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OSCILLATORY CRITERIA FOR n-ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL INEQUALITIES AND EQUATIONS WITH CONTINUOUS DISTRIBUTED DEVIATING ARGUMENTS

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Abstract

This paper deals with more general n-order nonlinear functional differential inequalities with continuous distributed deviating arguments and equations of this type. The author obtains some oscillatory criteria and generalizes and modifies some results given by Lu-San Chen and Chen-Chih Yeh.

§ 1. Introduction

In the last few years several results about the oscillatory criteria for n-order nonlinear functional differential inequalities with finite deviating arguments have been obtained, e. g. see [1, 2]. The purpose of this paper is to obtain some oscillatory criteria for more general n-order nonlinear functional differential inequalities with continuous distributed deviating arguments and to give some results for equations of this type. We generalize and modify some results in [2, 3].

In this paper we consider the following inequalities

$$u(t)\Big[L_n u(t)+\int_a^b F(t,\xi,u[G_1(t,\xi)],u[G_2(t,\xi)],\cdots,u[G_m(t,\xi)])\,d\sigma(\xi)-h(t)\Big]\leqslant 0, \tag{1.1}$$

and

$$u(t)\Big[L_n u(t)-\int_a^b F(t,\xi,u[G_1(t,\xi)],u[G_2(t,\xi)],\cdots,u[G_m(t,\xi)])\,d\sigma(\xi)-p(t)\Big]\geqslant 0, \tag{1.2}$$

where $n\geqslant 2$ and L_n is an operator defined by

$$L_0u(t)=u(t),\ L_iu(t)=\frac{1}{r_i(t)}(L_{i-1}u(t))',\quad r_i(t)>0\ (i=1,2,\cdots,n),\ r_n(t)=1. \tag{1.3}$$

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Also we study equations

$$L_n u(t)+\int_a^b F(t,\ \xi,\ u[G_1(t,\ \xi)],\ u[G_2(t,\ \xi)],\ \cdots,\ u[G_m(t,\ \xi)])\,d\sigma(\xi)=h(t), \tag{1.1}'$$

$$L_n u(t)-\int_a^b F(t,\ \xi,\ u[G_1(t,\ \xi)],\ u[G_2(t,\ \xi)],\ \cdots,\ u[G_m(t,\ \xi)])\,d\sigma(\xi)=h(t). \tag{1.2}'$$

A function $u(t)$ which satisfies some inequality as (1.1), (1.2) or some equation as (1.1)', (1.2)' is said to be oscillatory if it has an unbounded set of zeros but $u(t)\not\equiv 0$ for all $t\geqslant T$, where T is an appropriate constant. Otherwise it is called nonoscillatory.

Throughout this paper, we suppose:

(R_1) $F(t,\ \xi,\ x_1,\ \cdots,\ x_m)\in C[R_+\times[a,\ b]\times R^m,\ R]$, $F(t,\ \xi_1,\ x_1,\ \cdots,\ x_m)\not\equiv 0$ for all $t\geqslant t_1$, where t_1 is an appropriate constant. If $x_i>0$ $(i=1,\ 2,\ \cdots,\ m)$, then $F(t,\ \xi,\ x_1,\ \cdots,\ x_m)$ is positive and non-decreasing with respect to $x_1,\ \cdots,\ x_m$ for all $t\geqslant 0$, $\xi\in[a,\ b]$. If n is even, then

$$\begin{gathered}F(t,\ \xi,\ x_1,\ \cdots,\ x_m)\leqslant -F(t,\ \xi,\ x_1,\ \cdots,\ -x_m),\ \text{for all } x_i>0,\\ t\geqslant 0,\ \xi\in[a,\ b],\ (i=1,\ 2,\ \cdots,\ m).\end{gathered} \tag{1.4}$$

If n is odd, then

$$\begin{gathered}F(t,\ \xi,\ x_1,\ \cdots,\ x_m)\geqslant -F(t,\ \xi,\ -x_1,\ \cdots,\ -x_m),\quad \text{for all } x_i>0,\\ t\geqslant 0\ (i=1,\ 2,\ \cdots,\ m).\end{gathered} \tag{1.5}$$

(R_2) $r_i(t)\in C[R_+,\ R_+-\{0\}],\ \int^\infty r_i(t)\,dt=\infty\ (i=1,\ 2,\ \cdots,\ n-1).$ (1.6)

(R_3) $G_i(t,\ \xi)\in C[R_+\times[a,\ b],\ R],\ \lim\limits_{t\to+\infty}G_i(t,\ \xi)=\infty\ (\xi\in[a,\ b]).$

(R_4) The integrals in inequalities are Stieltjes integrals.

