Notice that (13) and the first equation of (11) do not depend on the parameter a. It means whether or not two curves defined by (11) have intersection point is not related to the concret value of the coefficient a.

It is our purpose to prove that two curves defined by (5) do not intersect when $W_3\neq 0$. This conclution has been proved in papers [3-5] for the case of $n=0$, $\pm\frac{1}{2}$, ± 1. When $n\neq 0$, $\pm\frac{1}{2}$, ± 1, for any $a\neq 0$ the two curves defined by (5) are the same with that defined by (8). Moreover, whether or not two curves have intersection point is not related to the concret value of a, and so are the coodinates of the intersection point, if it exists. The difference between (5) and (8) is only that when $a=0$ the first equation of (5) can't define a curve, but (8) can still define two curves which are continuous with regard to a. If two curves defined by $(5)_{a\neq 0}$

intersect, then, as a result of letting a tend to zero, the two curves defined by $(8)_{a=0}$ must intersect too. Therefore, we only need to prove two curves defined by $(8)_{a=0}$ don't intersect when $W_3\neq 0$ and $n\neq 0, \pm\frac{1}{2}, \pm 1$.

§ 2. Preliminary Study of Curves Γ_F and Γ_G

When $n\neq 0, \pm\frac{1}{2}, \pm 1$, system (8) can be transformed to (11). Let $a=0$. System (11) becomes (still use F, G to express the functions)

$$F(z_1)=F(z_2),\quad G(z_1)=G(z_2)\quad (0\leqslant z_0<z_1\leqslant 1\leqslant z_2<Z), \tag{16}$$

where

$$F(z)=\frac{\alpha_0+\alpha_1 z+\alpha_2 z^2}{z^{n+1}},\qquad G(z)=\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z^{2n}}, \tag{17}$$

the $\{\alpha_i\}$ are the same with (9), and

$$\begin{cases}\beta_0=-3(n-1)(n-2)(2n-1),\\ \beta_1=2n(n-1)(6n-7),\\ \beta_2=-n(2n-1)(3n-1).\end{cases} \tag{18}$$

We use Γ_F and Γ_G to express the two curves defined by (16) in the plane z_1-z_2. Γ_F and Γ_G have, obviously, a common end A (1, 1). We'll say Γ_F and Γ_G have or don't have the intersection point, that means the point is different from the point A. First, we discuss the relative position of Γ_F and Γ_G near the point A. Suppose Γ_F and Γ_G have the expressions $z_1=\varphi(z_2)$ and $z_1=\psi(z_2)$ (we can know from (17) that this supposition is reasonable near A). From $F(\varphi(z_2))=F(z_2)$ we have

$$\varphi'(z_2)=\frac{F'(z_2)}{F'(\varphi(z_2))}.$$

From (17) and (9) we have

$$\begin{cases}F'(1)=0,\\ F''(1)=n(n^2-1)(-5),\\ F'''(1)=n(n^2-1)[2(7n+6)],\\ F^{(k)}(1)=n(n^2-1)[(-1)^{k-1}(k-1)](n+2)(n+3)\cdots(n+k-2)\\ \qquad\cdot[(2k+1)n+(k+3)]\quad (k=4,5,\cdots).\end{cases}$$

Noting $z_1=\varphi(z_2)\to 1-0$ as $z_2\to 1+0$, by using the L'Hôpital's principle we have

$$\varphi'(1)=-1,$$

$$\varphi''(1)=\frac{4}{15}(7n+6),$$

$$\varphi'''(1)=-\frac{8}{75}(7n+6)^2,$$

$$\varphi^{(4)}(1)=-\frac{16}{25}(n+2)(2n+1)(13n+9)+\frac{128}{1125}(7n+6)^3.$$

Similarly

$$G'(1)=0,$$
$$G''(1)=2n(n-1)(2n-1)\cdot 5,$$
$$G'''(1)=2n(n-1)(2n-1)[-2(7n+6)],$$
$$G^{(k)}=2n(n-1)(2n-1)[(-1)^k(k-1)](2n+1)(2n+2)\cdots(2n+k-3)\cdot[(-3k+16)n+6(k-2)]\quad(k=4,5,\cdots),$$

and finaly

$$\begin{cases}\varphi^{(k)}(1)-\psi^{(k)}(1)=0, & k=1,2,3,4,5,\\ \varphi^{(6)}(1)-\psi^{(6)}(1)=\dfrac{576}{875}(2n+1)^2(3n-1)(n-1)(n-2). & \end{cases}\tag{20}$$

We needn't consider the values $n=-\frac{1}{2},\frac{1}{3},1,2\left(W_3=0\text{ when }n=\frac{1}{3}\text{ or }2\right)$. Let $I_1=\left(-\infty,-\frac{1}{2}\right)$, $I_2=\left(-\frac{1}{2},\frac{1}{3}\right)$, $I_3=\left(\frac{1}{3},1\right)$, $I_4=(1,2)$, $I_5=(2,+\infty)$. Then (20) shows Γ_F and Γ_G have fixed relative (up-low) position in a small neighbourhood of the point A when $n\in I_k$. Let us prove Γ_F and Γ_G have the same property at another end of the curves. From (17), (18) and (9) we can get the graphs of Γ_F and Γ_G (see Fig. 1). In the cases ④—⑦, obviously, the relative position of Γ_F and Γ_G at another end is fixed, and in the cases ①, ② and ⑧ it depend on the values of zero points of $F(z)$ and $G(z)$. From (17) we can calculate the zero points of $F(z)$ and $G(z)$:

$$\begin{cases}\xi_k=\dfrac{(n^2-1)(4n-3)+(-1)^k\sqrt{3(n^2-1)(n+3)(3n-1)}}{4n(n+1)(n-2)},\\ \eta_k=\dfrac{n(n-1)(6n-7)+(-1)^k\sqrt{2n(n-1)(2n+1)(4n-3)}}{n(2n-1)(3n-1)}\quad(k=1,2).\end{cases}\tag{21}$$

