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ON THE DIOPHANTINE EQUATION $\sum_{i=0}^{k}\frac{1}{x_i}=\frac{a}{n}$

SHEN ZUN (单 墫)*

Abstract

In this paper, the author proves the following result:

Let $E_{a,k}(N)$ denote the number of natural numbers $n\leqslant N$ for which equation

$$\sum_{0}^{k}\frac{1}{x_i}=\frac{a}{n}$$

is insolable in positive integers $x_i(i=0,1,\cdots,k)$. Then

$$E_{a,k}(N)\ll N\exp\{-C(\log N)^{1-\frac{1}{k+1}}\}$$

where the implied constant depends on a and k.

§ 1. Introduction

The main result in this paper is the following

Theorem 1. *Let $E_{a,k}(N)$ denote the number of natural numbers $n\leqslant N$ for which equation*

$$\sum_{0}^{k}\frac{1}{x_i}=\frac{a}{n}\tag{1}$$

is insoluble in positive integers $x_i(i=0,1,\cdots,k)$. Then

$$E_{a,k}(N)\ll N\exp\{-C(\log N)^{1-\frac{1}{k+1}}\},\tag{2}$$

where the implied constant depends on a and k.

(2) is better than the result of Viola[1], where the index $1-\frac{1}{k+1}$ is replaced by $1-\frac{1}{k}$. When $k=2$, (2) is just the same early result of Vaughan[2]. So we may assume k is an integer >2.

§ 2. Lemma

The ordered $k+1$–tuple $(y_1, y_2, \cdots, y_{k+1})$ is called an admissible $k+1$–factorization of v whenever $\prod_{1}^{k+1} y_i=v$, y_i positive integer, and y_{k+1} square free.

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* Department of Mathematics, China University of Science and Technology, Hefei, Anhui, China.

Lemma 1. *Let integers* $v_1 \geqslant v \geqslant (k-1)^8$. *Let* $(y_1, y_2, \cdots, y_{k+1})$ *and* $(y'_1, y'_2, \cdots, y'_{k+1})$ *be admissible* $k+1$*-factorization of* v. *If* $y_i \leqslant v^{\frac{3}{4^i}}, y'_i \geqslant v^{\frac{3}{4^i}}$ $(i=1, 2, \cdots, k-1)$ $v^{\frac{1}{k\cdot 4^{k+2}}} \leqslant y_k$, $y' \leqslant v^{\frac{1}{k\cdot 4^{k+1}}}$ *and there exists a natural number n satisfying*

$$\left.\begin{array}{l} y_k n \equiv -(y_1+y_2+\cdots+y_{k-1}) \\ y'_k n \equiv -(y'_1+y'_2+\cdots+y'_{k-1}) \end{array}\right\} \pmod{v_1}, \tag{3}$$

then $$y_i = y'_i \quad (i=1, 2, \cdots, k+1).$$

Proof From (3) we have

$$y_k(y'_1+y'_2+\cdots+y'_{k-1}) \equiv y'_k(y_1+y_2+\cdots+y_{k-1}) \pmod{v_1}. \tag{4}$$

Since

$$y_k(y'_1+y'_2+\cdots+y'_{k-1}) \leqslant y_k \cdot y'_1(k-1) \leqslant (k-1)\cdot \frac{v}{y_1 y_2}\cdot\frac{v}{y'_2}$$
$$\leqslant (k-1) v^{\frac{1}{16}} v^{1-\frac{3}{16}} = (k-1) v^{\frac{7}{8}} \leqslant v \leqslant v_1,$$
$$y'_k(y_1+y_2+\cdots+y_{k-1}) \leqslant v_1;$$

(4) becomes

$$y_k(y'_1+y'_2+\cdots+y'_{k-1}) = y'_k(y_1+y_2+\cdots+y_{k-1}). \tag{5}$$

If $y_k y'_1 \neq y'_k y_1$, then

$$|y_k y'_1 - y'_k y_1| = \left| y_k \cdot \frac{v}{y'_2 y'_3 \cdots y'_{k+1}} - y'_k \cdot \frac{v}{y_2 y_3 \cdots y_{k+1}} \right|$$
$$\geqslant \frac{v}{y'_2 y'_3 \cdots y'_{k+1} y_2 y_3 \cdots y_{k+1}} = \frac{y_1 y'_1}{v} \geqslant v^{\frac{1}{2}}.$$

