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QUALITATIVE ANALYSIS ON FITZHUGH'S NERVE CONDUCTION EQUATION*

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Abstract

The existence of limit cycles of Fitzhugh's nerve conduction equation (1) was studied in [1–4], but only a small range of parameter was considered. In this paper the authors establish a result (Theorem A) by using the qualitative method of O. D. E., which improves the results in [1–4] considerably.

By Fitzhugh's nerveconduction equation we mean the system

$$\begin{aligned} \frac{dx_1}{dt} &= \alpha + x_2 + x_1 - \frac{1}{3} x_1^3, \\ \frac{dx_2}{dt} &= \rho(a - x_1 - bx_2), \end{aligned} \tag{1}$$

where α, $a \in \mathbf{R}$, b, $\rho \in (0, 1)$. By the bifurcation theory some branching values for which system (1) creates limit cycle are investigated in [1–4]. For convenience of comparing their results with ours, we summarize those results in onr usual terminology as follows: Let v_1 and v_3 be the first and second focal values of the unique singular point respectively. Then (i) system (1) has a stable limit cycle if $0 < v_1 \ll 1$ and $v_3 \leqslant 0$; (ii) system (1) has an unstable limit cycle if $0 < -v_1 \ll 1$ and $v_3 > 0$. Here v_1 is an undetermined small parameter and the critical values of parameter corresponding to the existence and non-existence have not been found out completely. Moreover, although [3] pointed out that (1) might create two limit cycles, it did not indicate in detail when this would happen.

The main result of this paper is the following

Theorem A. (a) *If* $v_1 \leqslant 0$, *then system* (1) *has no limit cycle and the unique finite singular point is global asymptotic stable;* (b) *If either* $v_1 > 0$ *or* $v_1 = 0$, $v_3 > 0$, *then system* (1) *has exactly one stable limit cycle;* (c) *If* $0 < -v_1 \ll 1$ *and* $v_3 > 0$, *then system* (1) *has at least two limit cycles, one being stable and the other unstable. In case*

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$v_1<0$ with absolute value large enough, e. g. $v_1\leqslant -3(1-\rho b)$, system (1) has no limit cycle.

Obviously the results (i) and (ii) cited above are only part of (b) and (c). In Theorem A all the bifurcation values are discovered except for those corresponding to the existence of a semi-stable limit cycle. Furthermore we may suppose that there exist exactly two limit cycles in case (c), which remains to be confirmed,

§ 1. The Existence of Limit Cycle

It is easy to see that the coordinate x_1 of the finite singular point (x_1, x_2) of system (1) is determined by $S(x_1)\triangleq x_1^3+3\left(\frac{1}{b}-1\right)x_1-3\left(a+\frac{a}{b}\right)=0$. Since $S'(x_1)=3x_1^2+3\left(\frac{1}{b}-1\right)>0$, the equation $S(x_1)=0$ has a unique real root denoted by $x_{10}=x_1(a)$ and the coordinate x_2 of this singular point is $x_{20}=(a-x_{10})/b$. The characteristic equation at (x_{10}, x_{20}) is $\lambda^2-(1-\rho b-x_{10}^2)\lambda+\rho-\rho b+\rho b x_{10}^2=0$. Since $\rho-\rho b+\rho b x_{10}^2>\rho(1-b)>0$, (x_{10}, x_{20}) is an elementary singular point of index +1. Let $v_1=1-\rho b-x_{10}^2$, then the singular point is unstable if $v_1>0$ and stable if $v_1<0$. The case $v_1=0$ gives rise to the determination of center or focus with v_1, the first focal value.

Let $X=x_1-x_{10}$, $Y=x_2-x_{20}$. We can move the origin to the singular point and system (1) becomes

$$\begin{aligned}\frac{dX}{dt}&=(1-x_{10}^2)X+Y-x_{10}X^2-\frac{1}{3}X^3,\\ \frac{dY}{dt}&=-\rho X-\rho bY.\end{aligned}\tag{2}$$

We may assume $x_{10}\geqslant 0$, or otherwise by setting $X\to -X$, $Y\to -Y$ and $-x_{10}=x_1^*$ we will have $x_1^*>0$ without changing the form of (2).

[2] and [4] have calculated the focal values. But the calculation in [2] was wrong, which led to some mistakes in the result related to the case $v_3=0$. Our calculation yields the following lemma, which agrees with [4].

Lemma 1. *The first, second and third focal values of the singular point (0, 0) of (2) are*

$$v_1=1-b\rho-x_{10}^2,\ v_3=2b-1-b^2\rho,\ v_5=-\frac{10b\rho}{9\omega_0^2}<0,\tag{3}$$

respectively, where $\omega_0=\sqrt{\rho-b^2\rho^2}>0$.

