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QUOTIENTS OF STRONGLY *S*-DECOMPOSABLE OPERATORS

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Abstract

This paper shows that if $T\in C(X)$ is strongly S-decomposable, where S is a closed subset of $\sigma_e(T)$ and $S\neq C_\infty$, then for any (e) spectral maximal subspace Y of T, T^Y is a closed strongly $S\cap\sigma_e(T^Y)$-decomposable operator.

§ 1. Notations and Preliminaries

In this paper we denote the complex plane and its compactification by C and C_∞ respectively. If A is a subset of C_∞, we write $A^c=C_\infty\backslash A$.

We denote the classes of the closed and the bounded linear operators in Banach space X by $C(X)$ and $B(X)$. If X is a Banach space and Y is a closed subspace of X, then the quotient space of X modulo Y will be denoted by X/Y.

Given $T\in C(X)$, we denote the domain of T by $D(T)$, denote the resolvent set, the spectrum and the extended spectrum of T by $(\rho(T)$, $\sigma(T)$ and $\sigma_e(T)$ respectively. The local spectrum and the extended local spectrum will be denoted by $\sigma(x, T)$ and $\sigma_e(x, T)$. We have

$$\sigma_e(x, T)=\begin{cases}\sigma(x, T), & \text{if } \infty \text{ is a regular point of } \tilde{x}_1(\cdot).\\ \sigma(x, T)\cup(\infty), & \text{if } \infty \text{ is a singular point of } \tilde{x}_T(\cdot).\end{cases}$$

If Y is a closed subspace of X such that $T(Y\cap D(T))\subseteq Y$, then Y is called an invariant subspace of T. We denote by $\mathrm{INV}(T)$ the family of all invariant subspaces of T.

Given $Y\in\mathrm{INV}(T)$, the restriction of T to Y and the quotient operator of T induced in X/Y will be denoted by $T|Y$ and T^Y respectively.

Let $Y\in\mathrm{INV}(T)$, if for any $Z\in\mathrm{INV}(T)$, the inclusion $\sigma_e(T|Z)\subseteq\sigma_e(T|Y)$ implies $Z\subseteq Y$, then Y is called an (e) spectral maximal space of T. We deonte the family of all such subspaces by $SM_e(T)$.

Given $T\in C(X)$ and $\Delta\subseteq C_\infty$, we set

$$X(T, \Delta)=\cup(Y\in\mathrm{INV}(T),\ \sigma_e(T|Y)\subseteq\Delta).$$

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If T has the SVEP on S^c and $S\subseteq\Delta\subseteq C_\infty$, we set

$$X_T(\Delta)=(x\in X,\ \sigma_e(x,\ T)\subseteq\Delta).$$

Let F be a closed subset of C_∞ and S be a closed subset of F. A family of open sets $(G_0;\ G_1,\ \cdots,\ G_n)$ is called an open S-covering of F if $\bigcup_{i=0}^{n} G_i\supseteq F$ and $\overline{G}_i\cap S=\varnothing$ for $i=1,\ 2,\ \cdots,\ n$.

Let $T\in C(X)$ and S be a closed subset of $\sigma_e(T)$. If for any open S-covering $(G_0;\ G_1,\ \cdots,\ G_n)$ of $\sigma_e(T)$, there exists $(Y_i)_{i=0}^{n}\subseteq SM_e(T)$ such that $X=\sum_{i=0}^{n} Y_i$ and $\sigma_e(T|Y_i)\subseteq G_i$ for $i=0,\ 1,\ 2,\ \cdots,\ n$, then T is called S-decomposable.

If, in addition, we require $Y=\sum_{i=0}^{n} Y\cap Y_i$ for any $Y\in SM_e(T)$, then T is called strongly S-decomposable.

Obviously, if $T\in C(X)$ is an unbounded operator and S is a closed bounded subset of $\sigma_e(T)$, then T is S-decomposable if and only if T is $S\cup(\infty)$-decomposable. Hence when we discuss the S-decomposability of an unbounded operator, we can always suppose $\infty\in S$.

The theory of S-decomposable operators was studied by B. Nagy[1-3], I. Bacalu[5-8] and F. -H. Vasilescu[4] etc. B. Nagy[1] proved that $T\in C(X)$ is strongly S-decomposable if and only if for any $Y\in SM_e(T)$ $T|Y$ is $S\cap\sigma_e(T|Y)$-decomposable. In fact if T is strongly S-decomposable, then $T|Y$ is strongly $S\cap\sigma_e(T|Y)$-decomposable for any $Y\in SM_e(T)$. Moreover B. Nagy[1] proved that if $T\in C(X)$ is strongly S-decomposable and Y is a spectral maximal space of T such that $Y\supseteq X(\overline{G})$, where G is an open set containing S, then T^Y is strongly decomposable, i. e. strongly $S\cap\sigma_e(T^Y)$-decomposable.

