*Chin. Ann. of Math.*
7B (2) 1986

# THE NUMBER OF NONTRIVIAL SOLUTIONS TO HAMMERSTEIN NONLINEAR INTEGRAL EQUATIONS

GUO DAJUN (郭大钧)*

**Abstract**

In this paper, the author improves some results of Rabinowitz about the existence of infinitely many nontrivial solutions to Hammerstein nonlinear integral equations and gives some applications to the two-point boundary value problems for nonlinear ordinary differential equations.

## § 1. Introduction

In this paper, we improve some results obtained by P. H. Rabinowitz in [1]. Consider the Hammerstein nonlinear integral equation

$$\varphi(x)=\int_G k(x, y)f(y, \varphi(y))dy=A\varphi(x), \tag{1}$$

where $G$ is a measurable set in Euclidean space $R^N$ with $0<\text{mes } G<+\infty$ and $f(x, u)$ satisfies the Caratheodory condition, i. e. $f(x, u)$ is measurable with respect to $x$ on $G$ for every $u\in(-\infty, +\infty)$ and is continuous with respect to $u$ on $(-\infty, +\infty)$ for almost all $x\in G$. Suppose that kernel $k(x, y)$ is measurable on $G\times G$ and satisfies

$$\int_G\int_G |k(x, y)|^p\, dx\, dy<+\infty \tag{2}$$

for some $p>2$. Then the linear integral operator

$$K\varphi(x)=\int_G k(x, y)\varphi(y)dy \tag{3}$$

is completely continuous from $L^q(G)$ into $L^p(G)$, where $\frac{1}{p}+\frac{1}{q}=1$, $1<q<2<p$. The following conditions are used:

$(P_1)$ symmetric kernel $k(x,y)$ satisfies (2) and is strictly positive-definite on $L^q(G)$, i. e.

$$(K\varphi, \varphi)=\int_G\int_G k(x, y)\varphi(x)\varphi(y)dxdy>0,\ \forall\varphi\in L^q(G),\ \varphi\neq\theta, \tag{4}$$

Manuscript received February 13, 1984.
* Department of Mathematics, Shandong University, Jinan, Shandong, China.

where $\theta$ denotes the null-element;

($Q_1$) there exist $a>0$ and $b>0$ such that

$$|f(x, u)| \leqslant a+b|u|^{p-1}, \ \forall x \in G, \ -\infty<u<+\infty; \tag{5}$$

($Q_2$) there exist $0 \leqslant \tau < 1/2$ and $M>0$ such that

$$F(x, u)=\int_0^u f(x, v)dv \leqslant \tau u f(x, u), \ \forall |u| \geqslant M, \ x \in G; \tag{6}$$

($Q_3$) $\dfrac{f(x, u)}{u} \to 0$ as $u \to 0$ uniformly in $x \in G$; (7)

($Q_4$) $\dfrac{f(x, u)}{u} \to +\infty$ as $u \to \infty$ uniformly in $x \in G$; (8)

($Q_5$) $f(x, u)$ is odd in $u$, i. e.

$$f(x, -u)=-f(x, u), \ \forall x \in G, \ -\infty<u<+\infty. \tag{9}$$

P. H. Rabinowitz has proved:

(I) if ($P_1$), ($Q_1$)—($Q_4$) are satisfied, then the integral equation (1) has at least one nontrivial solution in $L^p(G)$ (see [1] Theorem 5.8);

(II) if ($P_1$), ($Q_1$)—($Q_5$) are satisfied, then the integral equation (1) has infinitely many nontrivial solutions in $L^p(G)$ (see [1] Theorem 5.9).

But many important kernels (for example, Green functions of some boundary value problems) do not satisfy condifion ($P_1$). We shall weaken condifion ($P_1$) to overcome this shortcoming. We shall also weaken conditions ($Q_3$) and ($Q_4$). In § 2 and § 3, we shall deal with the case of positive-definite kernels and quasi-positive-definite kernels respectively and in § 4 we shall give some applications to the two-point boundary value problems for nonlinear ordinary differential equations.

## § 2. Case of Positive-definite Kernels

**Definition 1.** (i) *$k(x, y)$ is called an $L^2$ kernel if $k(x, y)$ is measurable on $G \times G$,* mes$\{(x, y) \in G \times G \mid k(x, y) \neq 0\}>0$ *and*

$$\int_G \int_G [k(x, y)]^2 dx\, dy < +\infty; \tag{10}$$

(ii) *$L^2$ kernel $k(x, y)$ is said to be positive-definite (quasi-positive-definite) if $k(x, y)$ is symmetric and all its non-zero eigenvalues are positive (only has finite number of negative eigenvalues);*

(iii) *$L^2$ kernel $k(x, y)$ is said to be strictly positive-definite if it is symmetric and satisfies*

$$(K\psi, \psi)=\int_G \int_G k(x, y)\psi(x)\psi(y)dx\, dy>0, \ \forall \psi \in L^2(G), \ \psi \neq \theta. \tag{11}$$

**Remark 1.** It is well known (see [8]) that $L^2$ symmetric kernel $k(x, y)$ is positive-definite if and only if

$$(K\psi, \psi)=\int_G\int_G k(x, y)\psi(x)\psi(y)dx\,dy\geqslant 0, \quad \forall \psi\in L^2(G),$$

i. e. $K$ is a positive operator. Hence, observing $L^2(G)\subset L^q(G)$ (since mes $G<+\infty$ and $1<q<2$), we have

$k(x, y)$ satisfies $(P_1)\Rightarrow k(x, y)$ is strictly positive-definite

$\Rightarrow k(x, y)$ is positive-definite. (12)

**Lemma 1.** *Suppose that $k(x, y)$ is positive-definite. Denote the sequence of eigenvalues and corresponding sequence of orthonormal eigenfunctions of $k(x, y)$ by $\{\lambda_n|\lambda_1\geqslant\lambda_2\geqslant\cdots>0\}$ and $\{\psi_n\}$ respectively. Let $H_0=\{\psi\in L^2(G)\,|\,K^{\frac{1}{2}}\psi=\theta\}$ and $H_1=H_0^\perp$, where $K^{\frac{1}{2}}$ denotes the positive square root operator of $K$ and $H_0^\perp$ denotes the orthogonal complement (in $L^2(G)$) of $H_0$. Then*