But on the otherhand, from (5)

$$|y_k y'_1 - y'_k y_1| = |y'_k(y_2+y_3+\cdots+y_{k-1}) - y_k(y'_2+y'_3+\cdots+y'_{k-1})|$$
$$< \max(y'_k(y_2+y_3+\cdots+y_{k-1}),\ y_k(y'_2+y'_3+\cdots+y'_{k-1}))$$
$$\leqslant \max(y_2 y_3 \cdots y_{k-1} y'_k,\ y'_2 y'_3 \cdots y'_{k-1} y_k)$$
$$\leqslant \max\left(\frac{v y'_k}{y_1},\ \frac{v y_k}{y'_1}\right) \leqslant v^{1-\frac{3}{4}+\frac{1}{4}} = v^{\frac{1}{2}}.$$

It is a contradiction.

Now we assume that $y_k y'_1 = y'_k y_1$, $y_k y'_2 = y'_k y_2$, $\cdots$, $y_k y'_{s-1} = y'_k y_{s-1}$ $(s < k-1)$. Let $y_k^{1-s} \cdot u = \dfrac{v}{y_1 y_2 \cdots y_{s-1}} = y_s y_{s+1} \cdots y_{k+1}$. Since

$$y_s (y_1 y_2 \cdots y_{s-1})^{\frac{3}{4}} \geqslant v^{\frac{3}{4}\sum_{1}^{s-1}\frac{3}{4^i}+\frac{3}{4^s}} = v^{\frac{3}{4}},$$

we have

$$y_s \geqslant \left(\frac{v}{y_1 y_2 \cdots y_{s-1}}\right)^{\frac{3}{4}} = (u y_k^{1-s})^{3/4}.$$

If $y_k y'_s \neq y'_k y_s$, then

$$|y_k y'_s - y'_k y_s| = \left| \frac{y_k u y'^{1-s}_k}{y'_{s+1} y'_{s+2} \cdots y'_{k+1}} - \frac{y'_k u y_k^{1-s}}{y_{s+1} y_{s+2} \cdots y_{k+1}} \right|$$
$$\geqslant \frac{u y'^{1-s}_k y_k^{1-s}}{y'_{s+1} y'_{s+2} \cdots y'_{k+1} y_{s+1} y_{s+2} \cdots y_{k+1}} = \frac{y_s y'_s}{u}$$

But on the otherhand, from (5)

$$\begin{aligned}|y_k y'_s - y'_k y_s| &= |y'_k(y_{s+1}+y_{s+2}+\cdots+y_{k-1}) - y_k(y'_{s+1}+y'_{s+2}+\cdots+y'_{k-1})|\\ &<\max(y'_k(y_{s+1}+y_{s+2}+\cdots+y_{k-1}),\ y_k(y'_{s+1}+y'_{s+2}+\cdots+y'_{k-1}))\\ &\leqslant\max(y_{s+1}y_{s+2}\cdots y_{k-1}y'_k,\ y'_{s+1}y'_{s+2}\cdots y'_{k-1}y_k)\\ &\leqslant\max\left(\frac{uy'_k}{y_s}\,y_k^{1-s},\ \frac{uy_k}{y'_s}\,y_k'^{\,1-s}\right)\leqslant\frac{y_s y'_s}{u}.\end{aligned}$$

It is a contradiction again. Hence we have

$$y_k y'_1 = y'_k y_1,\ y_k y'_2 = y'_k y_2,\ \cdots,\ y_k y'_{k-1} = y'_k y_{k-1}.$$

i. e.

$$\frac{y_1}{y'_1}=\frac{y_2}{y'_2}=\cdots=\frac{y_k}{y'_k}. \tag{6}$$

From (6) and

$$y_1 y_2\cdots y_k y_{k+1} = y'_1 y'_2\cdots y'_k y'_{k+1}\ (=v)$$

we have

$$y_k^k y_{k+1} = y_k'^{\,k} y'_{k+1}.$$

Since y_{k+1} and y'_{k+1} are square free, it follows that

$$y_k = y'_k,\ y_{k+1} = y'_{k+1}. \tag{7}$$

Thus by (6) and (7)

$$y_1 = y'_1,\ y_2 = y'_2,\ \cdots,\ y_{k+1} = y'_{k+1}.$$

The proof of Lemma 1 is complete.