Moreover

$$(\xi_1-\eta_2)(\xi_2-\eta_1)=\frac{3n(2n+1)(3n-1)(n-1)-|n-1|\sqrt{6n(n+1)(n+3)(2n+1)(3n-1)(4n-3)}}{2n^2(n+1)(n-2)(3n-1)(2n-1)},$$

and

$$[3n(2n+1)(3n-1)]^2-6n(n+1)(n+3)(2n+1)(3n-1)(4n-3)=3n(3n-1)(2n+1)(n-2)(10n^2-3n-9).$$

When $n>2$ (the case ①), $\xi_2>1>\eta_1$ and $(\xi_1-\eta_2)(\xi_2-\eta_1)>0$ hold, so $\xi_1>\eta_2$ holds uniformly for $n\in I_5$. When $n<-3$ (the case ⑧), $\xi_2<1<\eta_1$ and $(\xi_1-\eta_2)(\xi_2-\eta_1)>0$ hold, so $\xi_1<\eta_2$ holds uniformly for $n<-3$. When $1<n<2$ (the case ②), $\xi_2<0$ and $\eta_1<0$ hold, $(\xi_1-\eta_2)(\xi_2-\eta_1)=0$ holds only for that n which is the positive zero point of the equation $10n^2-3n-9=0$. In this case the calculation shows $\xi_2=\eta_1$ but $\xi_1\neq\eta_2$, so $(\xi_1-\eta_2)$ keeps the same sign for $n\in I_4$. We'll prove in Section 9 that Γ_F and Γ_G do not intersect when $n=\frac{3}{2}\in I_4$, and (20) shows Γ_G is located up to Γ_F

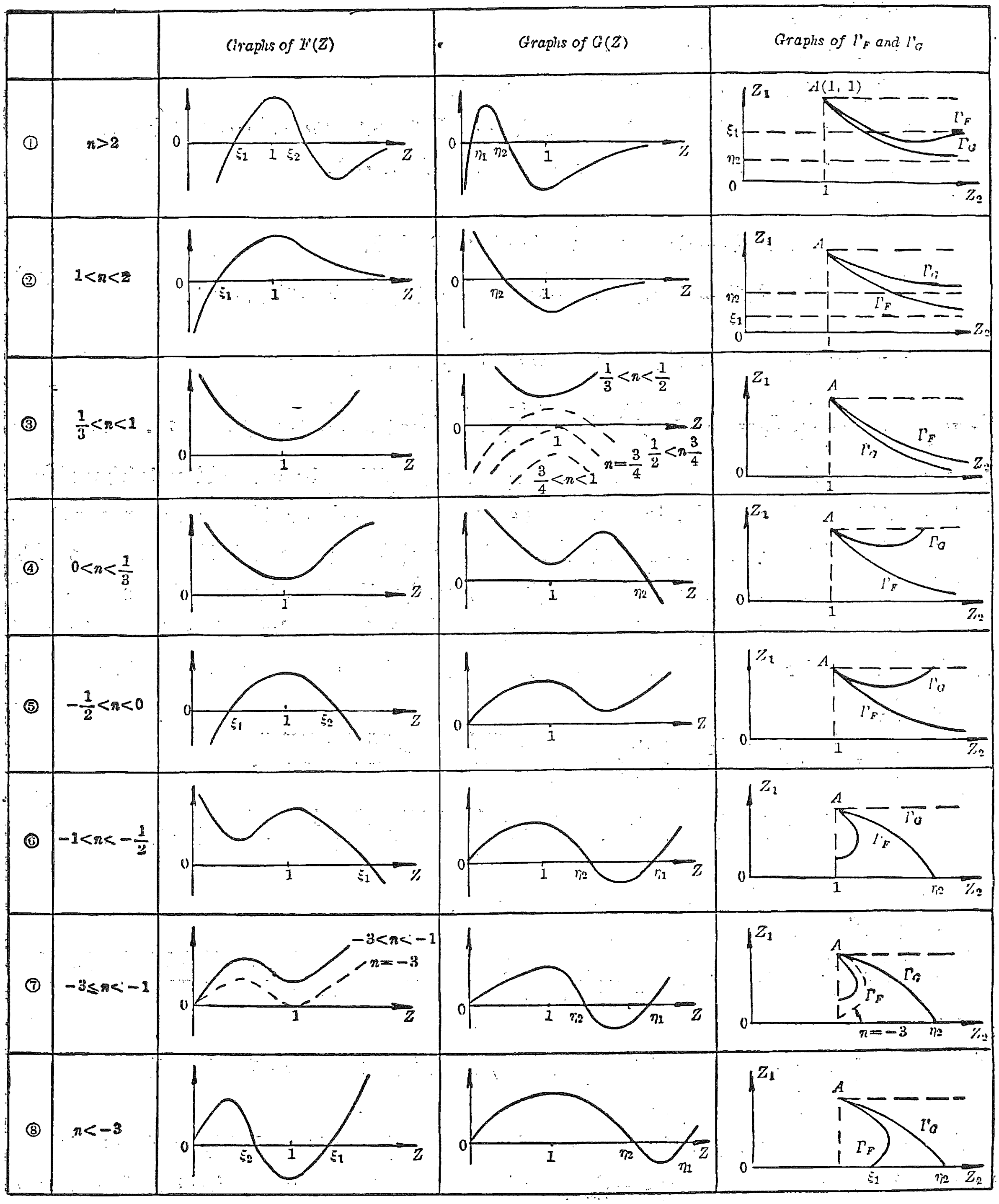

Fig. 1

near the point A, therefore $\xi_1<\eta_2$ holds uniformly for $n\in I_4$. Thus we have proved that Γ_F and Γ_G keep the same relative position at two ends uniformly for $n\in I_k$ except the case ③ which will be proved in Section 8 (in the cases ⑥, ⑦ and ⑥, z_1 and z_2 should be exchanged).