(i) $H_0=\{\psi\in L^2(G)\,|\,K\psi=\theta\}$;

(ii) $H_1=\overline{L\{\psi_1, \psi_2, \cdots\}}$, i. e. $H_1$ *is the closed subspace spanned by* $\psi_1, \psi_1, \cdots$;

(iii) $H_1\neq\{\theta\}$ *(hence, $K$ has at least one eigenvalue).*

*Proof* (i) follows immediately from the equality

$$(K\psi, \psi)=(K^{\frac{1}{2}}\psi, K^{\frac{1}{2}}\psi)\,\|K^{\frac{1}{2}}\psi\|^2.$$

To prove (ii), it is sufficient to show

$$\overline{L\{\psi_1, \psi_2, \cdots\}}^\perp=H_0. \tag{13}$$

It is well known that

$$K\psi=\sum_n\lambda_n(\psi, \psi_n)\psi_n, \quad \forall\psi\in L^2(G). \tag{14}$$

Therefore, $\psi\in\overline{L\{\psi_1, \psi_2, \cdots\}}^\perp$ implies $(\psi, \psi_n)=0$ ($n=1, 2, \cdots$), and by (14), $K\psi=\theta$, hence $\psi\in H_0$ by (i); conversely, $\psi\in H_0$ implies

$$\theta=K\psi=\sum_n\lambda_n(\psi, \psi_n)\psi_n,$$

and so

$$0=(K\psi, \psi_m)=\lambda_m(\psi, \psi_m) \quad (m=1, 2, \cdots),$$

hence, on account of $\lambda_m>0$, $\psi\in\overline{L\{\psi_1, \psi_2, \cdots\}}^\perp$.

Finally, if $H_1=\{\theta\}$, then $H_0=L^2(G)$. It follows from (i) that $K=\theta$ and therefore $k(x, y)=0$ p.p. on $G\times G$, in contradiction with Definition 1. Hence, (iii) holds and the Lemma is proved.

**Remark 2.** It follows from (ii) of Lemma 1 that a positive-definite kernel $k(x, y)$ is strictly positive-definite if and only if its sequence of orthonormal eigenfunctions $\{\psi_n\}$ is complete in $L^2(G)$.

In this paragraph, $\{\lambda_n\}$ ($\lambda_1\geqslant\lambda_2\geqslant\cdots>0$) and $\{\psi_n\}$ always denote the sequence of eigenvalues and the corresponding sequence of orthonormal eigenfunctions of the positive-definite kernel $k(x, y)$ respectively.

The following conditions are used:

$(P_2)$ $k(x, y)$ is positive-definite and satisfies (2) for some $p>2$;

($Q_3'$) there exist $\varepsilon_0>0$ and $\delta>0$ such that

$$\frac{f(x,u)}{u}\leqslant\frac{1}{\lambda_1+\varepsilon_0},\quad \forall\ 0<|u|<\delta,\ x\in G;\tag{15}$$

($Q_4'$) there exist $0<\varepsilon_0<\lambda_1$ and $R>0$ such that

$$\frac{f(x,u)}{u}\geqslant\frac{1}{\lambda_1-\varepsilon_0},\quad \forall |u|\geqslant R,\ x\in G;\tag{16}$$

($Q_3''$) $\frac{f(x,u)}{u}\to a_0(x)$ as $u\to 0$ uniformly in $x\in G$, where $a_0(x)$ satisfies $\sup_{x\in G} a_0(x)<\frac{1}{\lambda_1}$;

($Q_4''$) $\frac{f(x,u)}{u}\to a_1(x)$ as $u\to\infty$ uniformly in $x\in G$, where $a_1(x)$ satisfies $\inf_{x\in G} a_1(x)>\frac{1}{\lambda_1}$.

Obviously, $(P_1)\Rightarrow(P_2)$, $(Q_3)\Rightarrow(Q_3'')\Rightarrow(Q_3')$ and $(Q_4)\Rightarrow(Q_4'')\Rightarrow(Q_4')$.

**Theorem 1.** *If the conditions* $(P_2)$, $(Q_1)$, $(Q_2)$, $(Q_3')$ *and* $(Q_4')$ *are satisfied, then the integral equation* (1) *has at least one nontrivial solution in* $L^p(G)$.

*Proof* We have $A=Kf$, where $K$ is the linear integral operator (3) and $f$ is the operator $f\varphi(x)=f(x,\varphi(x))$. From $(P_2)$ and $(Q_1)$ we know (see[3, 6]) that $K$ is completely continuous from $L^2(G)$ into $L^2(G)$ and from $L^q(G)$ into $L^p(G)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and $f$ is bounded and continuous from $L^p(G)$ into $L^q(G)$; moreover, we have $K=HH^*$, where $H=K^{\frac{1}{2}}$ is completely continuous from $L^2(G)$ into $L^p(G)$ and $H^*$ denotes the adjoint operator of $H$, which is completey continuous from $L^q(G)$ into $L^2(G)$. It is well known (see[6]) that functional

$$\Psi(\psi)=\frac{1}{2}(\psi,\psi)-\int_G F(x,H\psi)dx,\ \forall\psi\in L^2(G)\tag{17}$$

is a $C^1$ functional in $L^2(G)$ and its Fréchet derivative is

$$\Psi'(\psi)=\psi-H^*fH\psi.\tag{18}$$

Let $H_0$ and $H_1$ be the closed subspaces of $L^2(G)$ in Lemma 1. By Lemma 1, $H_1\neq\{\theta\}$ and $H_1=H_0^\perp$. Since

$$(H^*fH\psi,h)=(fH\psi,Hh)=(fH\psi,K^{\frac{1}{2}}h)=0,\quad \forall\psi\in L^2(G),\ h\in H_0,\tag{19}$$

it follows that $H^*fH\psi\in H_1$ for all $\psi\in L^2(G)$, hence, we can regard $H^*fH$ as an operator from $H_1$ into $H_1$ and (18) holds again for all $\Psi\in H_1$ wLen we regard $\Psi$ as a functional only on $H_1$. In the following, we verify that the functional $\Psi$ (on $H_1$) satisfies all conditions of the Mountain Pass Lemma, i. e. the conditions $(I_1)$, $(I_1)$ and $(I_3)$ of Theorem 2.1 in [1] (this theorem is also true for finite-dimensional space, see[2]).

