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RINGS OF HILBERT MODULAR FORMS ON TOTALLY REAL NUMBER FIELDS WITH ODD DEGREE

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Abstract

E. Thomas and A. T. Vasquez proved the following result: For any totally real cubic number field K and subgroup Γ of modular type of $\mathrm{PSL}_2(O_K)$, the ring of Hilbert modular forms for Γ over K is not Gorenstein ring. In the present paper the author comes to the same conclusion for any totally real number field of odd degree $n\geqslant 3$.

§ 1. Introduction and Statement of Theorem

Let K be a totally real number field of degree n, O_K the ring of integers in K, $G=\mathrm{PSL}_2(O_K)$ the Hilbert (projective) modular group over K, f_i: $K\hookrightarrow R$ $(1\leqslant i\leqslant n)$ the n distinct embeddings of K into the field R of real numbers. For each $\alpha\in K$, let $\alpha^{(i)}=f_i(\alpha)$ $(1\leqslant i\leqslant n)$. Let H be the complex upper half plane. We define the action of $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G$ on H^n in following way: for $z=(z_1,\cdots,z_n)\in H^n$,

$$g(z)=\left(\frac{a^{(1)}z_1+b^{(1)}}{c^{(1)}z_1+d^{(1)}},\cdots,\frac{a^{(n)}z_n+b^{(n)}}{c^{(n)}z_n+d^{(n)}}\right).$$

Let Γ be a subgroup of G. A holomorphic function f: $H^n\to\mathbf{C}$ is called a modular form of weight 2k for Γ over K if

$$f(g(z))=\prod_{i=1}^{n}(c^{(i)}z_i+d^{(i)})^{2k}f(z)$$

for each $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$ and $z=(z_1,\cdots,z_n)\in H^n$ (if $n=1$, $K=Q$, we also must assume that f is "holomorphic at the cusps").

Let $(M_\Gamma)_k$ be the complex vector space of Hilbert modular forms of weight $2k$ for Γ over K. Then

$$M_\Gamma=\sum_{k\geqslant 0}(M_\Gamma)_k,\ (M_\Gamma)_0=\mathbf{C}$$

is a graded, finitely generated C-algebra which is called the ring of Hilbert modular forms for Γ over K. One of fundamental problems in modular form theory is to

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determine the structure of M_Γ for given K and Γ. When $n=1$, $K=Q$ and $\Gamma=G=\mathrm{PSL}_2(Z)$, it is a classical result that $M_G=\mathbf{C}[E_2, E_3]$ where E_i is the Eisenstein series of weight $2i$. When $n=2$ ($K=Q(\sqrt{D})$), only some scattered results have been obtained. More exactly speaking, Hirzebruch, Zagier and van der Geer determined the structure of M_G for the cases $D=5, 8, 13, 24$ in [2, 3, 4, 5]. Among them the first three rings are complete intersection rings, but the last one is not.

Generally, let $R=\sum_{k\geq 0} R_k$ be a graded, finitely generated C-algebra, $d=\dim R$ the Krull dimension of R, d_k the dimension of C-vector space R_k. We call the formal power series

$$H(R,\lambda)=\sum_{k\geq 0} d_k\lambda^k$$

the Hilbert series of the graded C-algebra R. It is well-known that $H(R,\lambda)$ is a rational function in λ and d equals the order of poles of $H(R,\lambda)$ at $\lambda=1$. (see the book [1] for detail). By using the Noether's normalization theorem, we know that there exist homogeneous elements $\theta_1, \cdots, \theta_d \in R$ of positive degree such that R is a finitely generated $\mathbf{C}[\theta_1,\cdots,\theta_d]$-module. If R is a free $\mathbf{C}[\theta_1,\cdots,\theta_d]$-module, then we call R a Cohen–Macaulay ring. If R is a Cohen–Macaulay ring and there exists $l\in\mathbf{Z}$ such that