§ 3. Continuous Variation of the Coefficient n. Further transformation of the Problem: Intersection ⇒ Contact

It should be pointed out that when $n=-1,\ 0,\ \frac{1}{2}$, the expression (16) can't define any curve. In these cases it is impossible to get (16) from $(8)_{a=0}$. Direct calculation shows the two curves defined by $(8)_{a=0}$ don't have any intersection point except the point A when $n=-1,\ 0,\ \frac{1}{2}$. Noting that for any $n\neq 0,\ \pm\frac{1}{2},\ \pm 1$ the curves defined by $(8)_{a=0}$ and by (16) are the same, we can take two curves of $(8)_{a=0}$ for the case of $n=-1,\ 0,\ \frac{1}{2}$ to define Γ_F and Γ_G. Thus, Γ_F and Γ_G are two families of planar smooth curves depending continuously on the parameter $n\in I_k$.

Now we can conclude that if there exist $\tilde{n}_k,\ \bar{n}_k\in I_k$ such that $\Gamma_F(\tilde{n}_k)$ and $\Gamma_G(\tilde{n}_k)$ intersect but $\Gamma_k(\bar{n}_k)$ and $\Gamma_G(\bar{n}_k)$ don't intersect, then there must exist an $\hat{n}_k\in I_k$ ($\tilde{n}_k\leqslant\hat{n}_k<\bar{n}_k$ or $\bar{n}_k<\hat{n}_k\leqslant\tilde{n}_k$) such that $\Gamma_F(\hat{n}_k)$ and $\Gamma_G(\hat{n}_k)$ have at least one contact point. But, in general, two families of smooth curves, defined in common interval and depending continuously on the same parameter, change relative position from intersection to non-intersection with the parameter's variation, the contact position might not occur because the intersection point might vanish at curve's end. However, this case can't appear for our system because Γ_F and Γ_G have fixed relative position at two ends (see section 2).

According to above discussion, once we prove Γ_F and Γ_G don't have any contact point for $n\in I_k$, and there exists an $\bar{n}_k\in I_k$ such that $\Gamma_F(\bar{n}_k)$ and $\Gamma_G(\bar{n}_k)$ don't intersect, we can conclude Γ_F and Γ_G don't intersect uniformly for $n\in I_k$. We can choose $\bar{n}_1=-1$, $\bar{n}_2=0$, and $\bar{n}_3=\frac{1}{2}$ (see the first paragraph of this section). The plan for the rest of this paper is as follows:

(a) In Sections 4—7 it will be proved that Γ_F and Γ_G don't have any intersection point except A for every $n\in I_k$ $(k=1—5)$.

(b) In Section 8 it will be proved that the relative position of Γ_F and Γ_G from intersection to non-intersection must pass contact state when n variates continuously in $I_3=\left(\frac{1}{3},\ 1\right)$. This is the case ③ in Fig. 1. Γ_F and Γ_G have the same asymptotic line $z_1=0$ as $z_2\to+\infty$, so the method in last section is unsuitable.

(c) In Sections 9 and 10 it will be proved that Γ_F and Γ_G don't intersect for $\bar{n}_4=\frac{3}{2}$ and $\bar{n}_5=3$ respectively.

§ 4. Further Transformation of Equations

In order to prove Γ_F and Γ_G don't contact, we prove the equations

$$\begin{cases} F(z_1)=F(z_2), \\ G(z_1)=G(z_2), \\ \dfrac{f(z_1)}{g(z_1)}=\dfrac{f(z_2)}{g(z_2)}, \end{cases} \quad (0\leqslant z_0<z_1<1<z_2<Z), \tag{22}$$

don't have any solution, where

$$\begin{cases} f(z)=F'(z)=-[(n+1)\alpha_0+n\alpha_1 z+(n-1)\alpha_2 z^2]z^{-(n+2)} \\ \qquad =n(n^2-1)(1-z)[2(n-2)z-(2n+1)]z^{-(n+2)}, \\ g(z)=G'(z)=-[-2n\beta_0+(2n-1)\beta_1 z+2(n-1)\beta_2 z^2]z^{-(2n+1)} \\ \qquad =2n(n-1)(2n-1)(1-z)[(3n-1)z-3(n-2)]z^{-(2n+1)}. \end{cases} \tag{23}$$

From the first and third equations of (22) we have

$$\frac{z_2^2[(1-3n)z_2+3(n-2)][2(2-n)z_1+(2n+1)]}{z_1^2[(1-3n)z_1+3(n-2)][2(2-n)z_2+(2n+1)]}=\left(\frac{z_2}{z_1}\right)^{n+1}=\frac{\alpha_0+\alpha_1 z_2+\alpha_2 z_2^2}{\alpha_0+\alpha_1 z_1+\alpha_2 z_1^2}.$$