Firstly, we verify $(I_1)$. $(Q_3')$ implies

$$F(x,u)\leqslant\frac{u^2}{2(\lambda_1+\varepsilon_0)},\quad \forall |u|<\delta,\ x\in G\tag{20}$$

and $(Q_1)$ implies

$$F(x, u) \leqslant a|u| + \frac{b}{p}|u|^p, \quad \forall x \in G, \ -\infty < u < +\infty. \tag{21}$$

Hence, there exists $b_1 > 0$ such that

$$F(x, u) \leqslant \frac{u^2}{2(\lambda_1 + \varepsilon_0)} + b_1 |u|^p, \quad \forall x \in G, \ -\infty < u < +\infty. \tag{22}$$

Thus, for $\Psi \in H_1$, we have

$$\begin{aligned}\int_G F(x, H\psi)\,dx &\leqslant \frac{1}{2(\lambda_1+\varepsilon_0)} \int_G [H\psi(x)]^2 dx + b_1 \int_G |H\psi(x)|^p dx \\ &= \frac{1}{2(\lambda_1+\varepsilon_0)} \|K^{\frac{1}{2}}\psi\|^2 + b_1 \|H\psi\|_p^p \\ &= \frac{1}{2(\lambda_1+\varepsilon_0)} (K\psi, \psi) + b_1 \|H\psi\|_p^p \\ &\leqslant \frac{1}{2(\lambda_1+\varepsilon_0)} \|K\| \cdot \|\psi\|^2 + b_1 \|H\|^p \cdot \|\psi\|^p \\ &= \frac{\lambda_1}{2(\lambda_1+\varepsilon_0)} \|\psi\|^2 + b_1 \|H\|^p \cdot \|\psi\|^p, \end{aligned} \tag{23}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(G)$ and $\|K\|$ denotes the norm of operator $K$ (from $L^2(G)$ into $L^2(G)$), which satisfies $\|K\| = \lambda_1$ (see [8]). From (17) and (23), we have

$$\Psi(\psi) \geqslant \frac{\varepsilon_0}{2(\lambda_1+\varepsilon_0)} \|\psi\|^2 - b_1 \|H\|^p \cdot \|\psi\|^p, \quad \forall \psi \in H_1. \tag{24}$$

Hence, on account of $p > 2$, there exists sufficiently small $r > 0$ such that

$$\begin{cases} \Psi(\psi) > 0, \ \forall \psi \in B_r \setminus \{\theta\}, \\ \inf\limits_{\psi \in \partial B_r} \Psi(\psi) = c_r > 0, \end{cases} \tag{25}$$

where $B_r = \{\psi \in H_1 \mid \|\psi\| < r\}$, i.e. condition $(I_1)$ is satisfied.

Secondly, we verify $(I_2)$. Consider the real continuous function

$$\begin{aligned}\Phi(t) = \Psi(t\psi_1) &= \frac{t^2}{2} \|\psi_1\|^2 - \int_G F(x, t\, H\psi_1)\,dx \\ &= \frac{t^2}{2} - \int_G F(x, t\sqrt{\lambda_1}\,\psi_1)\,dx, \end{aligned} \tag{26}$$

here we have used the known equality $H\psi_1 = K^{\frac{1}{2}}\psi_1 = \sqrt{\lambda_1}\,\psi_1$. Choosing $0 < t_1 < t_2 < \cdots$, $t_n \to +\infty$ and putting $D_n = \{x \in G \mid t_n \sqrt{\lambda_1}\,|\psi_1(x)| \geqslant R\}$, we have $G \setminus D_n = \{x \in G \mid t_n \sqrt{\lambda_1}\,|\psi_1(x)| < R\}$ and $D_n = D_n^{(1)} \cup D_n^{(2)}$, $D_n^{(1)} \cap D_n^{(2)} = \phi$, where $D_n^{(1)} = \{x \in G \mid t_n \sqrt{\lambda_1} \cdot \psi_1(x) \geqslant R\}$ and $D_n^{(2)} = \{x \in G \mid t_n \sqrt{\lambda_1}\psi_1(x) \leqslant -R\}$. By $(Q_4')$, we know

$$f(x, u) \geqslant \frac{u}{\lambda_1 - \varepsilon_0}, \quad \forall u \geqslant R, \ x \in G,$$

$$f(x, u) \leqslant \frac{u}{\lambda_1 - \varepsilon_0}, \quad \forall u \leqslant -R, \ x \in G,$$

hence

$$\int_G F(x, t_n\sqrt{\lambda_1}\psi_1)dx=\int_{D_n^{(1)}} dx\Big(\int_0^R+\int_R^{t_n\sqrt{\lambda_1}\psi_1(x)} f(x,v)dv\Big)$$
$$+\int_{D_n^{(2)}} dx\Big(\int_0^{-R}+\int_{-R}^{t_n\sqrt{\lambda_1}\psi_1(x)} f(x,v)dv\Big)$$
$$+\int_{G\setminus D_n} dx\int_0^{t_n\sqrt{\lambda_1}\psi_1(x)} f(x,v)dv$$
$$\geqslant\int_{D_n^{(1)}} dx\int_R^{t_n\sqrt{\lambda_1}\psi_1(x)}\frac{v\,dv}{\lambda_1-\varepsilon_0}-\int_{D_n^{(1)}} dx\int_0^R|f(x,v)|dv$$
$$-\int_{D_n^{(2)}} dx\int_{t_n\sqrt{\lambda_1}\psi_1(x)}^{-R}\frac{v\,dv}{\lambda_1-\varepsilon_0}-\int_{D_n^{(2)}} dx\int_{-R}^0|f(x,v)|dv$$
$$-\int_{G\setminus D_n} dx\int_{-R}^R|f(x,v)|dv$$
$$\geqslant\frac{1}{2(\lambda_1-\varepsilon_0)}\int_{D_n^{(1)}}(t_n^2\lambda_1[\psi_1(x)]^2-R^2)dx$$
$$-\int_{D_n} dx\int_0^R|f(x,v)|dv$$
$$+\frac{1}{2(\lambda_1-\varepsilon_0)}\int_{D_n^{(2)}}(t_n^2\lambda_1[\psi_1(x)]^2-R^2)dx$$
$$-\int_{D_n} dx\int_{-R}^0|f(x,v)|dv-\int_{G\setminus D_n} dx\int_{-R}^R|f(x,v)|dv$$
$$=\frac{1}{2(\lambda_1-\varepsilon_0)}\int_{D_n}(t_n^2\lambda_1[\psi_1(x)]^2-R^2)dx$$
$$-\int_G dx\int_{-R}^R|f(x,v)|dv$$
$$\geqslant\frac{\lambda_1 t_n^2}{2(\lambda_1-\varepsilon_0)}\int_{D_n}[\psi_1(x)]^2dx-M_1, \tag{27}$$