(R_5) $h(t)\in C[R_+,\ R]$ and there exists an oscillatory solution $p(t)$ such that

$$L_n p(t)=h(t),\ \lim_{t\to+\infty}p(t)=0, \tag{1.7}$$

where $R_+=[0,\ +\infty)$ and $R=(-\infty,\ +\infty)$.

For convenience, we define $W_i(t)\in C[R_+,\ R_+]$, $i=1,\ 2,\ \cdots,\ n-1$, as follows:

$$\begin{aligned}W_1(t)&=\int_0^t r_1(s)\,ds,\\ W_i(t)&=\int_0^t r_i(s)W_{i-1}(s)\,ds,\ i=2,\ 3,\ \cdots,\ n-1.\end{aligned} \tag{1.8}$$

In section 2 of this paper we give some results for inequalies of even order. In section 3 we give some results for inequalies of odd order. In section 4 we give some results for equations of this type. In section 5 we give some examples.

§ 2. Oscillatory Criteria for Inequalies (1.1), (1.2) of Even Order

In order to obtain our main results we need the following lemma, which is a simple generalization of results given by Philos[4]. The proof of this Lemma is omitted here.

Lemma. *Let u be a positive function defined on an interval $[T, \infty)$, $T\geqslant 0$. If $L_n u$ is of constant sign on $[T, \infty)$, then there exists a $T_0 \geqslant T$ and an integer K, $0\leqslant K \leqslant n$, with $n+K$* odd for *$L_n u\leqslant 0$ or $n+K$ even for $L_n u \geqslant 0$, such that*

$$\begin{aligned} &1\leqslant k\leqslant n-1,\ L_i u(t)>0, \quad i=0, 1, \cdots, k-1,\\ &(-1)^{k+i}L_i u(t)>0, \quad i=k, k+1, \cdots, n-1; \end{aligned} \tag{2.1}$$

$$\begin{aligned} &k=0,\ (-1)^i L_i u(t)>0, \quad i=0, 1, \cdots, n-1;\\ &k=n,\ L_i u(t)>0, \quad i=0, 1, \cdots, n-1. \end{aligned} \tag{2.2}$$

Theorem 1. *If n is even and*

$$\int^{\infty} W_{n-1}(t)\left[\int_a^b F(t, \xi, c, \cdots, c)\, d\sigma(\xi)\right] dt = \pm\infty \tag{2.3}$$

for any nonzero constant c, then every bounded solution of (1.1) is oscillatory.

Proof As in [2], we can see that if $u(t)$ is a bounded positive nonoscillatory solution of (1.1), then

$$L_n x(t) + \int_a^b F(t, \xi, u[G_1(t, \xi)], u[G_2(t, \xi)], \cdots, u[G_m(t, \xi)])\, d\sigma(\xi) \leqslant 0,$$

where $x(t) = u(t) - p(t)$ and $p(t)$ satisfies the condition (R_5). Obviously, we have $L_n x(t) < 0$ and $x(t) > 0$ for t large enough. By Lemma 1, we have $k=1, 3, \cdots, n-1$, but for $k\geqslant 3$, we can obtain $\lim\limits_{t\to+\infty} x(t) = +\infty$, this contradicts the fact that $x(t)$ is bounded.

So we only have $k=1$, and $(-1)^{i+1} L_i x(t) > 0$ for $i=1, 2, \cdots, n-1$. Specialy, when $i=1$, we have $x'(t) > 0$ for t large enough.

Therefore

$$A(t) = \sum_{i=1}^{n-1} (-1)^{i+1} W_i(t) L_i x(t) \geqslant 0,$$

$$\begin{aligned} x(t) &= K + A(t) - \int_T^t W_{n-1}(s) L_n x(s)\, ds \\ &\geqslant K + \int_T^t W_{n-1}(s)\left[\int_a^b F(s, \xi, u[G_1(s, \xi)], \cdots, u[G_m(s, \xi) d\sigma(\xi)\right] ds, \end{aligned}$$

where $K = x(T) - A(T)$. We have $u(t) = x(t) + p(t) \geqslant x(T) - |p(t)|$ for $t \geqslant T$. Because $\lim\limits_{t\to+\infty} p(t) = 0$, there exists $T^* \geqslant T$ such that $|p(t)| \leqslant \frac{1}{2} x(T)$ for $t\geqslant T^*$. So we have

$$u(t) \geqslant x(T) - \frac{1}{2} x(T) = \frac{1}{2} x(T) = C, \quad (t \geqslant T^*).$$

