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THE BEHAVIOR OF SOLUTIONS IN THE VICINITY OF A BOUNDED SOLUTION TO AUTONOMOUS DIFFERENTIAL EQUATIONS

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Abstract

By using the exponential dichotomy, this paper investigates the behavior of solutions in the vicinity of a bounded solution to the autonomous differential system

$$\frac{dx}{dt}=f(x). \tag{1}$$

Suppose $x=u(t)$ is a nontrivial bounded solution of system (1). By discussing the equivalent equations of system (1)

$$\frac{d\theta}{dt}=1+\bar{f}_1(\rho,\ \theta)$$
$$\frac{d\rho}{dt}=A(\theta)\rho+\bar{f}_2(\rho,\ \theta) \tag{2}$$

with respect to the moving orthonormal transformation

$$x=u(\theta)+s(\theta)\rho,$$

the author proves that if linear system corresponding to (2) admits exponential dichotomy, then the given bounded solution $x=u(t)$ should be periodic. The author also discusses the stadility of the obtained periodic solution. Finally, this paper discusses perturbation of the bounded solution of autonomous system (1).

§ 1. Introduction

We consider the n-dimensional autonomous system

$$\frac{dx}{dt}=f(x) \tag{1.1}$$

and the perturbed system

$$\frac{dx}{dt}=f(x)+\varepsilon F(x), \tag{1.2}$$

where $f(x)$, $F(x)$ are continuously differentiable in the domain D, an open set in R^n, and $n\geqslant 2$.

For system (1.1), the theory about the behavior of solutions in the vicinity of a periodic solution or equilibrium point has been well developed. There are also a few papers discussing the behavior of solutions in the vicinity of an almost periodic

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solution, [1,2]. However, the purpose of this paper is to discuss similar problems in the vicinity of a more common invariant set, the orbit of a bounded solution.

In section 2 we well introduce a simple method for the construction of a moving orthonormal system along a bounded orbit; then the equations of orbits with respect to the moving orthonormal system will be derived and some of their properties will be described.

In section 3 we will provide sufficient conditions for the existence of periodic solution to nonlinear autonomous system of arbitary dimension. It is well known that the classical answer to this question is given by the Poincare-Bendixson Theorem which holds only for 2-dimensional system. When the dimension of the system is larger than two, there is no common method. J. Cronin points out[3] that if an n-dimensional nonlinear autonomous system of ordinary differential equations has a bounded solution with a certain asymptotic stability property, this solution approaches a periodic solution. G. R. Sell points out[2] that under certain conditions the almost periodic solution of system (1.1) is a periodic one. In this paper we do not need the assumption of stability property. The main result is Theorem 3.1, which says that under certain conditions the bounded solution of system (1.1) is a periodic solution. We will also consider the stability of this obtained periodic solution.

In section 4 we will consider the perturbed system (1.2). We will obtain results similar to those in section 3.

§ 2. Moving Orthonormal Systems Along a Bounded Orbit

The search for orbits of autonomous systems is a problem in the geometry of the phase space. Therefore, it is convenient to use a moving coordinate system along an orbit.

Definition 1. *Solution $u(t)$ of system (1.1) is bounded if there exists a number $M>0$ such that if $t\in R$ and $u(t)$ is defined, then $u(t)\in B_M$, the closed ball in R^n with center O and radius M.*

Suppose $x=u(t)$ is a bounded solution of system (1.1) and there exists a neighborhood $U_\delta(u(t))$ of $u(t)$ such that there is no singular point in the set

$$\overline{U_\delta(u(t))}=\{x:\ \|x-u(t)\|\leqslant\delta\}\subset D,$$

then there exists a number $m>0$ such that

$$\|f(x)\|>m,\ x\in\overline{U_\delta(u(t))}, \tag{2.1}$$

where $\|\cdot\|$ is defined as

$$\|x\|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2},$$

$$x=\mathrm{col}(x_1, x_2, \cdots, x_n).$$

By the theory of linear equations and the conditions (1.2), (2.1), there exist n-1 n-dimensional non-zero vectors

$$\bar{e}_1(\theta), \bar{e}_2(\theta), \cdots, \bar{e}_{n-1}(\theta), \quad \theta\in R,$$

each of which is orthogonal to vector $f(v(\theta))$. By Schmidt's method, we have the orthonormal system

$$\{e_1(\theta), e_2(\theta), \cdots, e_{n-1}(\theta)\}$$

about the $(n-1)$-dimensional subspace perpendicular to the unit vector

$$v(\theta)=\frac{f(u(\theta))}{\|f(u(\theta))\|}.$$

The results are summarized as follows.