The proof is left out.

Lemma 2. *System (2) has only two singular points at infinity. $B(1, 0, 0)$ is an unstable elementary node; $A(0, 1, 0)$ is a high order singular point with index −1, or more specifically, a 4-branch saddle point. In a neighbourhood of A, all the positive* *half-orbits go far away from A except that two positive half-orbits on the equator approach A. Therefore all the positive half-orbits of (2) are bounded.*

Proof Let $U=Y/X$, $Z=1/X$, $dt=Z^2dT$, system (2) is transfromed into

$$\begin{aligned}\frac{dZ}{dT}&=Z\left[\frac{1}{3}+x_{10}Z-(1-x_{10}^2)Z^2-UZ^2\right],\\ \frac{dU}{dT}&=\frac{1}{3}U+x_{10}UZ-\rho Z^2-\rho bUZ^2-(1-x_{10}^2)UZ^2-U^2Z^2.\end{aligned}\tag{4}$$

The singular point $B(Z=0,\ U=0)$ with its homogeneous coordinate (1, 0, 0) is an unstable elementary node.

Again let $V=X/Y$, $Z=1/Y$, $dt=Z^2dT$. System (2) is transfromed into

$$\begin{aligned}\frac{dZ}{dT}&=Z^3(\rho V+\rho b)\triangleq P,\\ \frac{dV}{dT}&=Z^2+(\rho b+1-x_{10}^2)VZ^2-x_{10}V^2Z-\frac{1}{3}V^3+\rho T^2Z^2\triangleq Q.\end{aligned}\tag{5}$$

The singular point $A(Z=0,\ V=0)$ with its homogeneous coordinate (0, 1, 0) is a high order singular point. It follows from Poincaré's sphere index theorem that the index of A is -1.

Constructing a Dulac function $B(Z,\ V)=(2+4\rho b)V-1$, we have $(BP)'_Z+(BQ)'_V=Z^2+(x_{10}Z+V)^2+$higher term. Therefore, there is no closed orbit or singular closed orbit in some sufficient small neighbourhood of A, and consequently, no elliptic region is linked to A. From Bendixson index theorem $j=1+(e-h)/2$, we can see that besides the parabolic regions there are exactly four hyperbolas in the neighbourhood of A. Moreover there are at most two characteristic directions at A: $Z=0$, $V>0$ and $Z=0$, $V<0$. From the first expression in (5) we can see that dZ/dT and Z have the same sign when $V>-b$. Therefore no positive half-orbits which pass through the points in the half-plane $V>-b$ (except for the line $Z=0$) will enter into A, and thus there is just one positive half-orbit, i. e. the equator, entering into A along each characteristic direction. In addition, there are at least two negative half-orbits entering into A, which constitute the boundary of the hyperbolic region. Also B is an unstable node, so all the positive half-orbits are bounded. The lemma is thus proved.

Theorem 1. 1) *If $v_1>0$ or 2) if $v_1=0$, $v_3>0$, then system (2) has at least one stable limit cycle;* 3) *if $v_3>0$, $0<-v_1\ll 1$, then system (2) has at least two limit cycles, one being unstable while the other stable.*

Proof Both in case 1) and case 2), the unique singular point O is unstable. By Lemma 2 all the positive half-orbits are bounded. It follows from the generalized Poincaré–Bendixson annular region theorem that system (2) has at least one stable limit cycle. When the parameters run from $v_3>0$, $v_1=0$ to $v_3>0$, $0<-v_1\ll 1$, the stability of the singular point O changes from unstable to stable. Hence an unstable limit cycle occurs around the origin O. But the original stable limit cycle remains. Therefore the system has at least two limit cycles.

Remark 1. In case $x_{10}=0$ that system (2) has at least one limit cycle for $v_1+1-x_{10}^2-\rho b=1-\rho b>0$ is always satisfied. Therefore we only need to consider the case $x_{10}>0$ in the subsequent discussion on the non-existence of limit cycle.