$$H(R,\lambda^{-1})=(-1)^d\lambda^l H(R,\lambda),$$

then we call it a Gorenstein ring(this is not the original definition, but is equivalent with the original one, see Stanley [7]). At last, as a finitely generated C-algebra, R has the following form

$$R=\mathbf{C}[x_1,\cdots,x_s]/I,$$

where I is some ideal of the polynomial ring $\mathbf{C}[x_1,\cdots,x_s]$. Let r be the minimal number of generating elements of I. If $d=\dim R=s-r$, then we call R a complete intersection ring. It is well-known that

complete intersection⇒Gorenstein⇒Cohen–Macaulay.

For the properties and meanings of these types of ring in ring theory and algebraic geometry, see Stanley [6,7].

Now we can state more results in addition to above-mentioned Hirsebruch, Zagier and Van der Geer's results. Thomas and Vasquez[9] has recently proved that

(I) For $K=Q(\sqrt{D})$, $D\neq 12$, M_G is a complete intersection ring⇔$D=5$, 8, or 13.

A subgroup Γ of $G=\mathrm{PSL}_2(O_K)$ is called modular type if $\Gamma=G$ or Γ is torsion-free.

(II) Let Γ be a subgroup of modular type for a totally real cubic number field. Then the ring M_Γ is never Gorenstein ring (thus M_Γ is never complete intersection ring).

In the present paper we prove, that the result (II) hold for any totally real number field of odd degree $n\geqslant 3$. In other words, we are going to prove

Theorem. *For any totally real number field K of odd degree $n\geqslant 3$ and subgroup Γ of modular type of $G=PSL_2(O_k)$, M_Γ is never Gorenstein ring (thus M_Γ is never complete intersection ring).*

By the above definition for Gorenstein ring, we know that in order to prove the Theorem it is sufficient to prove the

Proposition (*). *Under the assumptions in Theorem, there does not exist $l\in\mathbf{Z}$ such that*

$$H(M_\Gamma, \lambda^{-1})=(-1)^{\dim M_\Gamma}\lambda^l H(M_\Gamma, \lambda).$$

To do that we need to deduce dimension formulas for dim $(M_\Gamma)_k$ $(k\geqslant 0)$ in § 2. Then we get sufficient information on Hilbert series $H(M_\Gamma, \lambda)$, so that we can prove the Proposition (*) and the Theorem in § 3.

§ 2. Dimension Formulas

From now on we let K be a totally real number field with odd degree $[K: Q]=n=2m+1\geqslant 3$, Γ a subgroup of modular type of $G=\mathrm{PSL}_2(O_k)$. Shimizu obtained the following dimension formula (see [9], (2, 1), (2, 2) and (2,5))

$$\dim (M_\Gamma)_k=h+\frac{(-1)^n(2k-1)^n e}{2^{n-1}}\zeta_k(-1)+\sum_\tau a(\tau)\gamma_k(\tau) \text{ for } k\geqslant 2, \tag{1}$$

where $e=[G: \Gamma]$, h=number of cusps of the fundamental domain H^n/Γ, $\zeta_k(s)$=the Dedekind zeta function for K. The last term $\sum_\tau$ comes from the fixed points on H^n/Γ.Let x be a fixed point on H^n/Γ (this means that the fixed subgroup $\Gamma_x\neq\{\pm I\}$), the $r=|\Gamma_x|\geqslant 2$ is also called the order of the fixed point x. With each fixed point x we associate a unique $(n+1)$-tuple

$$\tau=(r; 1, q_2, \cdots, q_n),$$

which is called the proper type of x. Here $q_2, \cdots, q_n$ are prime to r and viewed aselements in $\mathbf{Z}/r\mathbf{Z}$. Let

$a(\tau)$=the number of equivalent classes of fixed points with proper type τ.

$$\gamma_k(\tau)=\frac{1}{r}\sum_{\substack{\zeta^r=1,\\ \zeta\neq 1}}\zeta^{k(1+q_2+\cdots+q_n)}\Big/(1-\zeta)(1-\zeta^{q_2})\cdots(1-\zeta^{q_n}). \tag{2}$$

Since there are only finite number equivalent classes of fixed points, the sum $\sum_\tau$ on the right hand side of (1) is finite.