It is natural to ask whether the last conclusion holds in general case. Now we are going to solve this problem in the following section.

§ 2. The Main Theorem

To prove our main theorem we need the following four lemmas.

Lemma 1. *Let $T\in C(X)$ and S be a closed subset of $\sigma_e(T)$, then the following statements are equivalent:*

1) *T is strongly S-decomposable.*

2) *For any open S-covering $(G_0;\ G_1,\ \cdots G_n)$ of $\sigma_e(T)$, there is $(Y_i)_{i=0}^{n}\subseteq SM_e(T)$ such that $\sigma_e(T|Y_i)\subseteq G_i$ for $i=0,\ 1,\ \cdots,\ n$, and $Y=\sum_{i=0}^{n} Y\cap Y_i$ for every $Y\in SM_e(T)$ with $\sigma_e(T/Y)\supseteq S$.*

3) $T|Y$ is $S\cap\sigma_e(T|Y)$-decomposable for every $Y\in SM_e(T)$ with $\sigma_e(T|Y)\supseteq S$.

Proof 1) $\Rightarrow$ 2) is trivial.

2) $\Rightarrow$ 1). Suppose that $(G_0; G_1, \cdots, G_n)$ is an open S-covering of $\sigma_e(T)$. Take an open subset G_0' of C_∞ such that $S\subseteq G_0'\subseteq \overline{G}_0'\subseteq G_0$ and $\overline{G}_i\cap G_0'=\varnothing$ for $i=1, 2, \cdots, n$. Set $S_1=\overline{G}_0'$. For any $Y\in SM_e(T)$, set $Z=X(T, S_1\cup\sigma_e(T|Y))$. Then $Z\in SM_e(T)$, $Y\subseteq Z$ and $\bigcup_{i=0}^{n} G_i\supseteq S_1\cup\sigma_e(T|Y)\supseteq\sigma_e(T|Z)\supseteq S$. Let $Z_i=X(T, \overline{G}_i)$ for $i=0, 1, \cdots, n$. Then it follows from 2) that

$$X=\sum_{i=0}^{n} Z_i \quad\text{and}\quad Z=\sum_{i=0}^{n} Z\cap Z_i.$$

Obviously, $Z\cap Z_i=X(T, (S_1\cup\sigma_e(T|Y))\cap\overline{G}_i)=X(T, \overline{G}_i\cap\sigma_e(T|Y))\subseteq Y$ for $i=1, 2, \cdots, n$. Now we are going to prove $Y\subseteq\sum_{i=0}^{n} Y\cap Z_i$. If $x\in Y$, then $x\in Z$. Therefore there exist $x_i\in Z\cap Z_i$ such that $x=\sum_{i=0}^{n} x_i$. Since $x_i\in Z\cap Z_i\in Y$ for $i=1, 2, \cdots, n$, we have $x_0\in Y\cap Z_0$. Consequently $Y\subseteq\sum_{i=0}^{n} Y\cap Z_i$. The opposite inclusion is evident. Thus we obtain 1).

2) $\Rightarrow$ 3). Given $Y\in SM_e(T)$ with $\sigma_e(T|Y)\supseteq S$, assume that $(G_0; G, \cdots, G_n)$ is an open S-covering of $\sigma_e(T|Y)$. Take an open subset H_0 of C_∞ such that $\overline{H}_0\cap\sigma_e(T|Y)=\varnothing$ and $H_0\cup G_0\cup G_1\cup\cdots\cup G_n=C_\infty$. Let $G_0'=H_0\cup G_0$. Then $(G_0'; G_1, \cdots, G_n)$ is an open S-covering of $\sigma_e(T)$. Hence by 2) there exists $(Y_i)_{i=0}^{n}\subseteq SM_e(T)$ such that

$$Y=\sum_{i=0}^{n} Y\cap Y_i,\ \sigma_e(T|Y_0)\subseteq G_0' \text{ and } \sigma_e(T|Y_i)\subseteq G_i \text{ for } i=1, 2, \cdots, n.$$

Let $Z_0=X(T, \overline{G}_0')\cap Y$ and $Z_i=Y\cap Y_i$ for $i=1, 2, \cdots, n$. Since

$$\sigma_e(T|Y)\supseteq S,\ \sigma_e(T|X(T, \overline{G}_0'))\supseteq S \text{ and } \sigma_e(T|Y_i)\cap S=\varnothing \text{ for } i=1, 2, \cdots, n,$$

we have

$$Y=\sum_{i=0}^{n} Z_i \text{ and } \sigma_e(T|Z_i)\subseteq\overline{G}_i \text{ for } i=0, 1, \cdots, n.$$ Hence $T|Y$ is S-decomposable.