By calculation and reducing the non-zero common factor $(2n+1)^2(n-1)(z_2-z_1)$ we have

$$I(z_1, z_2)\equiv\delta_0(z_1+z_2)+\delta_1(z_1^2+z_1z_2+z_2^2)+\delta_2 z_1z_2+\delta_3 z_1z_2(z_1+z_2)+\delta_4(z_1z_2)^2=0, \tag{24}$$

where

$$\begin{cases} \delta_0=3n(n-2), \\ \delta_1=-n(3n-1), \\ \delta_2=-9(n-1)(n-2), \\ \delta_3=3(n-1)(3n-1), \\ \delta_4=-6(n+1)(n-2). \end{cases} \tag{25}$$

Thus (22) is equivalent to (see (13))

$$\begin{cases} F(z_1)=F(z_2), \\ H(z_1,z_2)=0, \\ I(z_1, z_2)=0. \end{cases} \quad (0\leqslant z_0<z_1<1<z_2<Z). \tag{26}$$

First, we manage to find the solutions of last two equations in (26). Then we study if it satisfys the first one in (26). Let

$$\begin{cases} z_1+z_2=u, \\ z_1\cdot z_2=v. \end{cases}$$

It transforms the domain $0<z_1<1<z_2<+\infty$ to $0<v<u-1$, and transforms the point (1, 1) to (2, 1). Between two domains it is a topological transformation. The last two equations of (26) become

$$\begin{cases} \delta_0 u+(\delta_2-\delta_1)v+\delta_1 u^2+\delta_3 uv+\delta_4 v^2=0, \\ \lambda_0 u+(\lambda_2-\lambda_1)v+\lambda_1 u^2+(\lambda_4-2\lambda_3)uv+(\lambda_5-\lambda_6)v^2 \\ \qquad +\lambda_3 u^3+\lambda_6 u^2 v+\lambda_7 uv^2+\lambda_8 v^3=0. \end{cases} \tag{27}$$

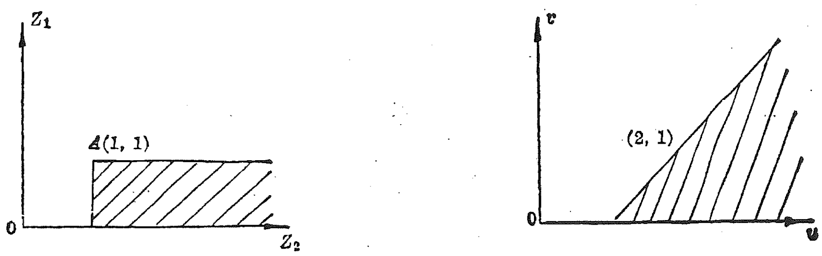

Fig. 2

Let us transform it again by polar coordinates

$$\begin{cases} u=\rho\cos\theta, \\ v=\rho\sin\theta. \end{cases} \tag{28}$$

The domain $0<v<u-1$ is transformed into domain

$$0<\theta<\frac{\pi}{4}, \quad \rho>\frac{1}{\cos\theta-\sin\theta}, \tag{29}$$

and (27) becomes

$$\begin{cases} \delta_0\cos\theta+(\delta_2-\delta_1)\cos\theta+(\delta_1\cos^2\theta+\delta_3\cos\theta\sin\theta+\delta_4\sin^2\theta)\rho=0, \\ \lambda_0\cos\theta+(\lambda_2-\lambda_1)\sin\theta+[\lambda_1\cos^2\theta+(\lambda_4-2\lambda_3)\cos\theta\sin\theta+(\lambda_5-\lambda_6)\sin^2\theta]\rho \\ \quad +(\lambda_3\cos^3\theta+\lambda_6\cos^2\theta\sin\theta+\lambda_7\cos\theta\sin^2\theta+\lambda_8\sin^3\theta)\rho^2=0. \end{cases} \tag{30}$$

§ 5. Decomposition of $p(k)$

We first consider the case $J(\theta)\equiv\delta_1\cos^2\theta+\delta_3\cos\theta\sin\theta+\delta_4\sin^2\theta=0$. In order to satisfy the first equation of (30), $\delta_0\cos\theta+(\delta_2-\delta_1)\sin\theta=0$ should hold in the same time. Noting $\delta_0\neq0$ we have

$$\delta_1(\delta_1-\delta_2)^2+\delta_3\delta_0(\delta_1-\delta_2)+\delta_4\delta_0^2=0.$$

By using (25) we deduce $n^2(7n-9)=0$, so $n=\frac{9}{7}$ and $\sin\theta=\frac{1}{\sqrt{5}}$, $\cos\theta=\frac{2}{\sqrt{5}}$. From the second equation of (30) we obtain $\rho=\sqrt{5}$. Therefore (29) don't hold for $n=\frac{9}{7}$.

Then we consider the case $J(\theta)\neq0$. Solving ρ from the first equation of (30), takeing it into the second one of (30) and reducing the common factor $3(n-2)\cdot(3n-1)$, we have

$$\sum_{i=0}^{5}A_i\cos^i\theta\sin^{5-i}\theta=0, \tag{31}$$

where

$$A_0=-8(n+1)^2(16n^3-72n^2+81n-27),$$
$$A_1=4(n+1)(80n^4-224n^3+15n^2+153n-54),$$
$$A_2=-2(160n^5-176n^4-341n^3+123n^2+117n-27),$$
$$A_3=n(160n^4-64n^3-239n^2-6n+45),$$

$$A_4 = -4n^2(2n+1)(5n^2-n-3),$$
$$A_5 = n^3(2n+1)^2.$$

Let $\operatorname{tg}\theta = k$. Then $0<k<1\left(0<\theta<\frac{\pi}{4}\right.$, see (29)$\left.\right)$, (31) becomes

$$P(k) \equiv \sum_{i=0}^{5} A_i k^{5-i} = (2k-1)^2 \sum_{i=0}^{3} B_i k^{3-i} = 0,$$

where

$$B_0 = -2(n-3)(n+1)^2(4n-3)^2,$$
$$B_1 = n(n+1)(4n-3)(12n^2-19n-15),$$
$$B_2 = -4n^2(2n+1)(3n^2-2n-3),$$
$$B_3 = n^3(2n+1)^2.$$

i) If $k=\frac{1}{2}$, then $\cos\theta = \frac{2}{\sqrt{5}}$, $\sin\theta = \frac{1}{\sqrt{5}}$. From the first equation of (30) we can obtain $\rho=\sqrt{5}\left(\text{using } n\neq\frac{9}{7}\right)$. (29) don't hold.