For $k=1$ Freitage proved that (see [9], (3,2))

$$\dim (M_\Gamma)_1=(-1)^n(\chi(\Gamma)-1)+h=1-\chi(\Gamma)+h, \tag{3}$$

where $\chi(\Gamma)$ is the arithmetic genus of Hilbert modular variety H^n/Γ, for which we have the Hirzebruch-vignèra's formula (see [8], Theorem 1,1)

$$\chi(\Gamma)=2^{-n}\Big[2e\zeta_k(-1)+\sum_{r\geq 2}a_r(\Gamma)\frac{r-1}{r}\Big], \tag{4}$$

where $a_r(\Gamma)$ = the number of Γ-equivalent classes of fixed points with order r. From the definition of $a(\tau)$ we know that

$$a_r(\Gamma)=\sum_{q_2,\cdots,q_n}a(\tau).\quad \tau=(r;\ 1,\ q_2,\cdots,\ q_n). \tag{5}$$

At last, for $k=0$ we have $\dim(M_\Gamma)_0=\dim \mathbf{C}=1$. Thus we have the dimension formula of $\dim(M_\Gamma)_k$ for each $k\geq 0$. But in order to prove our Theorem we need to clear up the sum $\sum_\tau$ in (1). Namely, we need to determine

(1) for which $r\geq 2$ the H^n/Γ has fixed point of order r;

(2) what kind of proper type $\tau=(r;\ 1,\ q_2,\cdots,q_n)$ a fixed point of order r may have and what $a(\tau)$ may be.

Now we answer these problems. At first, if Γ is torsion-free, by the definition we know that there is no any kind of fixed point on $H^n/\Gamma$and all $a(\tau)=0$, thus $\sum_\tau=0$. Thus the only non-trivial case is $\Gamma=G$.

Lemma 1. *Let K be a totally real number field with odd degree*

$$[K:Q]=n=2m+1\geq 3.$$

If H^n/G has a fixed point of order $r\geq 2$, then

(1) $r=2$ or p^l (p is an odd prime, $l\geq 1$), and $\varphi(p^l)\mid 2(2m+1)$.

(2) The proper type of a fixed point with order $r=p^l$ has the form

$$\tau=(p^l;1,\pm g,\ \pm g^2,\ \cdots,\ \pm g^{n-1}),$$

where g is a primitive root mod p^l. Moreover, $a(\tau)=\frac{1}{2^{n-1}}a_r(G)$ for each such kind of τ.

Proof (1) If x is a fixed point of order r, then $Q(\zeta_r+\zeta_r^{-1})$ is a subfield of K, $\zeta_r=e^{\frac{2\pi i}{r}}$(see [10], lemma 1.8). Thus $[Q(\zeta_r+\zeta_r^{-1}):Q]\mid[K:Q]=2m+1$. If $r>2$, it is well konwn that $r\not\equiv 2\pmod 4$ and $[Q\zeta_r+\zeta_r^{-1}):Q]=\frac{1}{2}\varphi(r)$, Then $\frac{1}{2}\varphi(r)$ is odd and r is a power of some odd prime number. So we proved (1).

(2) For $r=p^l$ ($p\geq 3$, $l\geq 1$) we can show that the proper type of x has the form $\tau=(p^l;\ 1,\ \pm g,\ \pm g^2,\ \cdots,\ \pm g^{n-1})$ by the argument in the proof of [9], proposition (2,10) ([9], pr oposition (2,10) is concerned with $n=3$, but the proof works for any odd $n=[K:Q]\geq 3$). The last assertion of(2) comes from the Prestel's results ([9],

(2.8) and (2.9)) directly.