3) $\Rightarrow$ 2). Suppose that $(G_0; G_1, \cdots, G_n)$ is an open S-covering of $\sigma_e(T)$. Since $X\in SM_e(T)$ and $\sigma_e(T|X)=\sigma_e(T)\supseteq S$, we see, by 3), that T is S-decomposable and therefore $X=\sum_{i=0}^{n} X(T, \overline{G}_i)$. If $Y\in SM_e(T)$ such that $\sigma_e(T|Y)\supseteq S$, then by 3) $T|Y$ is S-decomposable so that $Y=\sum_{i=0}^{n} Y(T|Y, \overline{G}_i)$. Since $Y(T, \overline{G}_i)=Y\cap X(T, \overline{G}_i)$, we have $Y=\sum_{i=0}^{n} Y\cap X(T, \overline{G}_i)$. Consequently 2) follows.

Lemma 2. *If $T\in C(X)$ is S-decomposable where S is a closed subset of $\sigma_e(T)$ and $S\neq C_\infty$, then for any $Y\in SM_e(T)$, T^Y is a closed operator.*

Proof If $S\cup\sigma_e(T|Y)\neq C_\infty$, then there is a bounded open subset $H\subseteq C_\infty$ and an open neighborhood G of ∞ such that $H\cup G=C_\infty$, $\overline{G}\neq C_\infty$ and $\overline{H}\cap(S\cup\sigma_e(T|Y))=\varnothing$. Obviously $S\cup\sigma_e(T|Y)\subseteq G$ and $H\neq\varnothing$. Since T is S-decomposable, we have $X=X(T, \overline{G})+X(T, \overline{H})$. It is easily seen that $Y+X(T, \overline{H})$ is closed because $X(T, \overline{H}\cup(S\cup\sigma_e(T|Y)))=X(T, S\cup\sigma_e(T|Y))+X(T, \overline{H})$ is closed. Therefore we have

$$X=X(T,\bar{G})+(X(T,\bar{H})+Y)$$

and so

$$X/Y=X(T,\bar{G})/Y+(X(T,\bar{H})+Y)/Y.$$

Using the fact that $T^Y|X(T,\bar{G})/Y\in C(X(T,\bar{G})/Y)$ and $T^Y|(X(T,\bar{H})+Y)/Y\in B((X(T,\bar{H})+Y)/Y)$, we obtain $T^Y\in C(X/Y)$.

If $S\cup\sigma_e(T|Y)=C_\infty$, then $\sigma_e(T|Y)\backslash S=C_\infty\backslash S$. Obviously, there is a bounded open subset H of C_∞ and an open neighborhood G of ∞ such that

$$H\cup G=C_\infty,\ \bar{H}\subseteq C_\infty\backslash S=\sigma_e(T|Y)\backslash S \text{ and } \bar{G}\neq C_\infty.$$

Since T is S-decomposable and $S\subseteq G$, we have $X=X(T,\bar{H})+X(T,\bar{G})$. By the inclusion $\sigma_e(T|X(T,\bar{H}))\subseteq\bar{H}\subseteq\sigma_e(T|Y)$ and the fact that $Y\in SM_e(T)$ we obtain $X(T,\bar{H})\subseteq Y$. By the boundedness of H we have $X(T,\bar{H})\subseteq D(T)$. Hence $X(T,\bar{H})\subseteq Y\cap D(T)$. For any $\lambda\in C_\infty\backslash\bar{G}$, $\lambda-T|X(T,\bar{G})\cap Y$ is evidently injective. Now we are going to show it is surjective. For any $y\in X(T,\bar{G})\cap Y$, since $\lambda\notin\bar{G}\supseteq\sigma_e(T|X(T,\bar{G}))$, there is an $x\in X(T,\bar{G})$ such that $(\lambda-T)x=y$. Because Y being an (e) spectral maximal subspace of T is T-absorbent and $\lambda\in\bar{H}\subseteq\sigma_e(T|Y)$, it follows that $x\in Y$. Therefore $\lambda-T|X(T,\bar{G})\cap Y$ is surjective. Hence $\sigma_e(T|X(T,\bar{G}))\cup\sigma_e(T|X(T,\bar{G})\cap Y)\subseteq\bar{G}\neq C_\infty$. Consequently T^Y is closed. Thus in any case we have proved $T^Y\in C(X/Y)$.