ii) If $n=\frac{3}{4}$, then $B_0=B_1=0$, $B_2>0$ and $B_3>0$. Equation $P(k)=0$ has only one positive root $k=\frac{1}{2}$. So (30) don't have solution in domain (29).

iii) If $n=3$, then $B_0=0$, $P(k)=27(2k-1)^2(12k-7)^2$. When $k=\frac{7}{12}$, $\cos\theta=\frac{12}{\sqrt{193}}$, $\sin\theta=\frac{7}{\sqrt{193}}$, $\rho=\frac{\sqrt{193}}{4}$, $u=3$ and $v=\frac{7}{4}$. So $z_1=\frac{1}{2}(3-\sqrt{2})$, $z_2=\frac{1}{2}(3+\sqrt{2})$. In this case $F(z_1)=F(z_2)$, but $G(z_1)\neq G(z_2)$ (by direct calculation) (22) don't hold.

iv) If $n\neq\frac{3}{4}$ and $n\neq 3$, then we can decompose $P(k)$ in this way

$$P(k) = B_0(2k-1)^2(k-k_n^*)(k-k_n^{**})^2, \tag{32}$$

when

$$k_n^* = \frac{n}{2(n-3)}, \quad k_n^{**} = \frac{n(2n+1)}{(n+1)(4n-3)}.$$

§ 6. The Case of $k=k_n^*$

In this case

$$\begin{aligned}\rho - \frac{1}{\cos\theta-\sin\theta} &= -\frac{\delta_0\cos\theta+(\delta_2-\delta_1)\sin\theta}{\delta_1\cos^2\theta+\delta_3\cos\theta\sin\theta+\delta_4\sin^2\theta} - \frac{1}{\cos\theta-\sin\theta} \\ &= -\frac{1}{\cos\theta}\left[\frac{\delta_0+(\delta_2-\delta_1)k}{\delta_1+\delta_3 k+\delta_4 k^2}+\frac{1}{1-k}\right]_{k=k_n^*} \\ &= \frac{1}{\cos\theta}\,\frac{90(n-3)}{(n-6)(2n^2-21n+9)}.\end{aligned} \tag{33}$$

Denote the roots of equation $2n^2-21n+9=0$ by n_{10} and n_{20} $(0<n_{10}<0.5,\ n_{20}>10)$. (33) shows $\rho>\frac{1}{\cos\theta-\sin\theta}$ only if $n<n_{10}$ or $n>n_{20}$ or $3<n<6$. On the other hand, in order to satisfy $0<k_n^*<1\left(0<\theta<\frac{\pi}{4}\right)$, it should holds that $n>6$ or $n<0$. Therefore, (29) holds only if $n>n_{20}$ or $n<0$.

When $k=k_n^*=\frac{n}{2(n-3)}$, we have $\rho=\frac{1}{\cos\theta}\frac{2(n-3)(2n-9)}{2n^2-21n+9}$ and

$$\begin{cases} u^*=z_1^*+z_2^*=\rho\cos\theta=\dfrac{2(n-3)(2n-9)}{2n^2-21n+9}, \\ v^*=z_1^*z_2^*=\rho\sin\theta=k_n^*u^*=\dfrac{n(2n-9)2}{2n^2-21n+9}. \end{cases}$$

So

$$z_j^*=\frac{u^*+(-1)^j\sqrt{u^{*2}-4v^*}}{2}=\frac{(n-3)(2n-9)+(-1)^j3\sqrt{(2n-9)(7n-9)}}{2n^2-21n+9},\quad (j=1,\ 2). \tag{34}$$

First, we prove $F(z_1^*)F(z_2^*)<0$ when $n_{30}<n<0$ and $n\neq-\frac{3}{11}$ (n_{30} is the negative root of equation $10n^2-3n-9=0$, $-0.82<n_{30}<-0.81$). Therefore the first equation of (26) doesn't hold. We only need to prove $(\xi_1-z_1^*)(\xi_2-z_2^*)>0$ (see the graph of $F(z)$ in Fig. 1). From (34) and (21) we have

$$(\xi_1-z_1^*)(\xi_2-z_2^*)$$
$$=\frac{3\sqrt{3(n^2-1)(n+3)(3n-1)(2n-9)(7n-9)}-(98n^3-63n^2-144n+81)}{2n(n+1)(n-2)(2n^2-21n+9)}.$$

The denomimator is positive for $-1<n<0$. Moreover

$$(3\sqrt{3(n^2-1)(n+3)(3n-1)(2n-9)(7n-9)})^2-(98n^3-63n^2-144n+81)^2$$
$$=-nQ(n)>0,\ \text{when } n_{30}<n<0 \text{ and } n\neq-\frac{3}{11},$$

where $Q(n)=(11n+3)^2(7n-9)(10n^2-3n-9)$ (see Fig 3). The double root of $Q(n)=0$, $n=-\frac{3}{11}$ corresponds to the case of $z_j^*=\xi_j$ ($j=1,\ 2$), which will be discussed in Section 7. When $n=n_{30}$, we have $F(z_1^*)\neq F(z_2^*)=0$.