Lemma 2 (Brun–Titchmarsh). *If $q\leqslant x^{\alpha}$, $0<\alpha<1$, $(q, l)=1$, then*

$$\pi(x;\ q,\ l)\ll\frac{x}{\varphi(q)\log x}.$$

Lemma 3 (Bombieri). *For any $A>0$, there is $B>0$ such that*

$$\sum_{q\leqslant x^{\frac{1}{2}}(\log x)^{-B}}\ \max_{y\leqslant x}\ \max_{(q,l)=1}\left|\pi(y;\ q,\ l)-\frac{\mathrm{li}\,y}{\varphi(q)}\right|\ll x(\log x)^{-A}.$$

Lemma 4[1]. *Let* $d_{k(n)}=\sum_{x_1x_2\cdots x_k=n}1$. *Then*

$$\sum_{n\leqslant x}\frac{d^l_{k(n)}}{\varphi(n)}\ll(\log x)^{k^l}.$$

Lemma 5 (Montgomery). *If $\omega(p)\ (0<\omega(p)<p)$ residue classes (mod p) are removed from the first N natural numbers for each prime $p\leqslant\sqrt{N}$, then the number Z of natural numbers which remain satisfies*

$$Z\leqslant\frac{4N}{\sum_{m\leqslant\sqrt{N}}\mu^2_{(m)}\prod_{p\mid m}\frac{\omega(p)}{p-\omega(p)}}.$$

§ 3. Theorem 2 and its Proof

Let

$$f_k(v,\ \xi)=\sum_{\substack{x_1x_2\cdots x_{k+1}=v\\ x_i\geqslant\xi^{3/4i}\\ (i=1,2,\cdots,k-1)\\ \xi^{\frac{1}{k\cdot 4^{k+2}}}\leqslant x_k\leqslant\xi^{\frac{1}{k\cdot 4^{k+1}}}}}|\mu_{(x_{k+1})}|$$

$$\omega_{a,k(p)}=\begin{cases} f_k\left(\frac{p+1}{a},\ \frac{p+1}{a}\right), & \text{if } p\equiv -1 \pmod a,\ p>a(k-1)^8. \\ 0, & \text{otherwise.}\end{cases}$$

In this section, we give the average order of the faction $\omega_{a,k(p)}$ i. e. the following

Theorem 2.

$$(\log\xi)^k \ll \sum_{p\leqslant\xi}\frac{\omega_{a,k(p)}}{p} \ll (\log\xi)^k.$$

Proof By partial summations, it suffices to prove that

$$\xi(\log\xi)^{k-1} \ll \sum_{p\leqslant\xi}\omega_{a,k(p)} \ll \xi(\log\xi)^{k-1}. \tag{8}$$

The upper bound. If $p\equiv -1 \pmod a$, $p\geqslant a(k-1)^8$, then

$$\omega_{a,k(p)} = \sum_{\substack{x_1x_2\cdots x_{k+1}=\frac{p+1}{a} \\ x_i\geqslant\left(\frac{p+1}{a}\right)^{3/4i} \\ (i=1,2,\cdots,k)}} |\mu_{(x_{k+1})}| \leqslant \sum_{\substack{x_1x_2\cdots x_{k+1}=\frac{p+1}{a} \\ x_1\geqslant\left(\frac{p+1}{a}\right)^{3/4}}} 1$$

$$= \sum_{\substack{r\mid\frac{p+1}{a} \\ r\leqslant\left(\frac{p+1}{a}\right)^{1/4}}} \sum_{x_2x_3\cdots x_{k+1}} 1 = \sum_{\substack{r\mid\frac{p+1}{a} \\ r\leqslant\left(\frac{p+1}{a}\right)^{1/4}}} d_{k(r)}.$$

Hence

$$\sum_{p\leqslant\xi}\omega_{a,k(p)} \leqslant \sum_{\substack{p\leqslant\xi \\ p\equiv-1 ((\text{mod} a)}} \sum_{\substack{r\mid\frac{p+1}{a} \\ r\leqslant\left(\frac{p+1}{a}\right)^{1/4}}} d_{k(r)} \leqslant \sum_{r\leqslant(\xi+1)^{1/4}} d_{k(s)}\pi_{(\xi;ar,-1)}.$$