Suppose that H^n/Γ has a fixed point of order r. For $r=2$ there is only one proper type $\tau=(2;1,1,\cdots,1)$ and $\gamma_k(\tau)=\frac{1}{2}\cdot\frac{(-1)^{k_n}}{(1-(-1))^n}=(-1)^k 2^{-(n+1)}$, so the fixed points of order 2 contribute $a_2(\Gamma)(-1)^k/2^{n+1}$ to the term $\sum_\tau$ of (1). For $r=p^l$ ($p\geqslant 3$, $l\geqslant 1$), from Lemma 1 we know that the fixed points of order p^l contribute $\frac{a_r(r)}{r\cdot 2^{n-1}}$ $A_k(r)$ to $\sum_\tau$, where

$$A_k(r)=\sum_{\substack{q_i=\pm g^{i-1}\\2\leqslant i\leqslant n}}\sum_{\substack{\zeta^r=1\\\zeta\neq 1}}\zeta^{k(1+q_2+\cdots+q_n)}/(1-\zeta)(1-\zeta^{q_2})\cdots(1-\zeta^{q_n}). \tag{6}$$

Therefore the formula (1) becomes (for $k\geqslant 2$)

$$\dim(M_\Gamma)_k=h-\frac{(2k-1)^n e}{2^{n-1}}\zeta_k(-1)+a_2(\Gamma)(-1)^k/2^{n+1}+\sum_{\substack{r=p^l\\\varphi(p^l)\mid 2n}}A_k(r)\frac{a_r(\Gamma)}{r\cdot 2^{n-1}}. \tag{7}$$

On the other hand, for $k=1$ we obtain from (3) and (4)

$$\dim(M_\Gamma)_1=1+h-\frac{e}{2^{n-1}}\zeta_k(-1)-\frac{1}{2^{n+1}}a_2(\Gamma)-\frac{1}{2^n}\sum_{r\geqslant 3}a_r(\Gamma)\frac{r-1}{r}. \tag{8}$$

But in the cases of $r=p^l$ ($p\geqslant 3$, $l\geqslant 1$), from (6) we know that

$$\begin{aligned}A_1(r)&=\sum_{\substack{q_i=\pm g^{i-1}\\2\leqslant i\leqslant n}}\sum_{\substack{\zeta^r=1\\\zeta\neq 1}}\frac{\zeta}{1-\zeta}\cdot\frac{\zeta^{q_2}}{1-\zeta^{q_2}}\cdots\frac{\zeta^{q_n}}{1-\zeta^{q_n}}\\
&=\sum_{\substack{\zeta^r=1\\\zeta\neq 1}}\frac{\zeta}{1-\zeta}\left(\frac{\zeta^g}{1-\zeta^g}+\frac{\zeta^{-g}}{1-\zeta^{-g}}\right)\left(\frac{\zeta^{g^2}}{1-\zeta^{g^2}}+\frac{\zeta^{-g^2}}{1-\zeta^{-g^2}}\right)\cdots\left(\frac{\zeta^{g^{n-1}}}{1-\zeta^{g^{n-1}}}+\frac{\zeta^{-g^{n-1}}}{1-\zeta^{-g^{n-1}}}\right)\\
&=\sum_{\substack{\zeta^r=1\\\zeta\neq 1}}\frac{\zeta}{1-\zeta}\left(\text{Since }\frac{\zeta^t}{1-\zeta^t}+\frac{\zeta^{-t}}{1-\zeta^{-t}}=-1\text{ and }2\mid n-1\right).\\
&=\sum_{1\leqslant l\leqslant\frac{r-1}{2}}\left(\frac{\zeta^l}{1-\zeta^l}+\frac{\zeta^{-l}}{1-\zeta^{-l}}\right)=-\frac{r-1}{2}.\end{aligned}$$

From this and (8) we know that formula (7) also holds for $k=1$ but plus one at the right hand side of (7).