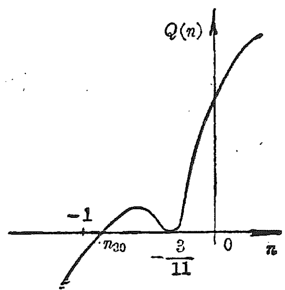

Fig. 3

Then, we prove $F(z_1^*)\neq F(z_2^*)$ when $n>n_{20}$ or $n<n_{30}$. Let

$$\Phi(n)=\left(\frac{z_2^*}{z_1^*}\right)^2\frac{\alpha_0+\alpha_1z_1^*+\alpha_2z_1^{*2}}{\alpha_0+\alpha_1z_2^*+\alpha_2z_2^{*2}},\qquad \Psi(n)=\left(\frac{z_1^*}{z_2^*}\right)^{n-1}.$$

Then $F(z_1^*)=F(z_2^*)$ is equivalent to $\Phi(n)=\Psi(n)$. From (9) and (34) we obtain (for $n>n_{20}$ or $n<n_{30}$)

$$\Phi(n)=\frac{|2n-3|\sqrt{|7n-9|}-3(n-1)\sqrt{|2n-9|}}{|2n-3|\sqrt{|7n-9|}+3(n-1)\sqrt{|2n-9|}},$$

$$\Psi(n)=\left(\frac{|n-3|\sqrt{|2n-9|}-3\sqrt{|7n-9|}}{|n-3|\sqrt{|2n-9|}+3\sqrt{|7n-9|}}\right)^{n-1}.$$

Note $\Phi'(n)<0$ and $\Psi'(n)>0$ for $n>n_{20}$, and moreover $\Phi(n_{20})>0$, $\Phi(+\infty)=\frac{2\sqrt{7}-3\sqrt{2}}{2\sqrt{7}+3\sqrt{3}}$, $\Psi(n_{20})=0$ and $\Psi(+\infty)=\lim_{n\to+\infty}\left(1-\frac{6\sqrt{7n-9}}{(n-3)\sqrt{2n-9}+3\sqrt{7n-9}}\right)^{n-1}=$

$e^{-3\sqrt{14}}<\Phi(+\infty)$. Therefore $\Phi(n)=\Psi(n)$ doesn't have any solution for $n>n_{20}$ (see Fig. 4).

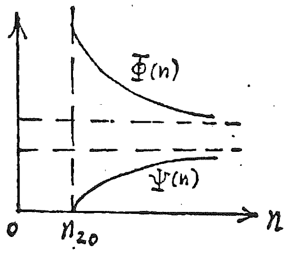

Fig. 4

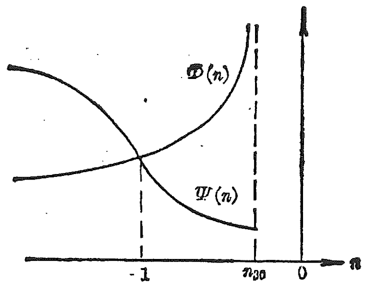

Fig. 5

Similarly, $\Phi'(n)>0$ and $\Psi'(n)<0$ for $n<n_{30}$, and $\Phi(n_{30}-0)=+\infty$, $\Phi(-\infty)=\frac{2\sqrt{7}+3\sqrt{2}}{2\sqrt{7}-3\sqrt{2}}$, $\Psi(n_{30}\leqslant 0)<+\infty$ and $\Psi(-\infty)=e^{3\sqrt{14}}>\Phi(-\infty)$. Therefore, $\Phi(n)=\Psi(n)$ has a unique root for $n<n_{30}$. It is easy to verify that this case corresponds to $n=-1$ (Fig. 5), but $\Gamma_F(-1)$ and $\Gamma_G(-1)$ don't intersect.

§ 7. The Case of $k=k_n^{**}$

By calculations (similar to (33)) we have

$$\rho-\frac{1}{\cos\theta-\sin\theta}=\frac{1}{\cos\theta}\,\frac{(n+3)(4n-3)}{2n(n-2)(2n^2-3)},$$

$$\operatorname{tg}\theta=k_n^{**}=\frac{n(2n+1)}{(n+1)(4n-3)}=-\frac{\alpha_0}{\alpha_1}\quad\text{and}\quad\rho=-\frac{1}{\cos\theta}\,\frac{\alpha_1}{\alpha_2}.$$

Thus, (29) becomes

$$\frac{(n+3)(4n-3)}{2n(n-2)(2n^2-3)}>0\quad\text{and}\quad 0<\frac{n(2n+1)}{(n+1)(4n-3)}<1.$$

So we only need to consider the domains $n>2$ or $n<-3$ or $-\frac{1}{2}<n<0$. On the other hand

$$\begin{cases} u^{**}=z_1^{**}+z_2^{**}=\rho\cos\theta=-\dfrac{\alpha_1}{\alpha_2}, \\ v^{**}=z_1^{**}z_2^{**}=\rho\sin\theta=u^{**}\cdot k_n^{**}=\dfrac{\alpha_0}{\alpha_2}. \end{cases}$$