Lemma 3. *Suppose that $T\in C(X)$ is a strongly S-decomposable operator, where S is a closed subset of $\sigma_e(T)$ and $S\neq C_\infty$. If T is unbounded, in addition, we suppose $\infty\in S$. Suppose $Y\in SM_e(T)$ and $\hat{Z}\in SM_e(T^Y)$ with $\sigma_e(T^Y|\hat{Z})\supseteq S\cap\sigma_e(T^Y)$. Let $Z=(x\in X,\ [x]_Y\in\hat{Z})$. Then*

$$Z\in SM_e(T)\quad\textit{and}\quad\sigma_e(T|Z)\supseteq S\cup\sigma_e(T|Y).$$

Proof By Lemma 2 T^Y is closed. Obviously $Z\in\mathrm{INV}(T)$, $Z/Y=\hat{Z}$ and $\sigma_e(T|Z)=\sigma_e((T|Z)^Y)\cup\sigma_e(T|Y)=\sigma_e(T^Y|\hat{Z})\cup\sigma_e(T|Y)\supseteq(S\cap\sigma_e(T^Y))\cup\sigma_e(T|Y)=(S\cup\sigma_e(T|Y))\cap(\sigma_e(T^Y)\cup\sigma_e(T|Y))=(S\cup\sigma_e(T|Y))\cap\sigma_e(T)=S\cup\sigma_e(T|Y)$.

Let $W=X(T,\sigma_e(T|Z))$. Then $W\in SM_e(T)$, $\sigma_e(T|W)\subseteq\sigma_e(T|Z)$ and $Y\subseteq Z\subseteq W$. Now we show $W\subseteq Z$. To this aim it is sufficient to prove $W/Y\subseteq Z/Y$. Since $Z/Y=\hat{Z}\in SM_e(T^Y)$, we only have to verify $\sigma_e(T^Y|W/Y)\subseteq\sigma_e(T^Y|\hat{Z})$. Assume that $\infty\notin\sigma_e(T^Y|\hat{Z})$. If T is bounded, then clearly $\infty\notin\sigma_e(T^Y|W/Y)$. If T is unbounded, by the hypothesis $\infty\in S$ and $\sigma_e(T^Y|\hat{Z})\supseteq S\cap\sigma_e(T^Y)$, we have $\infty\notin S\cap\sigma_e(T^Y)$ and consequently $\infty\notin\sigma_e(T^Y)$. Therefore $\infty\notin\sigma_e(T^Y|W/Y)$. Assume that there is a $\lambda_0\in C\backslash\sigma_e(T^Y|\hat{Z})$. If $\lambda_0\notin\sigma_e(T|Z)$, then $\lambda_0\notin\sigma_e(T|W)\cup\sigma_e(T|Y)\supseteq\sigma_e((T|W)^Y)=\sigma_e(T^Y|W/Y)$. If $\lambda_0\in\sigma_e(T|Z)$, since $\lambda_0\notin\sigma_e(T^Y|\hat{Z})$ and $\sigma_e(T|Z)=\sigma_e(T^Y|\hat{Z})\cup\sigma_e(T|Y)$ we have $\lambda_0\in\sigma_e(T|Y)$. Let $x\in W\cap D(T)$ such that $(\lambda_0-T^Y)[x]_Y=0$. Then $(\lambda_0-T)x\in Y$. Since Y being an (e) spectral maximal subspabe of T is T-absorbent and $\lambda_0\in\sigma_e(T|Y)$ we have $x\in Y$ and so $[x]_Y=0$. Hence $\lambda_0-T^Y|W/Y$ is injective. Now let us show $\lambda_0-T^Y|W/Y$ is surjective. If $\lambda_0\notin\sigma_e(T|W)$, then for any $y\in W$ there is an

$x\in W\cap D(T)$ such that $y=(\lambda_0-T)x$. Therefore $[y]_Y=(\lambda_0-T^Y)[x]_Y$. Consequently $\lambda_0-T^Y|W/Y$ is surjective. If $\lambda_0\in\sigma_e(T|W)$, we discuss in two cases: a). $\lambda_0\in S$. Since $\lambda_0\notin\sigma_e(T^Y|\hat{Z})$ and $\sigma_e(T^Y|\hat{Z})\supseteq S\cap\sigma_e(T^Y)$, we have $\lambda_0\notin\sigma_e(T^Y)$. By Theorem 3.15 in [9], we obtain $W/Y\in SM_e(T^Y)$ and $\sigma_e(T^Y|W/Y)\subseteq\sigma_e(T^Y)$ so that $\lambda_0\notin\sigma_e(T^Y|W/Y)$ and clearly $\lambda_0-T^Y|W/Y$ is surjective. b). $\lambda_0\notin S$. Since $\sigma_e(T|W)\subseteq\sigma_e(T|Z)=\sigma_e(T^Y|\hat{Z})\cup\sigma_e(T|Y)$ and $\lambda_0\notin\sigma_e(T^Y|\hat{Z})$, there exists an open neighborhood $N(\lambda_0)$ of λ_0 such that $\sigma_e(T/Z)\setminus N(\lambda_0)\supseteq\sigma_e(T^Y|\hat{Z})\supseteq\sigma_e(T|Z)\setminus\sigma_e(T|Y)$. Therefore $\sigma_e(T|W)\setminus N(\lambda_0)\supseteq\sigma_e(T|W)\setminus\sigma_e(T|Y)$. Let $G_0=(\lambda_0)^c$, $G_1=N(\lambda_0)$. Then (G_0, G_1) is an open covering of $\sigma_e(T|W)$ and $G_0\supseteq S$. Since T is strongly S-decomposable and $W\in SM_e(T)$, $T|W$ is $S\cap\sigma_e(T|W)$-decomposable. Hence there exist $Y_0, Y_1\in SM_e(T|W)$ such that