Theorem 2.1. *Suppose the autonomous system* (1.1), *in which* $f(x)$ *is continuously differentiable in the domain* D, *has a bounded orbit* C: $x=u(t)$, *satisfying* (2.1). *Then there is always a continuously differentiable moving orthogonal coordinate system*

$$\{v(\theta), e_1(\theta), e_2(\theta), \cdots, e_{n-1}(\theta)\}$$

along C, *where*

$$v(\theta)=\frac{f(u(\theta))}{\|f(u(\theta))\|}.$$

It is evident that when $n=2$, we have the moving orthogonal coordinate system

$$\{v(\theta), s(\theta)\},$$

where $s(\theta)=(-v_2(\theta), v_1(\theta))$, $v_1(\theta)$, $v_2(\theta)$ are the components of the vector $v(\theta)$.

Now we consider the same equations of system (1.1) with respect to the moving orthonormal system. Take the transformation

$$x=u(\theta)+s(\theta)\rho, \tag{2.2}$$

where

$$s(\theta)=(e_1(\theta), e_2(\theta), \cdots, e_{n-1}(\theta)),$$
$$\rho=\mathrm{col}(\rho_1, \rho_2, \cdots, \rho_{n-1}).$$

Substituting (2.2) into (1.1), we have

$$\frac{dx}{dt}=(u'(\theta)+s'(\theta)\rho)\frac{d\theta}{dt}+s(\theta)\frac{d\rho}{dt}=f(u(\theta)+s(\theta)\rho),$$

that is

$$(u'(\theta)+s'(\theta)\rho, s(\theta))\begin{pmatrix}\frac{d\theta}{dt}\\ \frac{d\rho}{dt}\end{pmatrix}=f(u(\theta)+s(\theta)\rho). \tag{2.3}$$

Because

$$|(u'(\theta), s(\theta))|^2 = \left|\begin{pmatrix} \|f(u(\theta))\|^2 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}\right| = \|f(u(\theta))\|^2 > m^2 > 0,$$

there exists a small number δ such that the $n \times n$ matrix

$$(u(\theta) + s'(\theta)\rho,\ s(\theta))$$

is not singular when $\|\rho\| < \delta$. Therefore, (2.2) is a regular transformation when $\|\rho\| < \delta$, and both of $\frac{d\theta}{dt}$ and $\frac{d\rho}{dt}$ can be expressed as the function of ρ, θ and t. Multiplying $v^*(\theta)$ and $s^*(\theta)$ on both side of (2.3), we have, respectively,

$$\frac{d\theta}{dt} = 1 + \bar{f}, (\rho, \theta) \tag{2.4}$$

and

$$\frac{d\rho}{dt} = A(\theta)\rho + \bar{f}_2(\rho, \theta). \tag{2.5}$$

Let us substitute (2.4) into (2.5), then we have

$$\frac{d\rho}{d\theta} = A(\theta)\rho + \bar{f}_2(\rho, \theta). \tag{2.6}$$

(2.4) and (2.6) is nothing but the equivalent equations of system (1.1) with respect to the moving orthonormal system, where

$$\bar{f}_1, \bar{f}_2 \in C^1;$$

$$\|\bar{f}_1(\rho, \theta)\| = O(\|\rho\|), \quad |\bar{f}_2(\rho,\theta)\| = O(\|\rho\|^2), \quad (\rho \to 0);$$

$A(\theta)$ is a continuous, bounded, $(n-1) \times (n-1)$ matrix defined in R.