§ 2. The Non-Existence of Limit Cycle

Theorem 2. *If $v_1=0$ and $v_3\leqslant 0$, then system (2) has no limit cycle.*

Proof By the transformation $x=X$, $y=\rho bX+Y$, system (2) is changed into a Liénard equation

$$\frac{dx}{dt}=y-F(x), \quad \frac{dy}{dt}=-g(x), \tag{6}$$

where

$$F(x)=-v_1x+x_{10}x^2+x^3/3, \tag{7}$$

$$g(x)=\rho\{[1-b(1-x_{10}^2)]x+bx_{10}x^2+bx^3/3\}. \tag{8}$$

We shall make use of the Lemma 2 quoted in [5]. For this purpose we are to see if the two curves

$$F(x)=F(y), \tag{9}$$

$$G(x)=G(y) \tag{10}$$

intersect in the region $D\triangleq\{(x, y); x<0, y>0\}$, where $G(x)$ is given by

$$G(x)\triangleq\int_0^x g(x)\,dx=\rho\left\{\frac{1}{2}[1-b(1-x_{10}^2)]x^2+\frac{1}{3}bx_{10}x^3+\frac{1}{12}bx^4\right\}.$$

By the transformation $\xi=-x-y$, $\eta=-x+y$, (9), (10) and D can be changed into

$$\eta^2=h(\xi), \tag{11}$$

$$b(2x_{10}-\xi)\eta^2=\psi(\xi) \tag{12}$$

and

$$D_1\triangleq\{(\xi, \eta); \xi+\eta>0, \xi-\eta<0\}, \tag{13}$$

respectively, where

$$h(\xi)=12v_1+12x_{10}\xi-3\xi^2, \tag{14}$$

$$\psi(\xi)=12[1-b(1-x_{10}^2)]\xi-6bx_{10}\xi^2+b\xi^3. \tag{15}$$

Noting that $v_1=0$ and $x_{10}>0$, we can see that (11) represents an ellipse which is symmetric with respect to ξ-axis and is located on the right side of η-axis. The part of the ellipse inside D_1 has its projection $0<\xi<3x_{10}$ on ξ-axis. On the other hand, by eliminating η from (11) and (12) we obtain $b\xi^2-6bx_{10}\xi+6v_3=0$. When $v_3\leqslant 0$, neither of its roots lies in the interval $(0, 3x)$. This implies that (11) and (12) do not intersect in D_1, or equivalently, (9) and (10) do not intersect in D. It follows from Lemma 2 in [5] that system (2) has no limit cycle.

Let us now turn to the case $v_1<0$.

Theorem 3. *If $3x_{10}^2+4v_1\leqslant 0$, i. e. $v_1\leqslant -3(1-\rho b)$, then system (2) has no limit cycle.*

Proof First we note that the expressions (6) through (15) in Theorem 2 remain valid for $v_1<0$. The ellipse (11) intersects the boundary of D_1, i. e. $\xi=\eta$, at two points. Their ξ-coordinates ξ_1 and ξ_2 are determined by the equation

$$\xi^2-3x_{10}\xi-3v_1=0. \tag{16}$$

If $3x_{10}^2+4v_1\leqslant 0$, then (16) has no real root or has a unique multiple root. Therefore (11) has no loci inside D_1, and hence (11) and (12) do not intersect in D_1. This implies that (2) has no limit cycle, which completes the proof.

If $3x_{10}^2+4v_1>0$, the situation is a little complicated and we have

Theorem 4. *Suppose $v_1<0$ and $3x_{10}^2+4v_1>0$. Denote*

$$\varphi(\xi)=b\xi^3-6bx_{10}\xi^2+6(2bx_{10}^2+\rho b^2-1)\xi+12bx_{10}v_1. \tag{17}$$

Let ξ_1^* be the smaller root of equation $\varphi'(\xi)=0$:

$$\xi_1^*=2x_{10}-\sqrt{2(1-\rho b^2)/b}, \tag{18}$$

and ξ_1 be the smaller root of equation (16):

$$\xi_1=\frac{1}{2}[3x_{10}-\sqrt{9x_{10}^2+12v_1}]. \tag{19}$$

Then system (2) has no limit cycle if any one of the following conditions (a) $\xi_1^*\leqslant\xi_1$, (b) $\xi_1^*>\xi_1$, $\varphi(\xi_1^*)\leqslant 0$, is satisfied.

Proof We are to prove that (9) and (10) do not intersect in D. To this end, we consider first the section of (10) inside D, or equivalently, the section of (12) inside D_1. From (15) we have $\psi(0)=0$, $\psi'(\xi)=3[b\xi^2-4bx_{10}\xi+4(1-b(1-x_{10}^2))]$, and it is easy to check that $\psi'(\xi)>0$. Therefore $\psi(\xi)$ is an increasing function with the same sign as ξ, and hence only for $0<\xi<2x_{10}$, can the section of (12) be inside D_1.