§ 3. The Proof of Theorem

For proving Theorem we need to write down the Hilbert series $H(M_\Gamma,\lambda)$. It is clear that

$$h\lambda+h\lambda^2+\cdots+h\lambda^n+\cdots=\frac{h\lambda}{1-\lambda}, \tag{9}$$

$$\sum_{k=1}^{\infty}a_2(\Gamma)\frac{(-1)^k\lambda^k}{2^{n+1}}=-\frac{a_2(\Gamma)}{2^{n+1}}\frac{\lambda}{1+\lambda}. \tag{10}$$

From formula (6) we can see that $A_k(r)=A_{k+r}(r)$, thus for each $r=p^l$ ($p\geqslant 3$, $l\geqslant 1$) and $\varphi(p^l)\mid 2n$ we have

$$\sum_{k=1}^{\infty} A_k(r)\lambda^k = \frac{Q_r(\lambda)}{1-\lambda^r}, \tag{11}$$

where

$$Q_r(\lambda) = A_1(r)\lambda + A_2(r)\lambda^2 + \cdots + A_r(r)\lambda^r. \tag{12}$$

At last, from elementary algebra we get

$$\sum_{k\geq 1}^{\infty} (2k-1)^n \lambda^k = \frac{P_n(\lambda)}{(1-\lambda)^{n+1}}, \tag{13}$$

where $P_n(\lambda)$ is a polynomial and $(1-\lambda)\nmid P_n(\lambda)$. From (9)–(13) and the statement at the end of § 2 we know that

$$H(M_\Gamma, \lambda) = 1 + \sum_{k=1}^{\infty} \dim (M_\Gamma)_k \lambda^k = \frac{h\lambda}{1-\lambda} + (1+\lambda) - \frac{e}{2^{n-1}} \zeta_k(-1) \frac{P_n(\lambda)}{(1-\lambda)^{n+1}}$$

$$- \frac{a_2(\Gamma)}{2^{n+1}} \frac{\lambda}{1+\lambda} + \sum_{\substack{r=p^l \\ a_r(\Gamma)\neq 0}} \frac{a_r(\Gamma)}{r\cdot 2^{n-1}} \frac{Q_r(\lambda)}{1-\lambda^r}. \tag{14}$$

We need further imformation about the polynomials $Q_r(\lambda)$ and $P_n(\lambda)$.

Lemma 2. $\lambda^{r+1} Q_r(\lambda^{-1}) = -Q_r(\lambda)$.

Proof From formula (9) we know that for $1\leq k\leq r$,

$$A_k(r) = \sum_{\substack{q_i=\pm g^{t-1} \\ 2\leq i\leq n}} \sum_{\substack{\zeta^r=1 \\ \zeta\neq 1}} \frac{\zeta^{k(1+q_2+\cdots+q_n)}}{(1-\zeta)(1-\zeta^{q_2})\cdots(1-\zeta^{q_n})}$$

$$= \sum_{\substack{q_i=\pm g^{t-1} \\ 2\leq i\leq n}} \sum_{\substack{\zeta^r=1 \\ \zeta\neq 1}} \frac{\zeta^{-k(1+q_2+\cdots+q_n)}}{(1-\zeta^{-1})(1-\zeta^{-q_2})\cdots(1-\zeta^{-q_n})}$$

$$= \sum_{\substack{q_i=\pm g^{t-1} \\ 2\leq i\leq n}} \sum_{\substack{\zeta^r=1 \\ \zeta\neq 1}} \frac{\zeta^{(r+1-k)(1+q_2+\cdots+q_n)}}{(\zeta-1)(\zeta^{q_2}-1)\cdots(\zeta^{q_n}-1)}$$

$$= (-1)^n \sum_{\substack{q_i=\pm g^{t-1} \\ 2\leq i\leq n}} \sum_{\substack{\zeta^r=1 \\ \zeta\neq 1}} \frac{\zeta^{(r+1-k)(1+q_2++q_n)}}{(1-\zeta)(1-\zeta^{q_2})\cdots(1-\zeta^{q_n})} = -A_{r+1-k}(r).$$

From this and the expression (12) for $Q_r(\lambda)$ we can complete the proof of Lemma 2.