where

$$M_1=\Big[\frac{R^2}{2(\lambda_1-\varepsilon_0)}+2aR+\frac{2b}{p}R^p\Big]\text{mes }G$$

is a constant. Putting $D=\{x\in G\,|\,\psi_1(x)\neq 0\}$, we have

$$\int_D[\psi_1(x)]^2dx=\int_G[\psi_1(x)]^2dx=\|\psi_1\|^2=1,$$

$D_1\subset D_2\subset\cdots$, $\bigcup_{n=1}^{\infty}D_n=D$ and mes $D_n\to$mes $D$. Hence, there exists $N_0>0$ such that

$$\int_{D_n}[\psi_1(x)]^2dx>\int_D[\psi_1(x)]^2dx-\frac{\varepsilon_0}{2\lambda_1}=1-\frac{\varepsilon_0}{2\lambda_1},\quad \forall n>N_0. \tag{28}$$

It follows from (26), (27) and (28) that

$$\Phi(t_n)=\frac{t_n^2}{2}-\int_G F(x,t_n\sqrt{\lambda_1}\psi_1)dx<-\frac{\varepsilon_0}{4(\lambda_1-\varepsilon_0)}t_n^2+M_1,\qquad \forall n>N_0, \tag{29}$$

hence $\Phi(t_n)\to-\infty\,(n\to\infty)$. On the other hand, (25) implies $\Phi(r)=\Psi(r\psi_1)\geqslant c_r>0$, it follows therefore from the continuity of $\Phi(t)$ that there exists $r<t^*<+\infty$ such that $\Phi(t^*)=\Psi(t^*\psi_1)=0$ and, thus, $(I_2)$ is satisfied by taking $e=t^*\psi_1(e\neq\theta,\ \Psi(e)=0)$.

Finally, we verify $(I_3)$. In fact, we can prove that $\Psi$ satisfies the (P. S.) condition. Let $\{h_n\}\subset H_1$ with $|\Psi(h_n)|\leqslant\beta$ $(n=1, 2, \cdots)$ and $\Psi'(h_n)\to\theta$. Putting $G_n=\{x\in G\mid |Hh_n(x)|\geqslant M\}$, we find from $(Q_1)$, (21) and $(Q_2)$ that

$$\beta\geqslant\Psi(h_n)=\frac{1}{2}\|h_n\|^2-\int_G F(x, Hh_n)\,dx$$
$$\geqslant\frac{1}{2}\|h_n\|^2-\int_{G_n} F(x, Hh_n)\,dx-M_2$$
$$\geqslant\frac{1}{2}\|h_n\|^2-\tau\int_{G_n} f(x, Hh_n)Hh_n\,dx-M_2$$
$$\geqslant\frac{1}{2}\|h_n\|^2-\tau\int_G f(x, Hh_n)Hh_n\,dx-M_3, \tag{30}$$

where $M_2$ and $M_3$ are constants independent of $n$. By virtue of (18), we have

$$(\Psi'(h_n), h_n)=(h_n-H^*fHh_n, h_n)=\|h_n\|^2-(fHh_n, Hh_n)$$
$$=\|h_n\|^2-\int_G f(x, Hh_n)Hh_n\,dx, \tag{31}$$

it follows from (30) and (31) that

$$\beta\geqslant\left(\frac{1}{2}-\tau\right)\|h_n\|^2+\tau(\Psi'(h)_n, h_n)-M_3$$
$$\geqslant\left(\frac{1}{2}-\tau\right)\|h_n\|^2-\tau\|\Psi'(h_n)\|\cdot\|h_n\|-M_3 \quad (n=1, 2, \cdots), \tag{32}$$

hence, $\{h_n\}$ is bounded. Since $H_1$ is a Hilbert space, there exists a subsequence $\{h_{n_k}\}\subset\{h_n\}$, which converges weakly to an element $h_0\in H_1$, hence, on account of the complete continuity of $H$, $\|Hh_{n_k}-Hh_0\|_p\to0$. It follows therefore from (31) and the continuity of operator $f$ that

$$\|h_{n_k}\|^2\to\int_G f(x, Hh_0)Hh_0\,dx=(fHh_0, Hh_0). \tag{33}$$

On the other hand, we have

$$(\Psi'(h_{n_k}), h_0)=(h_{n_k}-H^*fHh_{n_k}, h_0)=(h_{n_k}, h_0)-(fHh_{n_k}, Hh_0).$$

Letting $k\to\infty$, we obtain

$$\|h_0\|^2=(fHh_0, Hh_0). \tag{34}$$

From (33) and (34), we find $\|h_{n_k}\|\to\|h_0\|$, hence $\|h_{n_k}-h_0\|\to0$, and $(I_3)$ is satisfied.

By Theorem 2.1 in [1], $\Psi$ has a critical point $\psi^*\in H_1$ with $\psi^*\neq\theta$, i. e.

$$\Psi'(\psi^*)=\psi^*-H^*fH\psi^*=\theta.$$

Putting $\varphi^*=H\psi^*\in L^p(G)$, we find

$$A\varphi^*=Kf\varphi^*=HH^*fH\psi^*=H\psi^*=\varphi^*,$$

i. e. $\varphi^*$ is a solution of (1) in $L^p(G)$. If $\varphi^*=\theta$, i. e. $K^{\frac{1}{2}}\psi^*=\theta$, then $\psi^*\in H_0$. But $\psi^*\in H_1$, hence $\psi^*=\theta$. This is a contradiction. Thus, we have $\varphi^*\neq\theta$ and our theorem is proved.

**Remark 3.** Evidently, Theorem 1 is an improvement of Theorem 5.8 in [1].

**Theorem 2.** *If the conditions $(P_2)$, $(Q_1)$, $(Q_2)$, $(Q_3')$, $(Q_4)$ and $(Q_5)$ are* *satisfied and $k(x, y)$ has infinitely many eigenvalues, then the integral equation* (1) *has infinitely many nontrivial solutions in* $L^p(G)$.