On the other side, it is already known that the ξ-coordinate of the section of (11) inside D_1 must be in the interval (ξ_1, ξ_2), where ξ_1, and ξ_2 are the roots of (16). Therefore it is sufficient to show that (11) and (12) do not intersect when $\xi_1<\xi<2x_{10}$. Combining (11) and (12) to eliminate η, we obtain a cubic equation $\varphi(\xi)=0$, where $\varphi(\xi)$ is given by (17). $\varphi(\xi)$ may be rewritten into the form

$$\varphi(\xi)=b(\xi^2-3x_{10}\xi-3v_1)(\xi-3x_{10})+3(b+\rho b^2-2)\xi+3bx_{10}v_1, \tag{20}$$

or

$$\varphi(\xi)=\frac{1}{3}\varphi'(\xi)(\xi-2x_{10})+4(\rho b^2-1)\xi+4x_{10}(3b-1-2\rho b^2-bx_{10}^2). \tag{21}$$

From (20) and the hypotheses $0<b<1$, $0<\rho<1$, $\xi_1>0$, $x_{10}>0$, $v_1<0$, we have

$$\varphi(\xi_1)=3(b+\rho b^2-2)\xi_1+3bx_{10}v_1<0.$$

But from (21) we can see that

$$\varphi(2x_{10})=4x_{10}(3b-3-bx_{10}^2)<0,$$
$$\varphi(+\infty)=+\infty.$$

Thus, in the interval $(\xi_1, 2x_{10})$ only one of the following three cases for $\varphi(\xi)=0$ may happen: (i) on real root; (ii) a unique root of multiplicity two; (iii) two distinct real roots. Cases (i) and (ii) are equivalent to that (9) and (10) have no common point, or have just one tac-point but do not cross. In both cases system (2) has no limit cycle. But under the conditions of this theorem, only case (i) or case (ii) will happen. This completes the proof.

Corollary. *If $v_1<0$, $v_3\leqslant 0$, then* (2) *has no limit cycle.*

Proof By Theorem 3, we need only consider the case $3x_{10}^2+4v_1>0$. With simple manipulation it can be shown that $\xi_1^*\leqslant 2x_{10}-2\sqrt{1-\rho b}<\xi_1$, and thus the couclusion follows.

§ 3. The Uniqueness of the Limit Cycle

In this section we always assume $v_1>0$ or $v_1=0$ and $v_3>0$. Under these conditions we shall show that the Liénard equation (6) has at most one limlt cycle. Then by combining Theorem 1, the uniqueness of limit cycle is obtained.

The equation $F(x)=0$ (see (7)) has three roots: $\Delta_1=\frac{3}{2}\left[-x_{10}+\sqrt{x_{10}^2+\frac{4}{3}v_1}\right]\geqslant 0$, $\Delta_2=\frac{3}{2}\left[-x_{10}-\sqrt{x_{10}^2+\frac{4}{3}v_1}\right]<0$ and 0. The roots of $F'(x)=0$ are $\delta_1=-x_{10}+\sqrt{1-\rho b}$ and $\delta_2=-x_{10}-\sqrt{1-\rho b}$. It is easy to see $\delta_2<0\leqslant\delta_1$, and $\delta_1=0$ only when $v_1=0$.

Lemma 3. *System* (6) *has a unique stable limit cycle if $G(\Delta_2)\leqslant G(\Delta_1)$.*

Proof By Theorem 2 in [7], we need only to verify the following four couditions.

Since $xg(x)=\rho bx^2\left(\frac{x^2}{3}+x_{10}x+x_{10}^2+\frac{1}{b}-1\right)$, it follows immediately that

(1) $xg(x)>0$ $(x\neq 0)$, and $G(\pm\infty)=+\infty$.

The condition $G(\Delta_2)\leqslant G(\Delta_1)$ implies $v_1>0$. Hence the two roots of $F'(x)=0$ satisfy $\delta_2<0<\delta_1$, and we have

(2) $F'(x)<0$ when $\delta_2<x<\delta_1$; $F'(x)>0$ when $x<\delta_2$ or $x>\delta_1$.

Obviously we can see

(3) $F(+\infty)>F(-\infty)$.

Finally $\Delta_2<\delta_2<0$ means $G(\delta_2)<G(\Delta_2)$. Also we assert that $G(\delta_2)\geqslant G(\delta_1)$, for

$$G(\delta_2)-G(\delta_1)=\int_0^{-x_{10}-\sqrt{1-\rho b}}g(x)dx-\int_0^{-x_{10}+\sqrt{1-\rho b}}g(x)dx$$
$$=\frac{2}{3}\rho bx_{10}\sqrt{1-\rho b}\left(\frac{3}{b}-3+x_{10}^2\right)\geqslant 0.$$

Hence we obtain

(4) $\max[G(\delta_2), G(\delta_1)] \leqslant \min[G(\Delta_2), G(\Delta_1)]$.