Because $\lim\limits_{t\to+\infty} G_i(t, \xi)=+\infty$, for $\xi\in[a, b], i=1, 2, \cdots, m$, there exists $T^{**}>T^*$, when $t\geqslant T^{**}$, $G_i(t, \xi)\geqslant T^*$ for $\xi\in[a, b]$. Hence for $t\geqslant T^{**}$,

$$x(t)\geqslant K+\int_{T^*}^{t} W_{n-1}(s)\left[\int_a^b F(s, \xi, c, \cdots, c)d\sigma(\xi)\right]ds\to+\infty$$

as $t\to+\infty$, which contradicts the fact that $x(t)$ is bounded. For $u(t)$ is a bounded negative nonoscillatory solution of (1.1), we only need notice that (1.4) and (1.5) hold, we can similarly obtain a contradiction. Thus our proof is complete.

Remark. In [2], it is supposed that

$$\lim_{t\to+\infty} L_i p(t)=0 \quad \text{for} \quad i=0, 1, \cdots, n-1.$$

But from this one can not obtain $u'(t)=x'(t)+p'(t)\geqslant 0$, because there exists $p(t)$ and $x(t)$ such that $p'(x_n)<0$ and $|p'(t_n)|>x'(t_n)$ i. e. $u'(t_n)=x'(t_n)-|p'(t_n)|<0$ for some $\{t_n, n=1, 2, \cdots\}$. In fact there exists $x(t)$ such that $\lim\limits_{t_n\to+\infty} x'(t_n)=0$ for some $\{t_n, n=1, 2, \cdots\}$. So there is something wrong in the proof of Theorem 1 in [2]. In fact as in this paper we only need

$$\lim_{t\to+\infty} L_0 p(t)=\lim_{t\to+\infty} p(t)=0.$$

In this paper we give a modified and simple condition to complete the proof of Theorem 1.

Theorem 2. *If n is even and the condition (2.3) holds, then every bounded solution of (1.2) either oscillates or tends to zero as $t\to+\infty$.*

Proof If there exists $u(t)$ which is a bounded positive nonoscillatory solution of (1.2) and $\lim\limits_{t\to+\infty} u(t)\neq 0$, then as in [2], we can see that $L_n x(t)>0$, where $x(t)=u(t)-p(t)$ is positive eventually. So we have $(-1)^i L_i x(t)>0$ $(i=1, 2, \cdots, n-1)$ and $\sum\limits_{i=1}^{n-1}(-1)^{i+1}W_i(t)L_i x(t)\leqslant 0$ for t large enough. Hence

$$x(t)\leqslant K-\int_T^t W_{n-1}(s)\int_a^b F(s, \xi, u[G_1(s, \xi)], \cdots, u[G_m(s, \xi)]d\sigma(\xi)ds,$$

where $K=x(T)-\sum\limits_{i=1}^{n-1}(-1)^{i+1}W_i(T)L_i x(T)$. Notice $x'(t)<0$, so the limit $\lim\limits_{t\to+\infty} x(t)=C^*=x(\infty)$ exists, and $C^*\geqslant 0$. Also $C^*>0$, because if $C^*=0$, then $\lim\limits_{t\to+\infty} u(t)=\lim\limits_{t\to+\infty} x(t)+\lim\limits_{t\to+\infty} p(t)$, i. e. this leads to a contraduction to $\lim\limits_{t\to+\infty} u(t)\neq 0$. So for $t\geqslant T$ we have

$$u(t)=x(t)+p(t)\geqslant x(\infty)+p(t)\geqslant C^*-|p(t)|.$$

From $\lim\limits_{t\to+\infty} p(t)=0$ we can see that there exists T such that, for $t\geqslant T^*$, $|p(t)|<C^*/2$. Hence $u(t)>C^*/2=C$. So we have

$$x(t)\leqslant K-\int_T^t W_{n-1}(s)\left[\int_a^b F(s, \xi, c, \cdots, c)d\sigma(\xi)\right]ds\to-\infty$$

as $t\to+\infty$, which contradicts the boundedness of $x(t)$.

Likewise, we can prove that there cannot exist a bounded negative solution $u(t)$

of (1.2), $\lim_{t\to+\infty} u(t) \neq 0$. The proof of Theorem 2 is complete.

Remark. As the remark of Theorem 1, we easily see that from the condition $\lim_{x\to+\infty} L_i p(t)=0$ $(i=0, 1, \cdots, n-1)$, $u'(t)<0$ can not be obtained. So there is something wrong in the proof of Theorem 3 in [2]. In this paper we only need $\lim_{t\to+\infty} p(t)=0$ to obtain $u(t)>C\neq 0$ $(t\geqslant T^*)$, i. e. we give a modified and simple condition to complete the proof of Theorem 2.