$$\sigma_e(T|Y_0)\subseteq G_0,\ \sigma_e(T|Y_1)\subseteq G_1 \text{ and } W=Y_0+Y_1.$$

Thus for any $y\in W$ there exist $y_0\in Y_0$, $y_1\in Y_1$ such that $y=y_0+y_1$. Since $\lambda_0\notin\sigma_e(T|Y_0)$, there exists an $x_0\in Y_0\cap D(T)$ such that $(\lambda_0-T)x_0=y_0$. Because $N(\lambda_0)\cap\sigma_e(T|W)\subseteq\sigma_e(T|Y)$, we have $\sigma_e(T|Y_1)\subseteq\sigma_e(T|Y)$ and so $Y_1\subseteq Y$. Therefore $[y]_Y=[y_0]_Y=(\lambda_0-T^Y)[x_0]_Y$. Consequently $\lambda_0-T^Y|W/Y$ is surjective.

Thus we have proved $\sigma_e(T^Y|W/Y)\subseteq\sigma_e(T^Y|\hat{Z})$. Since $\hat{Z}\in SM_e(T^Y)$, we obtain $W/Y\subseteq\hat{Z}=Z/Y$ so that $W\subseteq Z$. The opposite inclusion is obvious. Hence $Z=W\in SM_e(T)$.

Lemma 4. *Suppose that $T\in C(X)$ is strongly S-decomposable, where S is a closed subset of $\sigma_e(T)$ and $S\neq C_\infty$. If T is unbounded, in addition, we suppose $\infty\in S$. Suppose $Y\in SM_e(T)$, K is a closed subset of C_∞, G is an open subset of C_∞ and $K\subseteq G\subseteq\bar{G}\subseteq S^c$. Then there exists a $Z\in SM_e(T)$ such that*

$$X(T, K)\subseteq Z,\ \sigma_e(T|Z)\subseteq\bar{G},\ Z+Y\in \mathrm{INV}(T)$$

and

$$\sigma_e(T^Y|(Z+Y)/Y)\subseteq\bar{G}\cup(S\setminus\sigma_e(T|Y)).$$

Proof. Let $V=X(T, S\cup\sigma_e(T|Y))$, $W=X(T, K\cup S\cup\sigma_e(T|Y))$. Then V, $W\in SM_e(T)$. Let H be an open subset of C_∞ such that $\bar{H}\cap K=\varnothing$ and $H\cup G=C_\infty$. Then $H\supseteq S$. Since T is strongly S-decomposable, there exist

$$V_0, V_1\in SM_e(T|V),\ W_0, W_1\in SM_e(T|W),\ Y_0, Y_1\in SM_e(T|Y)$$

such that

$$V_0+V_1=T,\ W_0+W_1=W,\ Y_0+Y_1=Y,$$

and

$$\sigma_e(T|V_0)\subseteq\bar{H}\cap\sigma_e(T|V),\quad \sigma_e(T|V_1)\subseteq\bar{G}\cap\sigma_e(T|V),$$
$$\sigma_e(T|W_0)\subseteq\bar{H}\cap\sigma_e(T|W),\quad \sigma_e(T|W_1)\subseteq\bar{G}\cap\sigma_e(T|W),$$
$$\sigma_e(T|Y_0)\subseteq\bar{H}\cap\sigma_e(T|Y),\quad \sigma_e(T|Y_1)\subseteq\bar{G}\cap\sigma_e(T|Y).$$