Lemma 3. *For* $n\geq 1$, $\lambda^{n+2} P_k(\lambda^{-1}) = P_n(\lambda)$ *and* $\deg P_n(\lambda) = n+1$.

Proof Let $\sum_{k=1}^{\infty} k^n \lambda^k = \frac{R_n(\lambda)}{(1-\lambda)^{n+1}}$. Then $R_n(\lambda)$ is a polynomial and $\deg R_n(\lambda) \leq n$ by elementary algebra. Differentiating both sides of above equality, we get the recursion formula for $R_n(\lambda)$:

$$R_1(\lambda) = \lambda,\ R_{n+1}(\lambda) = \lambda[R_n'(\lambda)(1-\lambda) + (n+1)R_n(\lambda)].$$

From this, it is easy to prove by induction that

$$\deg R_n(\lambda) = n, \quad \lambda^{n+1} R_n(\lambda^{-1}) = R_n(\lambda).$$

Moreover, from (13) we have

$$\frac{P_n(\lambda^2)}{(1-\lambda^2)^{n+1}} = \sum_{k=1}^{\infty} (2k-1)^n \lambda^{2k} = \lambda\left(\sum_{k=1}^{\infty} k^n\lambda^k - \sum_{k=1}^{\infty} (2k)^n \lambda^{2k}\right)$$

$$= \lambda\left(\frac{R_n(\lambda)}{(1-\lambda)^{n+1}} - \frac{2^n R_n(\lambda^2)}{(1-\lambda^2)^{n+1}}\right) = \frac{\lambda[(1+\lambda)^{n+1} R_n(\lambda) - 2^n R_n(\lambda^2)]}{(1-\lambda^2)^{n+1}}$$

Thus $P_n(\lambda^2) = \lambda[(1+\lambda)^{n+1} R_n(\lambda) - 2^n R_n(\lambda^2)]$ and $\deg R_n(\lambda) = n$

$$\Rightarrow \deg P_n(\lambda) = \frac{1}{2}(1+n+n+1) = n+1.$$

By using $\lambda^{n+1}R_n(\lambda^{-1})=R_n(\lambda)$ we get

$$\begin{aligned}\lambda^{2n+4}P_n(\lambda^{-2})&=\lambda^{2n+3}[\lambda^{-n-1}(1+\lambda)^{n+1}R_n(\lambda^{-1})-2^nR_n(\lambda^{-2})]\\&=\lambda^{n+2}(1+\lambda)^{n+1}\lambda^{-n-1}R_n(\lambda)-2^n\lambda^{2n+3}\lambda^{-2n-2}R_n(\lambda^2)\\&=\lambda[(1+\lambda)^{n+1}R_n(\lambda)-2^nR_n(\lambda^2)]=P_n(\lambda^2).\end{aligned}$$

Thus $\lambda^{n+2}P_n(\lambda^{-1})=P_n(\lambda)$.

Now we continue to examine the $H(M_\Gamma,\lambda)$. Let

$$M(\lambda)=(1-\lambda)^n(1-\lambda^2)\prod_{\substack{r\geq 3\\ a_r(\Gamma)\neq 0}}(1-\lambda^r)=1+\cdots+(-1)^\alpha\lambda^\beta. \tag{15}$$