§ 3. Periodic Solution

We consider equation (2.6). By discarding the terms of higher order, we have the equation for normal variation of the solution as follows

$$\frac{d\rho}{d\theta} = A(\theta)\rho. \tag{3.1}$$

Theoren 3. 1. *If $x = u(t)$ is a nontrivial bounded solution of (1.1) satisfying (2.1) and linear system (3.1) admits exponential dichotomy, then $u(t)$ is a periodic solution of system (1.1).*

Proof The proof is a consequence of the following three Lemmas.

Definition 2. *A bounded solution $u(t)$ of system (1.1) is isolated if there is a neighborhood of $u(t)$ such that the system has no other solution lying in this neighborhood.*

Lema 1. *If the linear system (3.1) admits exponential dichotomy, then any bounded solution $u(t)$ of system (1.1) is isolated.*

Proof Let B denote the set which is constructed by all continuous, bounded, $(n-1)$-dimensional vector functions defined in R. For $\rho\in B$, take

$$\|\rho\|=\sup_{\theta\in R}\|\rho(\theta)\|;$$

and for a given positive number $\delta_0<\delta$, let

$$B_{\delta_0}=\{\rho\in B \mid \|\rho\|\leqslant\delta_0\},$$

then B_{δ_0} is a bounded, closed set in Banach space B. By the condition of Lemma 1, we may suppose $X(\theta)$ is a fundamental solution matrix of linear system (3.1) such that

$$\begin{aligned}&\|X(\theta)PX^{-1}(s)\|\leqslant\beta\exp(-\alpha(\theta-s)), \quad \theta\geqslant s,\\ &\|X(\theta)(I-P)X^{-1}(s)\|\leqslant\beta\exp(\alpha(\theta-s)), \quad \theta\leqslant s,\end{aligned}\tag{3.2}$$

where α, β are positive constants, P is a projection such that $P^2=P$.

By equation (2.6), for arbitrary $(n-1)$-dimensional vector function $\rho\in B_{\delta_0}$, take the mapping

$$T:\ \rho\longrightarrow T\rho,$$

which is defined by

$$T\rho(\theta)=\int_{-\infty}^{\theta}X(\theta)PX^{-1}(s)\bar f_2(\rho(s),\ s)\,ds-\int_{\theta}^{+\infty}X(\theta)(I-P)X^{-1}(s)\bar f_2(\rho(s),\ s)\,ds.$$

Let δ_0' be sufficiently small such that

$$K(\delta_0'),\ 2\beta/\alpha=\bar\theta<1,$$

where the positive number $K(\delta_0')$ is the Lipschitz constant of $\bar f_2(\rho,\ \theta)$ with respect to ρ. We have

$$\begin{aligned}\|T\rho(\theta)\|&\leqslant\int_{-\infty}^{\theta}\beta\exp(-\alpha(\theta-s))\|\bar f_2(\rho(s),\ s)\|ds\\ &\quad+\int_{\theta}^{+\infty}\beta\exp(\alpha(\theta-s))\|\bar f_2(\rho(s),\ s)\|ds\\ &\leqslant\beta K(\delta_0')\Big[\int_{-\infty}^{\theta}\exp(-\alpha(\theta-s))\|\rho(s)\|ds+\int_{\theta}^{+\infty}\exp(\alpha(\theta-s))\|\rho(s)\|ds\Big]\\ &\leqslant\beta K(\delta_0')\cdot\delta_0\cdot\Big[\int_{-\infty}^{\theta}\exp(-\alpha(\theta-s))ds+\int_{\theta}^{+\infty}\exp(\alpha(\theta-s))ds\Big]\\ &=K(\delta_0')\cdot 2\beta/\alpha\cdot\delta_0<\delta_0.\end{aligned}$$

On the other hand, for $g\in B_{\delta_0}$, we have

$$\begin{aligned}\|T\rho(\theta)-T_g(\theta)\|&\leqslant\beta\int_{-\infty}^{\theta}\exp(-\alpha(\theta-s))\|\bar f_2(\rho(s),\ s)-\bar f_2(g(s),\ s)\|ds\\ &\quad+\beta\int_{\theta}^{+\infty}\exp(\alpha(\theta-s))\|\bar f_2(\rho(s),\ s)-\bar f_2(g(s),\ s)\|ds\\ &\leqslant\beta K(\delta_0')\Big[\int_{-\infty}^{\theta}\exp(-\alpha(\theta-s))\|\rho(s)-g(s)\|ds\\ &\quad+\int_{\theta}^{+\infty}\exp(\alpha(\theta-s))\|\rho(s)-g(s)\|ds\\ &\leqslant\beta K(\delta_0')\|\rho-g\|\cdot 2/\alpha\\ &\leqslant\bar\theta\|\rho-g\|.\end{aligned}$$