*Proof* We again use the notations in the proof of Theorem 1. Since $k(x, y)$ has infinitely many eigenvalues, it follows from (ii) of Lemma 1 that $H_1$ is infinite-dimensional. We verify that the functional $\Psi$ (on $H_1$) satisfies all conditions $(I_1)$, $(I_3)$, $(I_4)$ and $(I_5)$ of Theorem 2.8 in [1]. Conditions $(I_1)$ and $(I_3)$ have been already verified in the proof of Theorem 1. By $(Q_5)$ we have

$$F(x, -u)=F(x, u), \quad \forall x\in G, \ -\infty<u<+\infty,$$

hence, $\Psi(-\psi)=\Psi(\psi)$ for all $\psi\in H_1$ and $(I_4)$ is satisfied. Finally, we verify $(I_5)$. Suppose that $(I_5)$ is not satisfied. Then, there exist a finite-dimensional subspace $X$ of $H_1$ and $h_n^*\in X$, $\|h_n^*\|\to+\infty$, such that

$$\Psi(h_n^*)\geqslant 0 \quad (n=1, 2, \cdots). \tag{35}$$

Putting $t_n=\|h_n^*\|$ and $\psi_n^*=\frac{1}{t_n}h_n^*\in X$, we have $h_n^*=t_n\psi_n^*$, $\|\psi_n^*\|=1$ and $t_n\to+\infty$. Since $X$ is finite-dimensional, $\{\psi_n^*\}$ contains a convergent subsequence. Without loss of generality, we may assume that $\psi_n^*$ itself converges to some $\psi^*\in X$, i. e. $\|\psi_n^*-\psi^*\|\to 0$. Obviously, $\|\psi^*\|=1$. The continuity of operator $H$ implies

$$\|H\psi_n^*-H\psi^*\|_p=\left(\int_G |H\psi_n^*(x)-H\psi^*(x)|^p\,dx\right)^{\frac{1}{p}}\to 0, \tag{36}$$

hence, $\{H\psi_n^*(x)\}$ contains a subsequence, which converges to $H\psi^*(x)$ almost everywhere on $G$. Without loss of generality, we may assume that $H\psi_n^*(x)$ itself converges to $H\psi^*(x)$ almost everywhere on $G$. Obviously, $H\psi^*\neq\theta$ (otherwise, $K^{\frac{1}{2}}\psi^*=H\psi^*=\theta$ implies $\psi^*H_0$. But $\psi^*\in X\subset H_1$, hence $\psi^*=\theta$, in contradiction with $\|\psi^*\|=1$). Putting $v_n=H\psi_n^*$ $(n=1, 2, \cdots)$, $v_0=H\psi^*$, $G_0=\{x\in G\,|\,v_0(x)\neq 0$ and $v_n(x)\to v_0(x)\}$ and $a_0=\left(\int_{G_0}[v_0(x)]^2\,dx\right)^{\frac{1}{2}}$, we have mes $G_0>0$ and $a_0>0$. By $(Q_4)$, there exists $R>0$ such that

$$\frac{f(x, u)}{u}\geqslant\frac{8}{a_0^2}, \quad \forall |u|\geqslant R, \ x\in G. \tag{37}$$

Similar to (27), we can deduce the following inequality

$$\begin{aligned}\int_G F(x, Hh_n^*)dx&=\int_G F(x, t_nv_n)dx\\ &\geqslant\frac{4}{a_0^2}\,t_n^2\int_{D_n}[v_n(x)]^2dx-M^*,\end{aligned} \tag{38}$$

where $D_n=\{x\in G\,|\,t_n|v_n(x)|\geqslant R\}$ $(n=1, 2, \cdots)$ and

$$M^*=\left(\frac{4R^2}{a_0^2}+2aR+\frac{2bR^p}{p}\right)\text{mes}\,G$$

is a constant. Using Hölder inequality and observing (36), we find

$$\left|\left(\int_{D_n}[v_n(x)]^2\,dx\right)^{\frac{1}{2}}-\left(\int_{D_n}[v_0(x)]^2\,dx\right)^{\frac{1}{2}}\right|$$

$$\leqslant\left(\int_{D_n}[v_n(x)-v_0(x)]^2dx\right)^{\frac{1}{2}}\leqslant\left(\int_G|H\psi_n^*(x)-H\psi^*(x)|^2dx\right)^{\frac{1}{2}}$$

$$\leqslant(\mathrm{mes}\,G)^{\frac{1}{2}-\frac{1}{p}}\left(\int_G|H\psi_n^*(x)-H\psi^*x)|^p\,dx\right)^{\frac{1}{p}}\to 0\ (n\to\infty),$$

hence, there exists $N_1>0$ such that

$$\left(\int_{D_n}[v_n(x)]^2dx\right)^{\frac{1}{2}}>\left(\int_{D_n}[v_0(x)]^2dx\right)^{\frac{1}{2}}-\frac{a_0}{4},\quad \forall n>N_1. \tag{39}$$

Putting $D_n^*=\bigcap_{k=n}^{\infty}D_k$ and $D^*=\bigcap_{n=1}^{\infty}D_n^*$, we have $D_1^*\subset D_2^*\subset D_3^*\subset\cdots\subset D^*$, $D_n\supset D_n^*$ and mes $D_n^*\to\mathrm{mes}D^*$, hence, there exists $N_2>0$ such that

$$\left(\int_{D_n}[v_0(x)]^2dx\right)^{\frac{1}{2}}\geqslant\left(\int_{D_n^*}[v_0(x)]^2\mathrm{d}x\right)^{\frac{1}{2}}>\left(\int_{D^*}[v_0(x)]^2dx\right)^{\frac{1}{2}}-\frac{a_0}{4},\quad \forall n>N_2. \tag{40}$$

It is easy to see that $G_0\subset D^*$, hence

$$\left(\int_{D^*}[v_0(x)]^2dx\right)^{\frac{1}{2}}\geqslant\left(\int_{G_0}[v_0(x)]^2dx\right)^{\frac{1}{2}}=a_0. \tag{41}$$

It follows from (39), (40) and (41) that

$$\left(\int_{D_n}[v_n(x)]^2dx\right)^{\frac{1}{2}}>\frac{a_0}{2},\quad \forall n>N=\max\{N_1,\ N_2\}. \tag{42}$$

By (38) and (42), we have

$$\Psi(h_n^*)\leqslant\frac{1}{2}\|h_n^*\|^2-(t_n^2-M^*)=-\frac{t_n^2}{2}+M^*,\qquad \forall n>N,$$

hence $\Psi(h_n^*)\to-\infty\,(n\to\infty)$, which contradicts (35). Thus, $(I_5)$ is satisfied.