Comparing it with (21) we can know $z_j^{**}=\xi_j(j=1,\ 2)$. Therefore, the first equation of (22) holds constantly: $F(z_1^{**})=F(z_2^{**})=0$, and it is impossible to obtain (26) from (22). we rewrite the last two equations of (22) as

$$S(z_1,\ z_2)\equiv\frac{\beta_0+\beta_1z_1+\beta_2z_1^2}{\beta_0+\beta_1z_2+\beta_2z_2^2}=\left(\frac{z_1}{z_2}\right)^{2n}, \tag{35}$$

and

$$T(z_1,\ z_2)\equiv\frac{[2(2-n)z_2+(2n+1)][(1-3n)z_1+3(n-2)]}{[2(2-n)z_1+(2n+1)][(1-3n)z_2+3(n-2)]}\cdot\frac{z_1}{z_2}=\left(\frac{z_1}{z_2}\right)^{n}. \tag{36}$$

From (21) and (18) we can deduce

$$S(\xi_1, \xi_2)=\frac{s_1-s_2\sqrt{w}}{s_1+s_2\sqrt{w}}, \tag{37}$$

where

$$\begin{cases} s_1=3(3n-1)(14n^3+n^2-22n+3), \\ s_2=38n^2-37n-3, \\ w=3(n^2-1)(n+3)(3n-1). \end{cases}$$

Similarly

$$T(\xi_1, \xi_2)=\frac{t_1-t_2\sqrt{w}}{t_1+t_2\sqrt{w}}, \tag{38}$$

where

$$\begin{cases} t_1=3(3n-1)(13n^2-2n-23), \\ t_2=25(n-1), \end{cases}$$

and

$$\left.\frac{z_1}{z_2}\right|_{z_j=z_j}=\frac{\alpha_1-\sqrt{w}}{\alpha_1+\sqrt{w}}, \tag{39}$$

where $\alpha_1=-n(n-1)(2n+1)$. From (35) and (36) by using (37), (38) and (39), we have

$$\frac{s_1-s_2\sqrt{w}}{s_1+s_2\sqrt{w}}=\left(\frac{t_1-t_2\sqrt{w}}{t_1+t_2\sqrt{w}}\right)^2\left(\frac{\alpha_1-\sqrt{w}}{\alpha_1+\sqrt{w}}\right)^2 \overset{\text{let}}{=\!=} \frac{r_1-r_2\sqrt{w}}{r_1+r_2\sqrt{w}}. \tag{40}$$

Noting $w\neq 0$, from (40) we deduce

$$s_1r_2-s_2r_1=0. \tag{41}$$

It is easy to obtain

$$r_1=(t_1+t_2\alpha_1)^2w+(t_1\alpha_1+t_2w)^2,$$
$$r_2=2(t_1+t_2\alpha_1)(t_1\alpha_1+t_2w),$$
$$t_1+t_2\alpha_1=2(2n+1)(25n^3-27n^2-7n-3),$$
$$t_1\alpha_1+t_2w=6(n^2-1)(3n-1)(2n+1)(13n^2-12n-3).$$

Taking the above results into (41), we have

$$n(2n+1)^3(n-2)^3(n-1)^2(n+3)^2(n+1)(3n-1)(11n+3)=0.$$

In the domains which need to be cosidered $\left(n>2 \text{ or } n<-3 \text{ or } -\frac{1}{2}<n<0\right)$ there exists only one root $n=-\frac{3}{11}$. But, by direct calculation, $G(\xi_1)\neq G(\xi_2)$ when $n=-\frac{3}{11}$. Hence (22) doesn't have any solution for $k=k_n^{**}$.

Thus, we have proved completely that Γ_F and Γ_G don't contact for any $n\in I_k$ $(k=1—5)$.

§ 8. The Case of $\frac{1}{3}<n<1$

Now we return to the case ③ in Fig. 1 and prove that the relative position of

Γ_F and Γ_G must pass the contact state if they variate continuously from intersection to non-intersection with the variation of $n\in\left(\frac{1}{3}, 1\right)$. In this case Γ_F and Γ_G can be expressed by $z_1=\varphi(z_2)$ and $z_1=\psi(z_2)$ for $z_2\in[1, +\infty)$, and they have the same asymptotic line $z_1=0$ as $z_2\to+\infty$. It is known that $\Gamma_F\left(\frac{1}{2}\right)$ and $\Gamma_G\left(\frac{1}{2}\right)$ don't intersect (see the beginning of section 3). Now assume there exists an $\tilde{n}\in\left(\frac{1}{3}, 1\right)$ such that. $\Gamma_F(\tilde{n})$ and $\Gamma_G(\tilde{n})$ intersect. Take n_1, n_2: $\frac{1}{2}$, $\tilde{n}\in[n_1, n_2]\subset\left(\frac{1}{3}, 1\right)$. According to (20) there exists an $\epsilon>0$ such that Γ_G is located below to Γ_F uniformly for $n\in[n_1, n_2]$, $0<z_2-1<\epsilon$. If Γ_F and Γ_G intersect (not contact) for some n, then there exists at least one point (z_1, z_2) such that Γ_G crosses Γ_F upwards at this point, that is $\varphi'(z_2)<\psi'(z_2)$. Hence (z_1, z_2) satisfies

$$\begin{cases} F(z_1)=F(z_2), \\ G(z_1)=G(z_2), \\ \dfrac{f(z_2)}{f(z_1)}<\dfrac{g(z_2)}{g(z_1)}. \end{cases} \tag{42}$$

By using the deductive method, which was used from (22) to (30), and $\frac{1}{3}<n<1$, $1-z_1>0$ and $1-z_2<0$, we obtain from (42)

$$h(\theta)\equiv\delta_0\cos\theta+(\delta_2-\delta_1)\sin\theta+(\delta_1\cos^2\theta+\delta_3\cos\theta\sin\theta+\delta_4\sin^2\theta)\rho>0. \tag{43}$$