Therefore, by Lemmas 2 and 4

$$\sum_{p\leqslant\xi}\omega_{a,k(p)} \ll \frac{\xi}{\log\xi}\sum_{r\leqslant(\xi+1)^{1/4}}\frac{d_{k(r)}}{\varphi_{(r)}} \ll \xi(\log)^{k-1}.$$

The lower bound.

$$\sum_{p\leqslant\xi}\omega_{a,k(p)} \geqslant \sum_{\substack{\frac{\xi}{2}<p\leqslant\xi \\ p\equiv-1 (\text{mod } a)}} f_k\left(\frac{p+1}{a},\ \xi\right) = \sum_{\substack{\frac{\xi}{2}<p\leqslant\xi \\ p\equiv-1 (\text{mod } a)}} \sum_{\substack{x_1x_2\cdots x_{k+1}=\frac{p+1}{a} \\ x_i\geqslant^{3/4i} \\ (i=1,2,\cdots,k)}} |\mu_{(x_{k+1})}|$$

$$= \sum_{\substack{\frac{\xi}{2}<p\leqslant\xi \\ p\equiv-\xi (\text{mod } a)}} \sum_{\substack{x_1\mid\frac{p+1}{a} \\ x_1\geqslant\xi^{3/4}}} \sum_{\substack{x_2x_3\cdots x_{k+1}=\frac{p+1}{a} \\ x_1x_i\geqslant\xi^{3/4i} \\ (i=2,3,\cdots,k)}} |\mu_{(x_{k+1})}|$$

$$\geqslant \sum_{\substack{\frac{\xi}{2}<p\leqslant\xi \\ p\equiv-1 (\text{mod } a)}} \sum_{\substack{r\mid\frac{p+1}{a} \\ r\leqslant\frac{\xi^{1/4}}{2a}}} \sum_{\substack{x_2x_3\cdots x_{k+1}=r \\ x_i\geqslant\xi^{3/4i} \\ (i=2,3,\cdots,k)}} |\mu_{(x_{k+1})}|$$

$$= \sum_{r\leqslant\frac{\xi^{1/4}}{2a}} \sum_{\substack{x_1x_2\cdots x_k=r \\ x_i\geqslant\xi^{3/4i+1} \\ (i=1,2,\cdots k-1)}} |\mu_{(x_k)}| \sum_{\substack{\frac{\xi}{2}<p\leqslant\xi \\ p\equiv-1 (\text{mod } ar)}} 1$$

$$= \sum_{r\leqslant\frac{\xi^{1/4}}{2a}} f_{k-1}(r,\ \xi^{1/4})\left\{\pi(\xi;ar,\ -1)-\pi\left(\frac{\xi}{2};\ ar,\ -1\right)\right\},$$

hence

$$\sum_{p\leqslant\xi}\omega_{a,k(p)} \geqslant \left(l_i\xi - l_i\frac{\xi}{2}\right)\sum_{r\leqslant\frac{\xi^{1/4}}{2a}}\frac{f_{k-1}(r,\ \xi^{1/4})}{\varphi(ar)} + R \tag{9}$$

where

$$R=\sum_{r\le\frac{\xi^{1/4}}{2a}} f_{k-1}(r,\xi^{1/4})\left\{\left(\pi(\xi;\,ar,\,-1)-\frac{l_i\xi}{\varphi(ar)}\right)-\left(\pi\left(\frac{\xi}{2};\,ar,\,-1\right)-\frac{l_i\frac{\xi}{2}}{\varphi(ar)}\right)\right\}.$$

The Cauchy-Schwarz inequality yields

$$|R|\le\left(\sum_{r\le\frac{\xi^{1/4}}{2a}} f_{k-1}^2(r,\xi^{1/4})/\varphi(ar)\right)^{1/2}\left(\sum_{r\le\frac{\xi^{1/4}}{2a}}\varphi(ar)\left\{\left(\pi(\xi;\,ar,\,-1)-\frac{l_i\xi}{\varphi(ar)}\right)\right.\right.$$

$$\left.\left.-\left(\pi\left(\frac{\xi}{2};\,ar,\,-1\right)-\frac{l_i\frac{\xi}{2}}{\varphi(ar)}\right)\right\}^2\right)^{1/2}$$

$$\le\left(\sum_{r\le\xi^{1/4}}\frac{d_k^2(r)}{\varphi(r)}\right)^{1/2}\left(\sum_{r\le\frac{\xi^{1/4}}{2a}}\varphi(ar)\left\{\left(\pi(\xi;\,ar,\,-1)-\frac{l_i\xi}{\varphi(ar)}\right)\right.\right.$$

$$\left.\left.-\left(\pi\left(\frac{\xi}{2};\,ar,\,-1\right)-\frac{l_i\frac{\xi}{2}}{\varphi(ar)}\right)\right\}^2\right)^{1/2}. \qquad (10)$$