Also we see that $u(t)-p(t)$ is a decreasing function

§ 3. Oscillatory Criteria for inequalities (1.1), (1.2) of Odd Order

For inequalities (1.1), (1.2) of odd order, we have following results:

Theorem 3. *Suppose n is odd. If the condition* (2.3) *holds, then every bounded solution of* (1.1) *either oscillates or tends to zero as $t\to+\infty$.*

The proof of Theorem 3 is similar to that of Theorem 1.

Theorem 4. *Suppose n is odd. If the condition* (2.3) *holds, then every bounded solution of* (1.1) *either oscillates or tends to zero as $t\to+\infty$.*

The proof of Theorem 4 is similar to that of Theorem 2.

§ 4. Oscillatory Criteria for Equations (1.1)′, (1.2)′

Theorem 5. *Suppose n is even. If the condition* (2.3) *holds, then every bounded solation of* (1.1)′ *is oscillatory.*

Proof Notice that

$$L_n u(t)+\int_a^b F(t, \xi, u[G_1(t, \xi)], \cdots, u[G_m(t, \xi)])d\sigma(\xi)-h(t)\leqslant 0$$

has no eventually bounded positive solution and

$$L_n u(t)+\int_a^b F(t, \xi, u[G_1(t, \xi)], \cdots, u[G_m(t, \xi)])d\sigma(\xi)-h(t)\geqslant 0$$

has no eventually bounded negative solution. So we can see that every bounded solution of (1.1)′ is oscillatory. The proof of Theorem 5 is complete.

Similarly, we can prove following Theorems.

Theorem 6. *Suppose n is odd. If the condition* (2.3) *holds, then every bounded solution of* (1.1)′ *either oscillates or tends to zero as $t\to+\infty$.*

Theorem 7. *Suppose n is even. If the condition* (2.3) *holds, then every bounded solution of* (1.2)′ *either oscillates or tends to zero as $t\to+\infty$.*

Theorem 8. *Suppose n is odd. If the condition* (2.3) *holds, then every bounded solution of* (1.2)′ *is oscillatory.*

§ 5. Some Examples

Example 1.

$$u^{(3)}(t)+\int_{-1}^{1}\frac{\xi^2}{t}[u^3(t+\xi)+u^3(t-\xi)]d\xi=2(\sin t+\cos t)e^{-t}. \tag{5.1}$$

Obviously, $F(t, \xi, u[G(t, \xi)])$ satisfies hypothesis (R_1) and $p(t)=\sin te^{-t}$ satisfies $p^{(3)}(t)=2(\sin t+\cos t)e^{-t}$, $\lim_{t\to+\infty}p(t)=0$. Notice, $W_2(t)=\frac{t^2}{2}$,

$$\int^{\infty}\frac{t^2}{2}\int_0^1\frac{\xi^2}{t}(c^3+c^3)d\xi\,dt=\infty.$$

So the condition (2.3) is satisfied. By Theorem 6, we can see that every bounded solution of (5.1) either oscillates or tends to zero as $t\to+\infty$.

Example 2.

$$u^{(3)}(t)-\int_{-1}^{1}\frac{\xi^2}{t}[u^3(t+\xi)+u^3(t-\xi)]d\xi=2(\sin t+\cos t)e^{-t} \tag{5.2}$$

By Theorem 8, we can prove that every bounded solution of (5.2) is oscillatory.

Example 3.

$$u^{(4)}(t)+\int_{-1}^{1}\frac{\xi^2}{t}[u^3(t+\xi)+u^3(t-\xi)]d\xi=-4\sin te^{-t} \tag{5.3}$$

Notice $p(t)=\sin te^{-t}$ satisfies $p^{(4)}(t)=-4\sin te^{-t}$, $\lim_{t\to+\infty}p(t)=0$, $p(t)$ is oscillatory, $W_3(t)=\int_0^{\infty}\frac{t^3}{6}\int_0^1\frac{\xi^2}{t}(c^3+c^3)d\xi\,dt=\infty$, i. e. the condition (2.3) is satisfied. So by Theorem 5, we can see that every bounded solution of (5.3) is oscillatory.

Example 4.

$$u^{(4)}(t)-\int_{-1}^{1}\frac{\xi^2}{t}[u^3(t+\xi)+u^3(t-\xi)]d\xi=-4\sin te^{-t} \tag{5.4}$$

By Theorem 7, we can prove that every bounded solution of (5.4) either oscillates or tends to zero as $t\to+\infty$.

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