The lemma is then a consequence of Theorem 2 in [7].

Remark 2. $G(x)$ is an even function. If $x_{10}=0$, then $\Delta_2=-\Delta_1$. In this case $G(\Delta_1)=G(\Delta_2)$, and by Lemma 3 we can see that system (6) has a unique stable limit cycle. In what follows we are interested only in the case $x_{10}>0$.

Lemma 4. *Suppose $G(\Delta_2)>G(\Delta_1)$. Then* i) *the limit cycle of system (6) must encompass the point $(\Delta_1, 0)$;* ii) *System (6) has at most one limit cycle which contains the point $(\Delta_2, 0)$ inside or on it. If such a limit cycle does exist, it is stable.*

Proof. i) Let $x=x_i(z)$ represent the inverse function of $z=G(x)$ $((-1)^{i+1}x\geqslant 0)$. By Филиппов transformation $x=x_i(z)$ $(i=1, 2)$, system (6) is changed into

$$\frac{dz}{dy}=F_i(z)-y, \ z\geqslant 0, \ i=1, 2, \tag{E_i}$$

where $F_i(z)=F(x_i(z))$. In this way, each closed orbit L of system (6) can be divided into two arcs, L_1 and L_2, located on $x\geqslant 0$ and $x\leqslant 0$ respectively. L_i is an integral curve of equation (E_i) $(i=1, 2)$. Clearly, $G(\Delta_i)$ is a positive root of $F_i(Z)=0$. Hence $F_1(z)\leqslant 0<F_2(z)$ for $0<z<G(\Delta_1)$, that is to say, $F_1(z)$ and $F_2(z)$ do not intersect. Therefore if system (6) has a limit cycle, the limit cycle will never lie entirely in the strip region $-\infty<x\leqslant\Delta_1$, and thus it must contain the point $(\Delta_1, 0)$ within.

For ii) we need only to show that the limit cycle L must be stable if the point $(\Delta_2, 0)$ lies inside or on it. To do this, it is sufficient to show $\oint_L F'(x)dt>0$, which is equivalent to

$$\int_{L_1+L_2} F'(z)\,dy=\int_{L_1} F'(z)\,dy-\int_{L_2} F'(z)\,dy>0. \tag{23}$$

Let z_{N_i} be the rightest point on L_i $(i=1, 2)$. Clearly, $z_{N_i}\geqslant G(\Delta_i)$ (see Figure 1). If $z>G(\Delta_1)$, then $F_1(z)>0$ and $F_1'(z)>0$; If $0<z<G(\Delta_2)$, then $F_2(z)>0$; If $z>G(\Delta_2)$, then $F_2(z)<0$ and $F_2'(z)<0$. Therefore $F_1(z)\leqslant F_1(z_{N_1})$ when $0\leqslant z\leqslant z_{N_1}$ and $F_2(z)\geqslant F_2(z_{N_2})$ when $0\leqslant z\leqslant z_{N_2}$. By Lemma 1 in [8] we see easily that (23) holds, and the proof is completed.

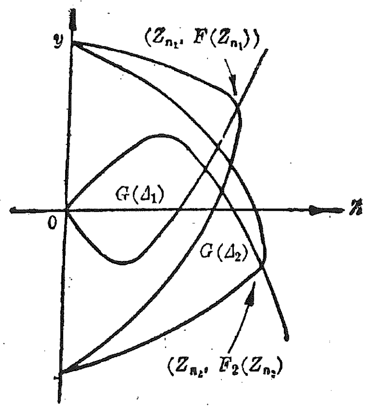

Fig. 1

Lemma 5. *Suppose $G(\Delta_2)>G(\Delta_1)$. Then system (6) has at most one limit cycle in the strip region $\Delta_2<x<+\infty$. The limit cycle must be stable if it exists.*

Proof We shall vesify the following four conditions alone, and this lemma can be derived immediately by applying Theorem 2 in [8].

1) $xg(x)>0$, for $x\neq 0$.

2) There exists an $a\geqslant 0$ such that $F_1(z)\leqslant 0\leqslant F_2(z)$ for $0\leqslant z\leqslant a$, but $F_1(z)\not\equiv$

$F_2(z)$ for $0<z\ll 1$; $F_1(z)>0$ for $z>a$; $F_2'(z)<0$ for $F_2(z)<0$.