By Lemma 4

$$\left(\sum_{r\le\xi}\frac{d_k^2(r)}{\varphi(r)}\right)^{1/2}\ll(\log\xi)^{k^2/2} \qquad (11)$$

and by Lemma 2 with $\alpha=\frac{1}{4}$

$$\varphi(s)\cdot\left|\pi(\xi;\,s,\,-1)-\frac{l_i\xi}{\varphi(s)}-\left(\pi\left(\frac{\xi}{2};\,s,\,-1\right)-\frac{l_i\frac{\xi}{2}}{\varphi(s)}\right)\right|\ll\frac{\xi}{\log\xi}. \qquad (12)$$

Lemma 3 gives, for any $A>0$,

$$\sum_{s\le\xi^{1/4}}\left|\pi(\xi;\,s,\,-1)-\frac{l_i\xi}{\varphi(s)}-\left(\pi\left(\frac{\xi}{2};\,s,\,-1\right)-\frac{l_i\frac{\xi}{2}}{\varphi(s)}\right)\right|\ll\xi(\log\xi)^{-A}. \qquad (13)$$

From (10), (11), (12) and (13) we obtain

$$|R|\ll\xi(\log\xi)^{-A_1}. \qquad (14)$$

For the main term in (9) we have

$$\sum_{r\le\frac{\xi^{1/4}}{2a}}\frac{f_{k-1}(r,\xi^{1/4})}{\varphi(ar)}=\sum_{r\le\frac{\xi^{1/4}}{2a}}\sum_{\substack{x_1x_2\cdots x_k=r\\ x_i\ge\xi^{3/4i+1}\\ (i=1,2,\cdots,k-1)}}\frac{|\mu(x_k)|}{\varphi(ar)}$$

$$=\sum_{\substack{x_1x_2\cdots x_k\le\frac{\xi^{1/4}}{2a}\\ x_i\ge\xi^{3/4i+1}\\ (i=1,2,\cdots,k-1)}}\frac{|\mu(x_k)|}{\varphi(ax_1x_2\cdots x_k)}\ge\sum_{\substack{x_1x_2\cdots x_k\le\frac{\xi^{1/4}}{2a}\\ x_i\ge\xi^{3/4i+1}\\ (i=1,2,\cdots,k-1)}}\frac{|\mu(x_k)|}{\varphi(x_k)ax_1x_2\cdots x_{k-1}}$$

$$=\frac{1}{a}\sum_{\substack{x_1x_2\cdots x_{k-1}\le\frac{\xi^{1/4}}{2a}\\ x_i\ge\xi^{3/4i+1}}}\frac{1}{x_1x_2\cdots x_{k-1}}\sum_{x_k\le\frac{\xi^{1/4}}{2ax_1x_2\cdots x_{k-1}}}\frac{|\mu(x_k)|}{\varphi(x_k)}.$$

Since

$$\frac{|\mu(x_k)|}{\varphi(x_k)}=|\mu(x_k)|\cdot\prod_{p|x_k}\frac{1}{p-1}=\frac{|\mu(x_k)|}{x_k}\prod_{p|x_k}\sum_{h=0}^{\infty}\frac{1}{p^h},$$

we have

$$\sum_{r\leqslant\frac{\xi^{1/4}}{2a}}\frac{f_{k-1}(r,\ \xi^{1/4})}{\varphi(ar)}\gg\frac{1}{a}\sum_{\substack{x_1x_2\cdots x_{k-1}\leqslant\frac{\xi^{1/4}}{2a}\\ x_i\geqslant\xi^{3/4^{i+1}}}}\frac{1}{x_1x_2\cdots x_{k-1}}\sum_{x_k\leqslant\frac{\xi^{1/4}}{2ax_1x_2\cdots x_{k-1}}}\frac{|\mu(x_k)|}{x_k}$$

$$\gg\sum_{x_1x_2\cdots x_{k-1}\leqslant\frac{\xi^{1/4}}{2a}}\frac{1}{x_1x_2\cdots x_{k-1}}\log\frac{\xi^{1/4}}{2ax_1x_2\cdots x_{k-1}}$$

$$\gg\log\xi\sum_{\substack{x_1x_2\cdots x_{k-1}\leqslant\frac{\xi^{1/4-\epsilon}}{2a}\\ x_i\geqslant\xi^{3/4^{i+1}}}}\frac{1}{x_1x_2\cdots x_{k-1}}$$