Therefore, T is a contraction mapping in B_{δ_0}. By the Fixed Point Theorem, we may suppose ρ_0 is its unique fixed point in B_{δ_0}. Thus, in the neighborhood $U_{\delta_0}(u(t))$ of $u(t)$, system (1.1) has the bounded solution

$$x=u(\theta)+s(\theta)\rho_0(\theta), \quad \theta\in R,$$

where

$$\|\rho_0\|=\sup_{\theta\in R}\|\rho_0(\theta)\|\leqslant\delta_0.$$

But, by $T0=0$, we have $\rho_0(\theta)=0$, $\theta\in R$. That is to say, system (1.1) has one and only one bounded solution $x=u(t)$ in the neighborhood $U_{\delta_0}(u(t))$ of $u(t)$. This completes the proof of Lemma 1.

Lemma 2. *If the bounded solution $x=u(t)$ of system (1.1) is isolated, then $x=u(t)$ is a recurrent solution*[6] *of system (1.1), and*

$$S[u(t)]=\overline{S[u(t)]},$$

where

$$S[u(t)]=\{x\in R^n \mid x=u(t),\ t\in R\}.$$

Proof. Let $\Omega[u(t)]$ denote the set of ω-limit points of $u(t)$. Since $x=u(t)$ is bounded on R, then $\Omega[u(t)]\neq\emptyset$. The set $\Omega[u(t)]$ is invariant, closed and compact, and hence contains a minimal set, which is closed and hence compact. Every trajectory of a compact minimal set is recurrent[6]. As shown above, $u(t)$ is isolated bounded solution, therefore $u(t)$ is recurrent. Then $\overline{S[u(t)]}$ is a compact minimal set and is constructed by recurrent trajectories. By the condition that $u(t)$ is isolated, we have

$$S[u(t)]=\overline{S[u(t)]}.$$

This completes the proof of Lemma 2.

Lemma 3. *If $x=u(t)$ is a nontrivial recurrent solution of system (1.1) and*

$$S[u(t)]=\overline{S[u(t)]},$$

then $x=u(t)$ is a periodic solution of system (1.1).

Proof Refer to Theorem 2.35 in [7].

Now we consider the stability of the periodic solution $x=u(t)$. Obviously, if system (3.1) admits only exponential dichotomy, then the trivial solution of equation (2.6) is usually of conditional stability. Hence $u(t)$ is usually of conditionally orbital stability (thus, the obtained periodic solution $u(t)$ may be unstable). If system (3.1) is of exponential asymptotic stability, then we can prove that the trivial solution of equation (2.6) is also of exponential asymptotic stability. Therefore, $u(t)$ is of asymptotically orbital stability. Thus, we have the following theorem.

Theorem 3. 2. *Suppose system (3.1) has exponential dichotomy and k is the rank of projection P in (3.2), we have*

(1) *if $k=n-1$ (that is, $P=I$), then $u(t)$ is of asymptotically orbital stability,*

(2) *if $0\leqslant k<n-1$, then $u(t)$ is of conditional orbital stability (that is, unstable).*

M. Urabe[4] points out that if $u(t)$ is periodic and the trivial solution of system (3.1) is expoentially asymptotically stable, then $u(t)$ is of asymptotically orbital stability with asymptotic phase. By Theorem 3.1 and Theorem 3.2, we have the following corollary.