3') $F(y)f(y)/g(y)$ is nondecreasing in $\Delta_1<y<+\infty$, where $f(y)=F'(y)$.

4') For every $c>0$, the curves $F(x)=F(y)$ and $G(x)=G(y)+c$ have at most on s-intersection point in the region $D'\triangleq\{(x, y); \Delta_2<x<0, \Delta_1<y\}$.

Condition 1) has already been checked in the proof of Lemma 3. For 2) it is enough to take $a=G(\Delta_1)$ (if $v_1=0$, take $a=0$). To verify 3), let us observe that

$$\frac{d}{dy}\left[\frac{F(y)f(y)}{g(y)}\right]=\frac{\rho(1-\rho b^2)yf^2(y)+F(y)\Phi(y)}{g^2(y)},$$

where

$$\begin{aligned}\Phi(y)&=\rho bf^2(y)+f'(y)g(y)-f(y)g'(y)\\&=\rho b\Big[\frac{2}{3}y^4+\frac{8}{3}x_{10}y^3+\Big(3x_{10}^2+\frac{1}{b}-1-v_1\Big)y^2\\&\quad-2x_{10}v_1y+v_1^2+v_1\Big(x_{10}^2+\frac{1}{b}-1\Big)\Big].\end{aligned}\tag{24}$$

Therefore it is sufficient to show $\Phi(0)\geqslant 0$ when $y>\Delta_1$. Through simple manipulation we have

$$\Phi'(y)=\rho b\Big[\frac{8}{3}y^3+8x_{10}y^2+2\Big(3x_{10}^2+\frac{1}{b}-1-v_1\Big)y-2x_{10}v_1\Big].$$

If $v_1=0$, then $\Delta_1=0$ and $\Phi(0)=0$. Also

$$\Phi'(y)=\rho by\Big[\frac{8}{3}y^2+8x_{10}y+2\Big(3x_{20}^2+\frac{1}{b}-1\Big)\Big].$$

The criterion of the quadratic expression in bracket $\frac{64}{3}\Big(1-\frac{1}{b}\Big)<0$. Hence $\Phi'(y)>0$ when $y>0$, which implies $\Phi(y)>0$ when $y>0$. If $v_1>0$, then it is easy to see that $\Phi'(y)=0$ has either one positive root and two negative roots or simply one positive root. Let y_3 denote the positive root of $\Phi'(y)=0$. we have $\Phi'(y)<0$ for $y\in(0, y_3)$ and $\Phi'(y)>0$ for $y\in(y_3, +\infty)$. In addition we may check

$$\begin{aligned}\Phi'(\Delta_1)&=\Big(6x_{10}^2+\frac{2}{b}-2+6v_1\Big)\Delta_1-2x_{10}v_1\\&=\Delta_1\Big[\Big(\frac{3}{4}x_{10}-\sqrt{x_{10}^2+\frac{4}{3}v_1}\Big)^2+\frac{47}{16}x_{10}^2+\frac{14}{3}v_1+\frac{2}{b}-2\Big]>0,\end{aligned}$$

from which we assert $y_3<\Delta_1$. Thus $\Phi'(y)>0$ when $y\geqslant\Delta_1$. Since

$$\Phi(\Delta_1)=\Big(x_{10}^2+\frac{1}{b}-1+v_1\Big)\Delta_1^2+v_1^2+v_1\Big(x_{10}^2+\frac{1}{b}-1\Big)>0,$$

we have $\Phi(y)>0$ when $y>\Delta_1$. Combining the above discussion we know that condition 3) holds.

We now proceed to condition 4'). By using the transformation $\xi=-x-y$, $\eta=-x+y$ as in section 2, the region D', the curves $F(x)=F(y)$ and $G(x)=G(y)+c$ are changed respectively into

$$\begin{aligned}&D_1'=\{(\xi, \eta); 0<\xi+\eta<-2\Delta_2, 2\Delta_1<\eta-\xi\},\\&\eta^2=h(\xi),\end{aligned}\tag{25}$$

and

$$b(2x_{10}-\xi)\eta^3=\psi(\xi)\eta-c, \tag{26}$$

for $h(\xi)$ and $\psi(\xi)$ see (14) and (15). Since $\psi(0)=0$,

$$\psi'(\xi)=3[b\xi^2-4bx_{10}\xi+4(1-b(1-x_{10}^2))]>0$$

and $\eta>\Delta_1\geqslant 0$, we can see that (26) makes sense only for $\xi>0$. Eliminating η^2 from (25) and (26) we have