$\left(\text{where } \epsilon \text{ is a positive number} < \frac{1}{4^k}\right)$

$$\gg\log\xi\sum_{\xi^{3/4^{i+1}}\leqslant x_i\leqslant\xi^{3/4^{i+1}+\epsilon_i}}\frac{1}{x_1x_2\cdots x_{k-1}}$$

$\left(\text{where } \epsilon_i \text{ are positive satisfing } \sum_{i=1}^{k-1}\epsilon_i<\frac{1}{4^k}-\epsilon\right)$

$$\gg(\log\xi)^k. \tag{15}$$

Since $l_i\xi-l_i\frac{\xi}{2}\gg\frac{\xi}{\log\xi}$, we obtain

$$\left(l_i\xi-l_i\frac{\xi}{2}\right)\sum_{r\leqslant\frac{\xi^{1/4}}{2a}}\frac{f_{k-1}(r,\ \xi^{1/4})}{\varphi(ar)}\gg\xi(\log\xi)^{k-1}. \tag{16}$$

(9), (14) and (16) give

$$\sum_{p\leqslant\xi}\omega_{a,k(r)}\gg\xi(\log\xi)^{k-1},$$

which proves the theorem.

§ 4. The Proof of Theorem 1

Let $p\equiv-1 \pmod a$ and $(y_1, y_2, \cdots, y_{k+1})$ be admissible $k+1$-factorization of $\frac{p+1}{a}$. If a natural number n satisfies

$$y_kn\equiv-(y_1+y_2+\cdots+y_{k-1}) \pmod p,$$

then

$$y_kn+(y_1+y_2+\cdots+y_{k-1})=y_0p=y_0(ay_1y_2\cdots y_{k+1}-1),$$

$$y_0+y_1+y_2+\cdots+y_{k-1}+y_kn=ay_0y_1\cdots y_{k+1},$$

$$\frac{a}{n}=\frac{y_0+y_1+\cdots+y_{k-1}+y_kn}{ny_0y_1\cdots y_{k+1}}=\frac{1}{x_0}+\frac{1}{x_1}+\cdots+\frac{1}{x_k}$$

with

$x_0=ny_1y_2\cdots y_{k+1},\ x_1=ny_0y_2\cdots y_{k+1},\ \cdots,\ x_{k-1}=ny_0y_1\cdots y_{k-2}y_ky_{k+1},\ x_k=y_0y_1\cdots y_{k-1}y_{k+1},$

i. e. the equation

$$\frac{a}{n}=\frac{1}{x_0}+\frac{1}{x_1}+\cdots+\frac{1}{x_k} \tag{1}$$

is soluble. Therefore by Lemma 1, there are at least $\omega_{a,k(p)}$ residue classes (mod p) such that, for any n belonging to one of them, equation (1) is soluble. It follows from Lemma 5 that

$$E_{a,k}(N)\leqslant\frac{4N}{\sum\limits_{m\leqslant\sqrt{N}}\mu^2_{(m)}\prod\limits_{p|m}\frac{\omega_{a,k(p)}}{q-\omega_{a,k(p)}}}. \tag{17}$$

Let $\omega_{a,k(m)}$ be, for any integer $m>0$, the completely multiplication function generated by $\omega_{a,k(p)}$. Then

$$\sum_{m\leqslant\sqrt{N}}\mu^2_{(m)}\prod_{p|m}\frac{\omega_{a,k(p)}}{p-\omega_{a,k(p)}}\geqslant\sum_{m\leqslant\sqrt{N}}\mu^2_{(m)}\prod_{p|m}\frac{\omega_{a,k(p)}}{p}=\sum_{m\leqslant\sqrt{N}}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m}.$$