Obviously, we can take

$$V_0=W_0=X(T, \bar{H}\cap(S\cup\sigma_e(T|Y))),$$
$$V_1=Y_1=X(T, \bar{G}\cap\sigma_e(T|Y)),$$

$$W_1=X(T, \bar{G}\cap(K\cup\sigma_e(T|Y)\cup S))=X(T, \bar{G}\cap(K\cup\sigma_e(T|Y)).$$

Put $Z=W_1$. Then $Z+Y\subseteq W$. Now in order to show $Z+Y\in \mathrm{INV}(T)$, it is sufficient to prove $Z+Y$ is closed. Let $x_n\in Z+Y$ such that $\|x_n\|\leqslant 1/2^n$. Then $x_n\in W$. Therefore there exist $x_n^{(0)}\in W_0$, $x_n^{(1)}\in W_1$ such that $x_n^{(0)}+x_n^{(1)}=x_n$ and $\|x_n^{(0)}\|+\|x_n^{(1)}\|\leqslant p\|x_n\|\leqslant p/2^n$, where $p>0$ is a constant. On the other hand there exist $z_n\in Z$ and $y_n\in Y$ such that $z_n+y_n=x_n$. Therefore

$$\begin{aligned} x_n^{(0)}-y_n=z_n-x_n^{(1)}&\in Z\cap X(T, (\bar{H}\cap(S\cup\sigma_e(T|Y)))\cup\sigma_e(T|Y)) \\ &=X(T, (\bar{G}\cap(K\cup\sigma_e(T|Y)))\cap((\bar{H}\cap(S\cup\sigma_e(T|Y))) \\ &\quad\cup\sigma_e(T|Y)))\subseteq X(T, \sigma_e(T|Y))=Y. \end{aligned}$$

Hence $(x_n^{(0)})_{n=1}^{\infty}\subseteq Y$. Let $\sum_{n=1}^{\infty}x_n^{(0)}=x^{(0)}$, $\sum_{n=1}^{\infty}x_n^{(1)}=x^{(1)}$. Then

$$x^{(0)}\in Y,\ x^{(1)}\in Z \quad\text{and}\quad \sum_{n=1}^{\infty}x_n=x^{(0)}+x^{(1)}\in Z+Y.$$

Consequently $Z+Y$ is closed. Now we show $\sigma_e(T|(Z+Y))\subseteq\bar{G}\cup\sigma_e(T|Y)\cup S$. If $\infty\notin\bar{G}\cup\sigma_e(T|Y)\cup S$, then $\infty\notin S$. By the hypothesis, T is bouned. Therefore $\infty\notin\sigma_e(T|(Z+Y))$. If $\lambda_0\in C\backslash(\bar{G}\cup\sigma_e(T|Y)\cup S)$, then $\lambda_0\notin\bar{G}\supseteq\sigma_e(T|Z)$ and $\lambda_0\notin\sigma_e(T|Y)$. Therefore both $\lambda_0-T|Z$ and $\lambda_0-T|Y$ are surjective. Consequently $\lambda_0-T|(Z+Y)$ is also surjective. Since $Z+Y\subseteq X(T, \bar{G}\cup\sigma_e(T|Y)\cup S)$, we see that $\lambda_0-T|(Z+Y)$ is injective. Thus $\lambda_0\notin\sigma_e(T|(Z+Y))$. Consequently $\sigma_e(T|(Z+Y))\subseteq\bar{G}\cup\sigma_e(T|Y)\cup S$. Finally we show $\sigma_e(T^Y|(Z+Y)/Y)\subseteq\bar{G}\cup(S\backslash\sigma_e(T|Y))$. By Lemma 2 T^Y is closed. Since $Z\subseteq D(T)$, we have $(Z+Y)/Y\subseteq D(T^Y)$, and so $T^Y|(Z+Y)/Y$ is a bounded operator. If $\lambda_0\in C\backslash(\bar{G}\cup(S\backslash\sigma_e(T|Y))$, then $\lambda_0\notin\bar{G}\supseteq\sigma_e(T|Z)$. For any $\hat{a}\in(Z+Y)/Y$, there is a $z\in Z$ such that $[z]_Y=\hat{a}$. Therefore there is an $x\in Z\cap D(T)$ satisfying $(\lambda_0-T)x=z$ and so $(\lambda_0-T^Y)[x]_Y=[z]_Y=\hat{a}$. Hence $\lambda_0-T^Y|(Z+Y)/Y$ is surjective. For any $z\in Z$ and $y\in Y$, if $(\lambda_0-T^Y)[z+y]_Y=0$, then $[(\lambda_0-T)z]_Y=0$ and so $(\lambda_0-T)z\in Y$. If $\lambda_0\in(\bar{G})^c\cap\sigma_e(T|Y)$, since Y being an (e) spectral maximal space of T is T-absorbent and $\lambda_0\in\sigma_e(T|Y)$ we have $z\in Y$. Consequently $[z+y]_Y=0$. Hence $\lambda_0-T^Y|(Z+Y)/Y$ is injective. If $\lambda_0\in(\bar{G})^c\cap\rho(T|Y)\cap S^c$, then there is an $x\in Y\cap D(T)$ such that