Corollary. *If the trivial solution of linear system* (3.1) *is exponentially asymptotically stable, then* $u(t)$ *is a periodic solution of asymptotically orbital stability with asymptotic phase.*

§ 4. Perturbation of Autonomous System

Now we consider the perturbed system (1.2) of the given autonomous system (1.1). By the method of section 2, we may obtain the equivalent equations of system (1.2)

$$\begin{cases} \dfrac{d\theta}{dt}=1+Q(\rho,\ \theta,\ \varepsilon), \\ \dfrac{d\rho}{dt}=A(\theta)\rho+G(\rho,\ \theta,\ \varepsilon) \end{cases} \tag{4.1}$$

with respect to the moving orthonormal system, where

$$Q,\ G\in C^1,$$

$$\|Q\|=O(\|\rho\|+\varepsilon),\quad \|G\|=O(\|\rho\|^2+\varepsilon)\quad (\rho\to 0,\ \varepsilon\to 0).$$

If $x=u(t)$ is a nontrivial bounded solution of (1.1) and linear system (3.1) admits exponential dichotomy, then $u(t)$ is periodic by section 3. Therefore, in the vicinity of $u(t)$, system (1.2) has unique periodic solution $x=\tilde{u}(t,\ \varepsilon)$. Thus, we have the following theorem.

Theorem 4.1. *If* $x=u(t)$ *is a nontrivial bounded solution of system* (1.1) *satisfying* (2.1) *and linear system* (3.1) *admits exponential dichotomy, then system* (1.2) *has unique periodic solution* $\tilde{u}(t,\ \varepsilon)$ *in the vicinity of* $x=u(t)$ *and*

$$\lim_{\varepsilon\to 0}\tilde{u}(t,\ \varepsilon)=u(t).$$

Proof We consider the equivalent equation (4.1) of system (1.2). Since linear system (3.1) admits exponential dichotomy, then by Theorem 3.1 $u(t)$ is a periodic solution of (1.1). Thus, $A(\theta)$, $G(\rho,\ \theta,\ \varepsilon)$ in equation

$$\frac{d\rho}{d\theta}=A(\theta)\rho+G(\rho,\ \theta,\ \varepsilon) \tag{4.2}$$

and $s(\theta)$ are all periodic functions with respect to θ. Therefore, for sufficiently small $\varepsilon>0$, system (4.2) has a unique periodic solution $\tilde{\rho}(\theta,\ \varepsilon)$ and

$$\tilde{\rho}(\theta,\ \varepsilon)=\int_{-\infty}^{\theta}X(\theta)PX^{-1}(s)G(\rho(s),\ t,\ \varepsilon)ds$$

$$-\int_{\theta}^{+\infty}X(\theta)(I-P)X^{-1}(s)G(\rho(s),\ s,\ \varepsilon)ds.$$

Thus, we have

$$\|\tilde{\rho}\|=O(\varepsilon),\ \varepsilon\to 0;$$

and system (1.2) has the periodic solution

$$x=\tilde{u}(\theta,\ \varepsilon)=u(\theta)+s(\theta)\tilde{\rho}(\theta,\ \varepsilon).$$

By the first equation of (4.1), we have

$$\theta=\theta(t,\ \varepsilon)=t+O(\varepsilon),\ \varepsilon\to 0.$$

Hence

$$x=\tilde{u}(t,\ \varepsilon)$$

and

$$\lim_{\varepsilon\to 0}\tilde{u}(t,\ \varepsilon)=u(t).$$

This complete the proof of Theorem 4.1.

In order to discuss the stability of the periodic solution $\tilde{u}(t,\ \varepsilon)$, we may use the "ROUGHNESS" of exponential dichotomy[5] to consider the equivalent equation of system (1.2) with respect to the new transformation

$$X=\tilde{u}(\theta,\ \varepsilon)+\tilde{s}(\theta,\ \varepsilon)\rho,$$

where $\tilde{s}(\theta,\ \rho)$ is the new $n\times(n-1)$ matrix which is constructed by $n-1$ orthogonal vectors with respect to $\tilde{u}(\theta,\ \varepsilon)$ (the method of construction is similar to that of section 2). Here, we give the result without proof.

Theoren 4.2. *Suppose system* (3.1) *admits exponential dichotomy and* k *is the rank of projection* P *in* (3.2). *We have*

(1) *if* $k=n-1$, *then* $\tilde{u}(t,\ \varepsilon)$ *is of asymptotically orbital stability with asymptotic phase;*

(2) *if* $0\leqslant k<n-1$, *then* $\tilde{u}(t,\ \varepsilon)$ *is of conditional orbital stability.*

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