Since $h(0)=\delta_0+\delta_1\rho$ and $\delta_0=3n(n-2)<-\frac{5}{3}$, $\delta_1\leqslant-n_1(3n_1-1)<0$ when $\frac{1}{3}<n_1\leqslant n\leqslant n_2<1$. Hence there exists a $\bar{\theta}>0$ such that $h(\theta)<0$ for $0<\theta\leqslant\bar{\theta}$. Therefore, in a sector $0<\theta\leqslant\bar{\theta}$ on the u–v plane (see Fig. 2) it is impossible for $\bar{\Gamma}_G$ to cross $\bar{\Gamma}_F$ upwards ($\bar{\Gamma}_F$ and $\bar{\Gamma}_G$ are images of Γ_F and Γ_G respectively). $\bar{\Gamma}_F$ and $\bar{\Gamma}_G$ have the same asymptotic live $v=0$ as $u\to+\infty$. So for $n\in[n_1, n_2]$ there exists a $T>0$ such that $\bar{\Gamma}_F$ and $\bar{\Gamma}_G$ keep in the sector $0<\theta\leqslant\bar{\theta}$ for $u>T$. Now let n variate from $\tilde{n}$ to $\frac{1}{2}$. Then the intersection points of $\bar{\Gamma}_F$ and $\bar{\Gamma}_G$ can vanish only in the interval $2+\bar{s}<u<T$. Hence their critical state must be the contact position of $\bar{\Gamma}_F$ and $\bar{\Gamma}_G$ as well as of Γ_F and Γ_G correspondingly.

§ 9. $\Gamma_F\left(\frac{3}{2}\right)$ and $\Gamma_G\left(\frac{3}{2}\right)$ Do Not Intersect

Suppose it is the contrary. Then (16) has a solution. When $1<n<2$, (16) is equivalent to

$$\begin{cases} G(z_1)=G(z_2), \\ H(z_1, z_2)=0. \end{cases} \tag{44}$$

Take $n=\frac{3}{2}$. It is easy to obtain $G(z)=\frac{1+2z-7z^2}{z^3}$. Hence the first equation of (44) becomes

$$(z_1+z_2)^2-z_1z_2+2z_1z_2(z_1+z_2)-7(z_1z_2)^2=0. \tag{45}$$

Calculating $H(z_1, z_2)$ from (13) and (14) $\left(\text{let } n=\frac{3}{2}\right)$, and using the transformations in Section 4, we can transform (45) and the second equation of (44) into

$$\begin{cases} -\sin\theta+(\cos^2\theta+2\cos\theta\sin\theta-7\sin^2\theta)\rho=0, \\ -2\cos\theta+9\sin\theta+(-4\cos^2\theta-18\cos\theta\sin\theta+45\sin^2\theta)\rho \\ \quad +(14\cos^3\theta-35\cos^2\theta\sin\theta-10\cos\theta\sin^2\theta+50\sin^3\theta)\rho^2=0. \end{cases} \tag{46}$$

Solving the first equation to obtain ρ, and taking it into the second one of (46), we deduce (let $k=\operatorname{tg}\theta$)

$$(2k-1)^3(2k+1)(11k+2)=0.$$

According to (29), we only need to consider $k=\frac{1}{2}$. But from (29) and (46)

$$\rho=\frac{\sin\theta}{\cos^2\theta+2\cos\theta\sin\theta-7\sin^2\theta}>\frac{1}{\cos\theta-\sin\theta}.$$

It shows $k>\frac{1}{2}$. Therefore, (46) doesn't have any positive solution for ρ satisfying (29).

§ 10. $\Gamma_F(3)$ and $\Gamma_G(3)$ Do Not Intersect

Suppose it is the contrary. Then the following equations (instead of (16)) have a solution.

$$\begin{cases} F(z_1)=F(z_2), \\ H(z_1, z_2)=0. \end{cases} \tag{47}$$

When $n=3$, $F(z)=-6(7-12z+4z^2)z^{-4}$. From the first one of (47) we obtain

$$7(z_1+z_2)(z_1^2+z_2^2)-12z_1z_2(z_1^2+z_1z_2+z_2^2)+4(z_1+z_2)(z_1z_2)^2=0. \tag{48}$$

The discussion is similar to the last section. Transform (48) and the second one of (47) to

$$\begin{cases} 14\cos\theta\sin\theta-12\sin^2\theta-(7\cos^3\theta-12\cos^2\theta\sin\theta+4\cos\theta\sin^2\theta)\rho=0, \\ 35\cos\theta+34\sin\theta-(154\cos^2\theta-248\cos\theta\sin\theta+80\sin^2\theta)\rho \\ \quad +(140\cos^3\theta-480\cos^2\theta\sin\theta+560\cos\theta\sin^2\theta-224\cos^3\theta)\rho^2=0. \end{cases} \tag{49}$$

Then we obtain

$$(2k-1)^3(12k-7)^2(28k^2-64k+35)=0.$$

Its four different roots are $k_1=\frac{1}{2}$, $k_2=\frac{7}{12}$, $k_{3,4}=\frac{16\pm\sqrt{11}}{14}$. The case of $k=\frac{7}{12}$ has been discussed in Section 5, iii). From the first equation of (49) we know that in order to satisfy $0<\theta<\frac{\pi}{4}$ and $\rho>0$, we need to have $k<\frac{3-\sqrt{2}}{2}$. Hence k_3, k_4 are not fit. Finally, from the first equation of (49) and (29) we obtain $k>\frac{1}{2}$. So k_1 is not fit too.

Summing up above ten sections, we obtain

Theorem. *There is no limit cycle around a weak focus of order 3 for any quadratic system.*

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