$$(\lambda_0-T)x=(\lambda_0-T)z,\ \text{i. e.}\ (\lambda_0-T)(z-x)=0.$$

Since $(\bar{G})^c\cap\rho(T|Y)\cap S^c\subseteq\rho(T|(Z+Y))$, we have $z-x=0$, i. e. $z=x\in Y$. Therefore $[z+y]_Y=0$ and so $\lambda_0-T^Y|(Z+Y)/Y$ is injective.

Thus we have proved that if $\lambda_0\notin(((\bar{G})^c\cap\sigma_e(T|Y))\cup((\bar{G})^c\cap\rho(T|Y)\cap S^c))^c=\bar{G}\cup(S\backslash\sigma_e(T|Y))$, then $\lambda_0-T^Y|(Z+Y)/Y$ is injective.

Our proof is complete.

Theorem 5. *If $T\in C(X)$ is strongly S-decomposable, where S is a closed subset of $\sigma_e(T)$ and $S\neq C_\infty$, then for any $Y\in SM_e(T)$, T^Y is a closed strongly $S\cap\sigma_e(T^Y)$-decomposable operator.*

Proof Without loss of generality we can suppose $\infty\in S$ if T is an unbounded operator. By Lemma 2 T^Y is closed. By Lemma 1, in order to prove this theorem it is sufficient to show $T^Y|\hat{X}_1$ is $S\cap\sigma_e(T^Y)$-decomposable for any $\hat{X}_1\in SM_e(T^Y)$ with $\sigma_e(T^Y|\hat{X}_1)\supseteq S\cap\sigma_e(T^Y)$. Let $X_1=(x\in X, [x]_Y\in\hat{X}_1)$. By Lemma 3, $X_1\in SM_e(T)$ and $\sigma_e(T|X_1)\supseteq S\cup\sigma_e(T|Y)$. Hence $T|X_1$ is strongly S-decomposable and $Y\in SM_e(T|X_1)$. Set $T_1=T|X_1$ and $S_1=S\cap\sigma_e(T^Y)$. Then we have only to prove T_1^Y is $S\cap\sigma_e$ (T^Y)-decomposable.

Assume that $(G_0; G_1, \cdots, G_n)$ is an open $S\cap\sigma_e(T^Y)$-covering of $\sigma_e(T_1^Y)$. Obviously $(S\backslash G_0)\cap\sigma_e(T^Y)=S\cap\sigma_e(T^Y)\backslash G_0\cap\sigma_e(T^Y)=\varnothing$ and $S\backslash G_0$ is closed. If $S\backslash G_0$ is bounded, then there is a bounded open set G_0' such that $G_0'\supseteq S\backslash G_0$ and $\overline{G}_0'\cap\sigma_e(T^Y)=\varnothing$. If $S\backslash G_0$ is unbounded, then $\sigma_e(T^Y)$ is bounded (Otherwise $\infty\in(S\backslash G_0)\cap\sigma_e(T^Y)=\varnothing$, this is a contradiction). Hence there exists a neighborhood G_0' of ∞ such that

$$G_0'\supseteq S\backslash G_0 \quad\text{and}\quad G_0'\cap\sigma_e(T^Y)=\varnothing.$$

Take an open subset G_0'' of C_∞ such that

$$\overline{G}_0''\cap\sigma_e(T_1^Y)=\varnothing \quad\text{and}\quad G_0''\cup G_0\cup\cdots\cup G_n=C_\infty.$$

Put $H_0=G_0\cup G_0'\cup G_0''$. Then $H_0\supseteq S$. Let H_0' be an open subset of C_∞ such that $S\subseteq H_0'\subseteq\overline{H}_0'\subseteq H_0$. Set $H_i=G_i\backslash\overline{H}_0'$ for $i=1, 2, \cdots, n$. Then $(H_i)_{i=0}^n$ is an open S-covering of C_∞. Obviously we can take another open S-covering $(H_i^*)_{i=0}^n$ of C_∞ such that

$$H_0\supseteq\overline{H}_0^*\supseteq H_0^*\supseteq S \text{ and } H_i\supseteq\overline{H}_i^* \text{ for } i=1, 2, \cdots, n.$$

Since T_1 is strongly S-decomposable, we have $X_1=\sum_{i=0}^n X_1(T_1, H_i^*)$.