Hence, in view of (17), Theorem 1 is proved if we show that

$$\sum_{m\leqslant\sqrt{N}}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m}\gg\exp\{C(\log N)^{1-\frac{1}{k+1}}\}. \tag{18}$$

Since $\omega_{a,k(m)}\ll m^{\varepsilon}$ for any $\varepsilon>0$, the Dirichlet series

$$F_{a,k(s)}=\sum_{m=1}^{\infty}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m^s}$$

converges for Re $s>1$. Also

$$\sum_p\left(\frac{\omega_{a,k(p)}}{p}\right)^2<\infty,$$

hence for any $\varepsilon>0$,

$$F_{a,k(1+\varepsilon)}=\prod_p\left(1+\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}\right)=\exp\left\{\sum_p\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}+0_{(1)}\right\}=\exp\sum_p\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}. \tag{19}$$

It follows from Theorem 2 that

$$\sum_p\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}\geqslant\sum_{p\leqslant x}\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}\geqslant x^{-\varepsilon}\sum_{p\leqslant x}\frac{\omega_{a,k(p)}}{p}\gg x^{-\varepsilon}(\log x)^k;$$

taking $x=\exp\frac{1}{\varepsilon}$ we obtain

$$\sum_p\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}\gg\varepsilon^{-k}. \tag{20}$$

Next

$$\sum_p\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}=\sum_{n=0}^{\infty}\sum_{2^n\leqslant p<2^{n+1}}\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}\leqslant\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{2^n\leqslant p<2^{n+1}}\frac{\omega_{a,k(p)}}{p}$$

$$=\sum_{n=0}^{\infty}(2^{-\varepsilon n}-2^{-\varepsilon(n+1)})\sum_{p<2^{n+1}}\frac{\omega_{a,k(p)}}{p}=(1-2^{-\varepsilon})\sum_{n=0}^{\infty}2^{-\varepsilon n}\sum_{p<2^{n+1}}\frac{\omega_{a,k(p)}}{p};$$

hence by Theorem 2

$$\sum_p\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}\ll(1-2^{-\varepsilon})\sum_{n=0}^{\infty}(n+1)^k2^{-\varepsilon n}.$$

Sence

$$\sum_{n=0}^{\infty}(n+1)^k2^{-\varepsilon n}\ll\varepsilon^{-k-1}$$

we obtain

$$\sum_p\frac{\omega_{a,k(p)}}{p^{1+\varepsilon}}\ll\varepsilon^{-k}. \tag{21}$$

Moreover

$$\sum_{m\leqslant\sqrt{N}}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m}\geqslant\sum_{m\leqslant\sqrt{N}}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m^{1+\varepsilon}}=F_{a,k}(1+\varepsilon)-\sum_{m>\sqrt{N}}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m^{1+\varepsilon}}$$

$$\geqslant F_{a,k}(1+\varepsilon)-\sum_{m=1}^{\infty}\left(\frac{m}{\sqrt{N}}\right)^{\varepsilon/2}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m^{1+\varepsilon}}=F_{a,k}(1+\varepsilon)-N^{-\varepsilon/4}F_{a,k}\left(1+\frac{\varepsilon}{2}\right)$$

From (19), (20), (21) and (22) we deduce that for any sufficiently small ε,

$$\sum_{m\leqslant\sqrt{N}}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m}\geqslant C_1\exp(C_2\varepsilon^{-k})-C_3N^{-\frac{\varepsilon}{4}}\exp(C_4\varepsilon^{-k})$$

$$=C_1\exp(C_2\varepsilon^{-k})-C_3\exp\left(-\frac{\varepsilon}{4}\log N+C_4\varepsilon^{-k}\right).$$

Putting $\varepsilon=(4C_4)^{\frac{1}{k+1}}(\log N)^{-\frac{1}{k+1}}$ we obtain.

$$-\frac{\varepsilon}{4}\log N-C_4\varepsilon^{-k}=0.$$

Hence the above choice for ε gives, if N is large,

$$\sum_{m\leqslant\sqrt{N}}\frac{\mu^2_{(m)}\omega_{a,k(m)}}{m}\geqslant C_1\exp\{C_2(\log N)^{1-\frac{1}{k+1}}\}-C_3\gg\exp\{C_2(\log N)^{1-\frac{1}{k+1}}\},$$

and (18) is proved.

References

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