By Theorem 2.8 and Corollary 2.9 in [1], we know that $\Psi$ has infinitely many nontrivial critical points $\psi_n^{**}$ $(n=1,\ 2,\ \cdots)$ in $H_1$. Similar to the proof of Theorem 1, $\varphi_n^*=H\psi_n^{**}\in L^p(G)\,(n=1,\ 2,\ \cdots)$ are nontrivial solutions of equation (1); moreover, we have $\varphi_n^*\neq\varphi_m^*$, $\forall n\neq m$ (Since, if $\varphi_n^*=\varphi_m^*$ for some $n$, $m$ with $n\neq m$, then $K^{\frac{1}{2}}(\psi_n^{**}-\psi_m^{**})=\theta$, and therefore $\psi_n^{**}-\psi_m^{**}\in H_0$; but $\psi_n^{**}-\psi_m^{**}\in H_1$, hence $\psi_n^{**}-\psi_m^{**}=\theta$, and it is a contradiction). Thus, our proof is complete.

**Remark 4.** Evidently, Theorem 2 is an improvement of Theorem 5.9 in [1].

**Remark 5.** It is easy to give some elementary functions $f(x,\ u)$, which satisfy all conditions of Theorem 2; for example

$$f(x,\ u)=\sum_{k=1}^{n}a_ku^{2k-1},\ n\geqslant 2,\ a_n>0,\ a_1<\frac{1}{\lambda_1}, \tag{43}$$

$$f(x,\ u)=\frac{u^7}{1+u^4}-u^{\frac{1}{3}} \tag{44}$$

(for function (43) we take $p=2n$ and for (44) take $p=4$). We only verify the condition $(Q_2)$. For function (43), we have

$$\int_0^u f(x, v)dv = \sum_{k=1}^{n} \frac{a_k}{2k} u^{2k} \leqslant \sum_{k=1}^{n} \frac{a_k}{2n-1} u^{2k} = \frac{1}{2n-1} uf(x, u)$$

for sufficiently large $|u|$, hence we may choose $\tau = \frac{1}{2n-1}\left(0<\tau\leqslant\frac{1}{3}\right)$; for function (44), we have

$$\int_0^u f(x, v)dv = \frac{u^4}{4} - \frac{1}{4}\ln(1+u^4) - \frac{3}{4}u^{\frac{4}{3}}$$
$$\leqslant \frac{u^4}{3} - \frac{u^4}{3(1+u^4)} - \frac{1}{3}u^{\frac{4}{3}} = \frac{1}{3}uf(x, u)$$

for sufficiently large $|u|$, hence we may choose $\tau = \frac{1}{3}$.

**Remark 6.** It is easy to see from the proof of Theorems 1 and 2 that the condition (2) in $(P_2)$ can be replaced by the weaker condition: operator $K$ is completely continuous from $L^q(G)$ into $L^p(G)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$.

**Remark 7.** We must point out that Theorem 2.8 and Corollary 2.9 in [1] hold only when Banach space $E$ is infinite-dimensional. For example, it is easy to verify that the real function (i.e. functional on $R^1$)

$$f(u) = u^2\left(e^{-u^2} - \frac{1}{2}\right), \quad \forall u \in R^1 \tag{45}$$

satisfies all conditions $(I_1)$, $(I_3)$, $(I_4)$ and $(I_5)$ (also $(I_2)$) of Theorem 2.8 in [1], but $f$ has only two nontrivial critical points. Thus, the hypothesis "$k(x, y)$ has infinitely many eigenvalues" in Theorem 2 can not be omitted.

## § 3. Case of Quasi-Positive-Definite Kernels

The following conditions are used:

$(P_3)$ $k(x, y)$ is quasi-positive-definite and satisfies (2) for some $p>2$;

$(Q_3^*)$ there exist $\varepsilon_0>0$ $(\varepsilon_0<-\lambda^*)$ and $\delta>0$ such that

$$\frac{f(x, u)}{u} \leqslant \frac{1}{\lambda^*+\varepsilon_0}, \quad \forall 0<|u|<\delta,\ x\in G, \tag{46}$$

where $\lambda^*$ denotes the largest negative eigenvalue of $k(x, y)$;

$(Q_4^*)$ there exist $\eta>0$ and $R>0$ such that

$$\frac{f(x, u)}{u} \geqslant \eta, \quad \forall |u| \geqslant R,\ x\in G. \tag{47}$$

**Theorem 3.** *If the conditions* $(P_3)$, $(Q_1)$, $(Q_2)$, $(Q_3^*)$ *and* $(Q_4^*)$ *are satisfied, then the integral equation* (1) *has at least one nontrivial solution in* $L^p(G)$.

*Proof* Let the sequence of eigenvalues of $k(x, y)$ be $\{-\lambda_0, -\lambda_1, \cdots, -\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\}$, where $0<\lambda_0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_m$, $\lambda_{m+1}\geqslant\lambda_{m+2}\geqslant\cdots>0$, hence $\lambda^*=-\lambda_0$. Choose a number $\mu$ satisfying $0<\mu<\lambda_0$ and put $K_1=KR_1$, where $R_1=(K+\mu I)^{-1}$ and $I$ denotes the identical operator. By Lemme 1.1 in [3], we know that $K_1$ is also a linear integral operator generated by some $L^2$ kernel $k_1(x, y)$ with sequence of eigenvalues $\left\{\frac{\lambda_0}{\lambda_0-\mu}, \frac{\lambda_1}{\lambda_1-\mu}, \cdots, \frac{\lambda_m}{\lambda_m-\mu}, \frac{\lambda_{m+1}}{\lambda_{m+1}+\mu}, \frac{\lambda_{m+2}}{\lambda_{m+2}+\mu}, \cdots\right\}$. Since all these eigenvalues are positive, $k_1(x, y)$ is a positive-definiet kernel with largest eigenvalue $\lambda_0^* = \frac{\lambda_0}{\lambda_0-\mu}$. Moreover, from the proof of Theorem 1.1 in [3], we know that $K_1$ is also a completely continuous operator from $L^q(G)$ into $L^p(G)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$.