$$2\eta\varphi(\xi)+c=0, \tag{27}$$

where $\varphi(\xi)$ is given by (17). Note that $\varphi(-\infty)=-\infty$, $\varphi(+\infty)=+\infty$, $\varphi(0)=12x_{10}v_1>0$, $\varphi(3x_{10})=3x_{10}(4b+2\rho b^2-6-bx_{10}^2)<0$, $\varphi(2x_{10})=4x_{10}(3b-3-bx_{10}^2)<0$. This means that $\varphi(\xi)=0$ has two and only two positive roots $\xi_2>\xi_1>0$, and $\varphi(\xi)<0$ when $\xi\in(\xi_1, \xi_2)$ (see Figure 2). On the other hand, $\xi=-x-y<-(\Delta_1+\Delta_2)=3x_{10}$ in D_1' and $3x_{10}<\xi_2$. Therefore we need only verify that (25) and (27) have only one common point in the region $D_1''=\{(\xi, \eta);\ \xi_1<\xi<3x_{10},\ \eta>\Delta_1\}$. In D_1'' we rewrite (25) and (27) as

$$\eta=\eta_1(\xi)=\sqrt{h(\xi)}, \tag{25'}$$

and

$$\eta=\eta_2(\xi)=-\frac{c}{2\varphi(\xi)}. \tag{27'}$$

Clearly, $\eta=\eta_1(\xi)$ is the upper-half part of an ellipse with $(2x_{10},\ 0)$ as its center, $2\sqrt{3}\sqrt{x_{10}^2+v_1}$ and $2\sqrt{x_{10}^2+v_1}$ as its half long-axis and half short-axis. Thus $\eta_1'(\xi)>0$ when $\xi_1<\xi<2x_{10}$; $\eta_1'(\xi)<0$ when $2x_{10}<\xi<3x_{10}$; $\eta_1''(\xi)<0$ when $\xi<\xi<3x_{10}$.

Further $\eta_2'(\xi)=\dfrac{c\varphi'(\xi)}{2\varphi^2(\xi)}$, and the equation $\varphi'(\xi)=0$ has two roots

$$\xi_1^*=2x_{10}-\sqrt{2(1-\rho b^2)/b} \quad\text{and}\quad \xi_2^*=2x_{10}+\sqrt{2(1-\rho b^2)/b}.$$

When $\xi_1^*<\xi<\xi_2^*$, we have $\varphi'(\xi)<0$. It is easy to see $\xi_1^*<\xi_1$, $3x_{10}<\xi_2^*$, and so $\eta_2'(\xi)<0$ when $\xi_1<\xi<3x_{10}$. Also, $\eta_2''(\xi)=\dfrac{c[\varphi(\xi)\varphi''(\xi)-2\varphi'^2(\xi)]}{2\varphi^3(\xi)}$, $\varphi''(\xi)=6b\xi-12bx_{10}$. Solving the equation $\varphi''(\xi)=0$ we obtain $\xi=2x_{10}$. Thus $\varphi''(\xi)>0$ when $2x_{10}<\xi<3x_{10}$, and consequently $\eta_2''(\xi)>0$ when $2x_{10}\leqslant\xi<3x_{10}$.

We will now discuss separately in three situations.

i) Let $c_0=-2\varphi(3x_{10})\sqrt{h(3x_{10})}$. If $c=c_0$, then the curves $\eta=\eta_1(\xi)$ and $\eta=\eta_2(\xi)$ intersect at $\xi=3x_{10}$. In addition, from the fact that

$$[\eta_1'(\xi)-\eta_2'(\xi)]_{\xi=3x_{10}}=\frac{144v_1(\rho b^2-1)}{2\sqrt{h(3x_{10})}\,\varphi(3x_{10})}>0,$$

and $\eta_1''(\xi)<0$, $\eta_2''(\xi)>0$ when $2x_{10}\leqslant\xi<3x_{10}$, we know $\eta_1'(\xi)>\eta_1'(3x_{10})>\eta_2'(3x_{10})>\eta_2'(\xi)$ and then $\eta_1(\xi)<\eta_2(\xi)$ for $2x_{10}\leqslant\xi<3x_{10}$. Moreover we can see $\eta_1'(\xi)>0>\eta_2'(\xi)$ for $\xi\in(\xi_1,\ 2x_{10})$. Thus in this interval we have $\eta_2(\xi)>\eta_1(\xi)$, which means that the curves $\eta=\eta_1(\xi)$ and $\eta=\eta_2(\xi)$ will not intersect again when $\xi_1<\xi<3x_{10}$ (see Figure 3).