Then $\alpha=n+1+\sum_{\substack{2\nmid r\\ a_r(\Gamma)\neq 0}}1$, $\beta=\deg M(\lambda)=n+2+\sum_{\substack{2\nmid r\\ a_r(\Gamma)\neq 0}}r\equiv\alpha+1 \pmod 2$ and from (14) we know that

$$H(M_\Gamma,\lambda)=N(\lambda)/M(\lambda),$$

where $N(\lambda)$ is the polynomial

$$\begin{aligned}N(\lambda)=&h\frac{\lambda M(\lambda)}{1-\lambda}+(1+\lambda)M(\lambda)-\frac{e}{2^{n-1}}\zeta_k(-1)P_n(\lambda)\frac{M(\lambda)}{(1-\lambda)^{n+1}}\\&-\frac{a_2(\Gamma)}{2^{n+1}}\frac{\lambda M(\lambda)}{1+\lambda}+\sum_{\substack{2\nmid r\\ a_r(\Gamma)\neq 0}}\frac{a_r(\Gamma)}{r\cdot 2^{n-1}}\frac{Q_r(\lambda)M(\lambda)}{1-\lambda^r}.\end{aligned} \tag{16}$$

Since $\deg P_n(\lambda)=n+1$, $\deg Q_r(\lambda)<r$, $P_n(0)=Q_r(0)=0$, from (16) we know that $\deg N(\lambda)=\beta+1$, the constant term and leading term of $N(\lambda)$ are 1 and $(-1)^\alpha\lambda^{\beta+1}$ respectively, and both come from the term $(1+\lambda)M(\lambda)$ of the right hand side of (15). Therefore from (15) and (16), $N(\lambda)$ can be written as

$$N(\lambda)=h(\lambda+\cdots+(-1)^{\alpha+1}\lambda^\beta)+(1+c_1\lambda+\cdots+c_\beta\lambda^\beta+(-1)^\alpha\lambda^{\beta+1}), \tag{17}$$

where the first term is $h\dfrac{\lambda M(\lambda)}{1-\lambda}$ and the second term is the sum of remaining terms of right hand side of (16). From (15) we know that $\lambda^\beta M(\lambda^{-1})=(-1)^\alpha M(\lambda)$. Then from Lemmas 2,3 we know that each term (denoted by $L(\lambda)$) of (16) except the first one satisfies the relation

$$\lambda^{\beta+1}L(\lambda^{-1})=L(\lambda)(-1)^\alpha.$$

So does the sum $1+c_1\lambda+\cdots+c_\beta\lambda^\beta+(-1)^\alpha\lambda^{\beta+1}$ of these terms. From this we have

$$c_1=(-1)^\alpha c_\beta. \tag{18}$$

Now we can complete the proof of Theorem easily. From $(1-\lambda)\nmid P_n(\lambda)$, formula (14) and the well-known fact $\zeta_k(-1)\neq 0$, we see that $H(M_\Gamma,\lambda)$ has a pole of order $n+1$ at $\lambda=1$, i. e. $\dim M_\Gamma=n+1\equiv 0 \pmod 2$. As we said at the end of § 1, in order to prove Theorem it is sufficient to prove the proposition (*) in § 1, i. e. we have to prove that there is no $l\in\mathbf{Z}$ such that

$$H(M_\Gamma,\lambda^{-1})=\lambda^l H(M_\Gamma,\lambda). \tag{19}$$

Suppose that there is $l\in\mathbf{Z}$ such that (19) hold. From (17) we have

$$H(M_\Gamma,\lambda)=\frac{1+(c_1+h)\lambda+\cdots+(c_\beta+(-1)^{\alpha+1}h)\lambda^\beta+(-1)^\alpha\lambda^{\beta+1}}{M(\lambda)}.$$

From (19) and $\lambda^\beta M(\lambda^{-1})=(-1)^\alpha M(\lambda)$ we know that l must be equal to 1 and $c_1+h=(-1)^\alpha(c_\beta+(-1)^{\alpha+1}h)=(-1)^\alpha c_\beta-h$. But from (18) we know that $c_1=c_\beta(-1)^\alpha$, therefore $h=0$ which is impossible since h=the number of cusps on $H^n/\Gamma\geqslant 1$. So we complete the proof of Theorem.

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