Now we consider the Hammerstein integral equation

$$\varphi(x) = \int_G k_1(x, y) f_1(y, \varphi(y))\, dy, \tag{48}$$

where

$$f_1(x, u) = u + \mu f(x, u), \tag{49}$$

and prove that $k_1(x, y)$ and $f_1(x, u)$ satisfy all conditions of Theorem 1. In fact, $(Q_1)$ for $f(x, u)$ implies $(Q_1)$ for $f_1(x, u)$, and, observing Remark 6, $(P_2)$ is satisfied for $k_2(x, y)$. By $(Q_2)$ for $f(x, u)$, we have

$$F_1(x, u) = \int_0^u f_1(x, v)\, dv = \frac{u^2}{2} + \mu F(x, u) \leqslant \frac{u^2}{2} + \mu\tau u f(x, u), \quad \forall |u| \geqslant M,\ x \in G. \tag{50}$$

Choosing $\tau_1$ such that $\tau < \tau_1 < \frac{1}{2}$ and $\left(\frac{1}{2}-\tau_1\right)/\mu(\tau_1-\tau) < \eta$, we find by $(Q_4^*)$

$$\frac{uf(x, u)}{u^2} = \frac{f(x, u)}{u} \geqslant \eta > \left(\frac{1}{2}-\tau_1\right)\Big/\mu(\tau_1-\tau), \quad \forall |u| \geqslant R,\ x \in G. \tag{51}$$

It follows from (50) and (51) that

$$\begin{aligned} F_1(x, u) &\leqslant \frac{u^2}{2} + \mu\tau u f(x, u) < \tau_1 u^2 + \mu\tau_1 u f(x, u) \\ &= \tau_1 u f_1(x, u),\ \forall |u| \geqslant M_1 = \max\{M, R\},\ x \in G, \end{aligned} \tag{52}$$

hence, $(Q_2)$ is satisfied for $f_1(x, u)$. Since

$$1 - \frac{\mu}{\lambda_0 - \varepsilon_0} < 1 - \frac{\mu}{\lambda_0} = \frac{\lambda_0 - \mu}{\lambda_0} = \frac{1}{\lambda_0^*} < 1,$$

we can choose a sufficiently small number $\varepsilon_1$ such that $0 < \varepsilon_1 < \lambda_0^*$ and

$$1 - \frac{\mu}{\lambda_0 - \varepsilon_0} < \frac{1}{\lambda_0^* + \varepsilon_1} < \frac{1}{\lambda_0^* - \varepsilon_1} < 1. \tag{53}$$

Thus, $(Q_3^*)$ and $(Q_4^*)$ imply

$$\begin{aligned} \frac{f_1(x, u)}{u} = 1 + \mu\frac{f(x, u)}{u} \leqslant 1 + \frac{\mu}{\lambda^* + \varepsilon_0} = 1 - \frac{\mu}{\lambda_0 - \varepsilon_0} < \frac{1}{\lambda_0^* + \varepsilon_1}, \\ \forall 0 < |u| < \delta,\ x \in G \end{aligned} \tag{54}$$

and

$$\frac{f_1(x, u)}{u} = 1 + \mu\frac{f(x, u)}{u} \geqslant 1 + \mu\eta > 1 > \frac{1}{\lambda_0^* - \varepsilon_1}, \quad \forall |u| \geqslant R,\ x \in G, \tag{55}$$

hence, $(Q_3')$ and $(Q_4')$ are satisfied for $f_1(x, u)$.

By Theorem 1 we know that the integral equation (48) has at least one nontrivial solution $\varphi^*(x)\in L^p(G)$ and from the proof of Theorem 1.1 in [3] we know that $\varphi^*(x)$ is also a solution of equation (1). Our proof is complete.

**Theorem 4.** *If conditions* $(P_3)$, $(Q_1)$, $(Q_2)$, $(Q_3^*)$, $(Q_4)$ *and* $(Q_5)$ *are satisfied and* $k(x, y)$ *has infinitely many eigenvalues, then the integral equation* (1) *has infinitely many nontrivial solutions in* $L^p(G)$.

*Proof* We again use the notations in the proof of Theorem 3. We prove that $k_1(x, y)$ and $f_1(x, u)$ satisfy all conditions of Theorem 2. In fact, since the sequence $\{-\lambda_0, -\lambda_1, \cdots, -\lambda_m, \lambda_{m+1}, \lambda_{m+2}, \cdots\}$ is infinite, the sequence $\left\{\frac{\lambda_0}{\lambda_0-\mu}, \frac{\lambda_1}{\lambda_1-\mu}, \cdots, \frac{\lambda_m}{\lambda_m-\mu}, \frac{\lambda_{m+1}}{\lambda_{m+1}+\mu}, \frac{\lambda_{m+2}}{\lambda_{m+2}+\mu}, \cdots\right\}$ is also infinite, i. e. $k_1(x, y)$ has infinitely many eigenvalues. Moreover, $(Q_4)$ and $(Q_5)$ for $f(x, u)$ imply $(Q_4)$ and $(Q_5)$ for $f_1(x, u)$ respectively:

$$\frac{f_1(x, u)}{u}=1+\mu\frac{f(x, u)}{u}\to=\infty \text{ as } u\to\infty \text{ uniformly in } x\in G,$$

$$f_1(x, -u)=-u+\mu f(x, -u)=-u-\mu f(x, u)=-f_1(x, u),$$

$$\forall x\in G, -\infty+u<+\infty.$$

Conditions $(P_2)$, $(Q_1)$, $(Q_2)$ and $(Q_3')$ (for $k_1(x, y)$ and $f_1(x, u)$) have been (already proved) in the proof of Theorem 3, hence, by virtue of Theorem 2, the equation (48) has infinitely many nontrivial solutions $\varphi_n^*(n=1, 2, \cdots)$ in $L^p(G)$ and from the proof of Theorem 1.1 in [3] these $\varphi_n^*$ $(n=1, 2, \cdots)$ are also solutions of equation (1). Our theorem is proved.