By Lemma 4, there exists $(Z_i)_{i=1}^n\subseteq SM_e(T_1)$ such that

$$Z_i+Y\in\mathrm{INV}(T_1),\ \sigma_e(T_1|Z_i)\subseteq\overline{H}_i,\ X(T_1, \overline{H}_i^*)\subseteq Z_i,$$

and

$$\sigma_e(T_1^Y|(Z_i+Y)/Y)\subseteq\overline{H}_i\cup(S\backslash\sigma_e(T|Y)).$$

Put $Z_0=X_1(T_1, \overline{H}_0^*\cup\sigma_e(T|Y))$. Then $Y\subseteq Z_0$ and $X_1=\sum_{i=0}^n Z_i=Z_0+\sum_{i=0}^n(Z_i+Y)$.

Therefore

$$X_1/Y=Z_0/Y+\sum_{i=1}^n(Z_i+Y)/Y.$$

By Lemma 4, we have $\sigma_e(T_1^Y|(Z_i+Y)/Y)\subseteq\overline{H}_i\cup(S\backslash\sigma_e(T|Y))$ for $i=1, 2, \cdots, n$. Hence

$$(Z_i+Y)/Y=\hat{W}_i\oplus\hat{V}_i,$$

where $\hat{W}_i$ and $\hat{V}_i$ are invariant subspaces of T_1^Y and satisfy $\sigma_e(T_1^Y|\hat{W}_i)\subseteq\overline{H}_i\subseteq\overline{G}_i$, $\sigma_e(T_1^Y|\hat{V}_i)\subseteq S\backslash\sigma_e(T|Y)$ respectively.

Since $Z_0=X_1(T_1, \overline{H}_0^*\cup\sigma_e(T|Y))\in SM_e(T_1)$, we see that $T_1|Z_0=T|Z_0$ is strongly $S\cap\sigma_e(T|Z_0)$-decomposable. Therefore for any $z\in Z_0$ there exists a $u\in$

$X_1(T_1, \overline{H}_0)\cap Z_0$ and a $v\in Y$ such that $z=u+v$. If $\infty\notin\overline{H}_0$, then $\infty\notin S$. Hence T is bounded and so $\infty\notin\sigma_e(T_1^Y|Z_0/Y)$. If $\lambda\in C\backslash\overline{H}_0$, then there is an $x\in X_1(T_1, \overline{H}_0)\cap Z_0\cap D(T)$ such that $(\lambda-T_1)x=u$ and so $(\lambda-T_1^Y)[x]_Y=[u]_Y=[z]_Y$. Hence $\lambda-T_1^Y|Z_0/Y$ is surjective. Now we show $\lambda-T_1^Y|Z_0/Y$ is injective. Assume $x\in Z_0\cap D(T)$ such that $(\lambda-T_1^Y)[x]_Y=0$. Then $(\lambda-T)x\in Y$. If $\lambda\in\sigma_e(T|Y)$, since Y being an (e) spectral maximal space of T is T-absorbent, we have $x\in Y$ and so $[x]_Y=0$. If $\lambda\in(C\backslash\overline{H}_0)\backslash\sigma_e(T|Y)$, then there is an $x'\in Y\cap D(T)$ such that $(\lambda-T)x'=(\lambda-T)x$, i. e. $(\lambda-T)(x'-x)=0$. Since $x'-x\in Z_0$ and $\lambda\in\rho(T|Z_0)$, we have $x'-x=0$. Therefore $x=x'\in Y$ and consequently $[x]_Y=0$. Hence $\lambda-T_1^Y|Z_0/Y$ is injective. Thus we have proved $\sigma_e(T_1^Y|Z_0/Y)\subseteq\overline{H}_0$. On the other hand, since $\sigma_e(T|Z_0)\supseteq S$, we have

$$\sigma_e(T_1^Y|Z_0/Y)\supseteq\sigma_e(T_1|Z_0)\backslash\sigma_e(T|Y)\supseteq S\backslash\sigma_e(T|Y).$$

Hence $\sigma_e(T_1^Y|\hat{V}_i)\subseteq\sigma_e(T_1^Y|Z_0/Y)$ and so $\hat{V}_i\subseteq Z_0/Y$ for $i=1,2,\cdots,n$. Thus

$$X_1/Y=Z_0/Y+\sum_{i=1}^{n}\hat{W}_i,\ Z_0/Y\in SM_e(T_1^Y),$$

$$\hat{W}_i\in SM_e(T_1^Y),\ \sigma_e(T_1^Y|Z_0/Y)\subseteq\overline{G}_0 \text{ and } \sigma_e(T_0^Y|\hat{W}_i)\subseteq\overline{G}_i \text{ for } i=1,2,\cdots,n.$$

Our proof is finished.

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