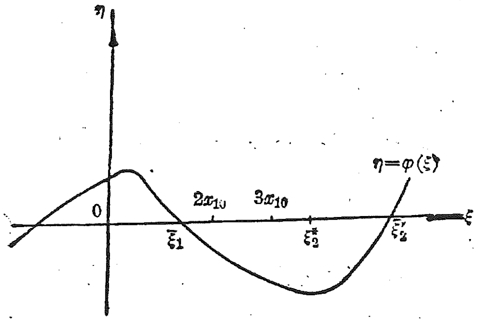

Fig. 2

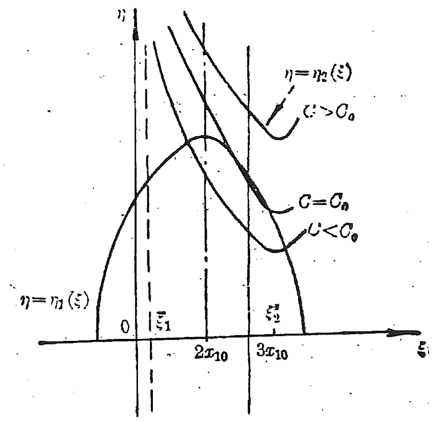

Fig. 3

ii) For $c>c_0$, it is obvious that $\eta=\eta_1(\xi)$ and $\eta=\eta_2(\xi)$ do not intersect.

iii) As for $c<c_0$, in a way similar to i), it can be shown that $\eta=\eta_1(\xi)$ and $\eta=\eta_2(\xi)$ intersect only once in either $2x_{10}\leqslant\xi<3x_{10}$ or $\bar{\xi}_1<\xi<2x_{10}$.

To sum up, we have proved that the curves $\eta=\eta_1(\xi)$ and $\eta=\eta_2(\xi)$ have at most one common point in D_1'', or equivalently, $F(x)=F(y)$ and $G(x)=G(y)+c$ have at most one common point in D' for every $c>0$. The proof is then completed.

Combining Lemmas 3—5 and Theorem 1, we have

Theorem 5. *If either $v_1>0$ or $v_1=0$ and $v_3>0$, then system* (2) *has a unique limit cycle and the limit cycle is stable.*

Proof Note that system (2) is equivalent to system (6). If $G(\Delta_2)\leqslant G(\Delta_1)$, then Theorem 5 follows immediately from Lemma 3. If $G(\Delta_2)\geqslant G(\Delta_1)$, then by Theorem 1 we know that ther exists at least one stable limit cycle. But Lemma 4 and Lemma 5 indicate that there is at most one such limit cycle that the point $(\Delta_2, 0)$ lies inside or on it, and there is at most one such limit cycle that the point $(\Delta_2, 0)$ lies outside it. If they do exist, they must be stable. Therefore these two limit cycles will never exist simultaneously, that is to say, system (2) has a unique stable limit cycle, and the proof is thus completed.

§ 4. Conclusion Remark

So far we have proved Theorems 1—5. Combining these results, we obtain Theorem A.

We know that the coordinate $x_{10}=x_1(\alpha)$ of the singular point satisfies

$$x_{10}^3+3\left(\frac{1}{b}-1\right)x_{10}-3\left(\alpha+\frac{a}{b}\right)=0, \quad \text{so} \quad \alpha=\frac{1}{3}x_{10}^3+\left(\frac{1}{b}-1\right)x_{10}-\frac{a}{b}\triangleq\tilde{\alpha}(x_{10})$$

and $\dfrac{d\alpha}{dx_{10}}=x_{10}^2+\left(\dfrac{1}{b}-1\right)>0$. This means that $\alpha=\tilde{\alpha}(x_{10})$ is a monotone increasing function with respect to x_{10} and $\alpha\to\pm\infty$ as $x_{10}\to\pm\infty$. Therefore there is a one-to-one correspondence between $\Omega=\mathbf{R}\times\mathbf{R}\times(0,1)\times(0,1)$ in parameter space (a, α, b, ρ) and some region Ω' in parameter space (a, x_{10}, v_1, v_3). The conditions in Theorem A are stated in terms of parameters (a, x_{10}, v_1, v_3). Evidently, it can be stated in terms of parameters (a, α, b, ρ) as well. For example the condition $v_1>0$ or $-\sqrt{1-b\rho}<x_{10}<\sqrt{1-b\rho}$ can be changed into an equivalent form $\tilde{\alpha}(-\sqrt{1-b\rho})<\alpha<\tilde{\alpha}(\sqrt{1-b\rho})$ and so forth.

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