**Remark 8.** It is easy to give some elementary functions $f(x, u)$, which satisfy all conditions of Theorem 4; for example, function (43) $\left(\text{replacing condition } a_1<\frac{1}{\lambda_1} \text{ by } a_1<\frac{1}{\lambda^*}\right)$ and function (44) are such elementary functions.

**Remark 9.** Theorems 2 and 4 show that using variational method in critical point theory we can obtain the existence of infinitely many solutions of integral equation (1), which can not be deduced by topological method; but, when we investigate the number of nontrivial solutions of equation (1) by topological method, the positive-definite property or quasi-positive-definite property of $k(x, y)$ is not required (see [4, 5]).

## § 4. Applications

In this paragraph, we give some applications of Theorems 1 and 2 to the two-point boundary value problem for the nonlinear ordinary differential equation

$$\begin{cases}-\frac{d^2u}{dx^2}=f(x, u),\ 0\leqslant x<1;\\ u(0)=u(1)=0.\end{cases} \tag{56}$$

**Theorem 5.** *If $f(x, u)$ is continuous in $0\leqslant x\leqslant 1$, $-\infty<u<+\infty$ and satisfies $(Q_1)$ (for some $p>2$), $(Q_2)$, $(Q_3')$ and $(Q_4')$ with $G=[0, 1]$ and $\lambda_1=\frac{1}{\pi^2}$, then the problem (56) has at least one nontrivial solution in $C^2[0, 1]$.*

**Theorem 6.** *If $f(x, u)$ is continuous in $0\leqslant x\leqslant 1$, $-\infty<u<+\infty$ and satisfies $(Q_1)$ (for some $p>2$), $(Q_2)$, $(Q_3')$, $(Q_4)$ and $(Q_5)$ with $G=[0, 1]$ and $\lambda_1=\frac{1}{\pi^2}$, then the problem (56) has infinitely many nontrivial solutions in $C^2[0, 1]$.*

*Proof of Theorems 5 and 6* It is well known that the solution (in $C^2[0, 1]$) of problem (56) is equivalent to the solution (in $C[0, 1]$) of the Hammerstein integral equation

$$u(x)=\int_0^1 G(x, y)f(y, u(y))dy, \tag{57}$$

where $G(x, y)$ is the corresponding Green function

$$G(x, y)=\begin{cases} x(1-y), & x\leqslant y, \\ y(1-x), & x>y. \end{cases}$$

It is also well known that the eigenvalues of $G(x, y)$ are $\left\{\frac{1}{n^2\pi^2}\right\}$ $(n=1, 2, \cdots)$ and the corresponding orthonormal eigenfunctions are $\{\sqrt{2}\sin n\pi x\}$ $(n=1, 2, \cdots)$, hence, $G(x, y)$ is a positive-definite kernel with infinitely many eigenvalues. By the continuity of $G(x, y)$, we have

$$\int_0^1\int_0^1 |G(x, y)|^p dx\, dy<+\infty,$$

hence, $G(x, y)$ satisfies condition $(P_2)$. It follows therefore from Theorems 1 and 2 that equation (57) has at least one nontrivial solution in $L^p[0, 1]$ in the case of Theorem 5 and has infinitely many nontrivial solutions in $L^p[0, 1]$ in the case of Theorem 6. We remain to prove that every solution $u^*(x)\in L^p[0, 1]$ of equation (57) must belong to $C[0, 1]$. In fact, $(Q_1)$ implies $f(x, u^*(x))\in L^q[0, 1]$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, it follows therefore by the continuity of $G(x, y)$ that the function

$$u^*(x)=\int_0^1 G(x, y)\, f(y, u^*(y))dy$$

belongs to $C[0, 1]$. Our proof is complete.

**Remark 10.** Using Theorem 6 to functions (43) and (44) in partioular, we cbtain the following conclusions:

(a) two-point boundary value problem

$$\begin{cases} -\dfrac{d^2u}{dx^2}=\sum\limits_{k=1}^{n} a_k u^{2k-1}, \quad 0\leqslant x\leqslant 1, \\ u(0)=u(1)=0 \end{cases} \tag{58}$$

(where $n\geqslant 2$, $a_n>0$ and $a_1<\pi^2$) has infinitely many nontrivial solutions in $C^2[0, 1]$

(b) two-point boundary value problem

$$\begin{cases} -\dfrac{d^2u}{dx^2}=\dfrac{u^7}{1+u^4}-u^{\frac{1}{3}}, & 0\leqslant x\leqslant 1, \\ u(0)=u(1)=0 \end{cases} \tag{59}$$

has infinitely many nontrivial solutions in $C^2[0, 1]$.

**Remark 11.** Since the sequence of orthonormal eigenfunctions $\{\sqrt{2}\sin n\pi x\}$ $(n=1, 2, \cdots)$ of $G(x, y)$ is not complete in $L^2[0, 1]$, by Remark 2, $G(x, y)$ is not strictly positive-definite and therefore by (12) $G(x, y)$ does not satisfy condition $(P_1)$; moreover, condition $(Q_3')$ is weaker than $(Q_3)$ (for example, function (44) sati sfies $(Q_3')$ but does not satisfy $(Q_3)$ and function (43) satisfies $(Q_3')$ but not $(Q_3)$ when $a_1\neq 0$). Hence, Theorems 5 and 6 (in particular, conclusions (a) and (b) mentioned above) can not be deduced from the results (I) and (II) of P. H. Rabinowitz mentioned at the beginning of this paper.

## References

[1] Ambrosetti, A. & Rabinowitz, P. H., Dual variational methods in critical point theory and applications, *J. Functional Anal.*, **14** (1973), 349—381.
[2] Nirenberg, L., Variational and topological methods in nonlinear problems, *Bull. Amer. Math. Soc.*, **4** (1981), 267—300.
[3] Guo Dajun, *Acta Math. Sinica*, **16** (1966), 137—149 (in Chinese).
[4] Guo Dajun, *Acta Math. Sinica*, **22** (1979), 584—594 (in Chinese).
[5] Guo Dajun, *Chin Ann of Math.*, **A4**:5 (1983), 645—656 (in Chinese).
[6] Красносельский, М. А., Топологические методы в теории нелинейных интегральных уравнений, Москва, 1956.
[7] Вайнберг, М. М., Вариационные методы исследования несинейных операторов, Моспва, 1956.
[8] Zaanen, A. C., Linear analysis, Amsterdam, 1953.