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PERFECT PARTITIONS

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Abstract

In this paper, the author gives a necessary and sufficient condition on the parts of a partition α for α to be perfect and a necessary and sufficient condition on n for which there exists a non-trivial perfect n-partition. Besides, the author gives some recurrence formulas and computing methods for the number $\operatorname{Per}(n)$ of perfect n-partitions and lists the values of $\operatorname{Per}(n)$ up to 100 and for some other cases of n.

A partition of n is said to be perfect when it contains just one partition of every number up to n[1]. For any number n, the n-partition $(1, 1, \cdots, 1)$ is always perfect, and for any odd number $n=2k+1$, the n-partition $(1, \underbrace{2, \cdots, 2}_{k})$ is always perfect too. These two kinds of perfect partitions are called trivial perfect partitions. For arbitrary n, a non-trivial perfect n-partition does not always exist. For example, when $n=4$, 6 or 10 there is no non-trivial perfect n-pariition. In this paper we discuss conditions for the existence of a non-trivial perfect n-partition, obtain some recurrence formulas for the number of perfect n-partitions $\operatorname{Per}(n)$, and give some methods for calculating $\operatorname{Per}(n)$.

Theorem 1. *The n-partition $\alpha=(a_1, a_2, \cdots, a_s)$ is perfect iff*

$$a_1=1,\ a_t=a_{t-1} \text{ or } a_1+a_2+\cdots+a_{t-1}+1, \quad 1<t\leqslant s. \tag{1}$$

Proof Induction on s. There is nothing to say for $s=1$. When $s=2$, (a_1, a_2) is perfect iff $a_1=1$, $a_2=1$ or 2, i.e., $a_2=a_1$ or a_1+1. By induction, $(a_1, a_2, \cdots, a_m)$ $(m<s)$ is perfect iff

$$a_1=1,\ a_t=a_{t-1} \text{ or } a_1+a_2+\cdots+a_{t-1}+1, \quad 1<t\leqslant m.$$

Suppose that $a_1, a_2, \cdots, a_s$ satisfy condition (1). Then $(a_1. a_2, \cdots, a_t)$ $(1\leqslant t\leqslant s-1)$ are all perfect. If $a_s=1$, then $\alpha=(1, 1, \cdots, 1)$ is perfect obviously. If $a_s>1$, we may suppose $1\leqslant a_r<a_{r+1}=\cdots=a_s=a$ $(r\leqslant s-1)$, $a_{r+1}=a_1+a_2+\cdots+a_r+1$. For any number m not exceeding n, we write

$$m=ka+m_0 \quad (0\leqslant m_0<a,\ 0\leqslant k\leqslant s-r).$$

When $m_0=0$, $(\underbrace{a, \cdots, a}_{k})$ is the unique m-subpartition of α. If $m_0>0$, any m_0-

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partition cannot have a_{r+1} as a part. Hence the unique m_0-subpartition β of $(a_1, \cdots, a_r)$ is also the unique m_0-subpartition of α, and the subpartition

$$(\beta, \underbrace{a, \cdots, a}_{k})$$

is the unique m-subpartition of α. Therefore α is perfect.

Suppose $\alpha=(a_1, \cdots, a_s)$ is perfect. Then $a_1=1$. Assume $a_1=\cdots=a_v=1<a_{v+1}$, $v<s$. Since $a_{v+1}\leqslant a_{v+2}\leqslant\cdots\leqslant a_s$, and α has $(v+1)$-subpartition, a_{v+1} must be equal to $v+1=a_1+\cdots+a_v+1$. Assume that $a_1=1$, $a_t=a_{t-1}$ or $a_1+a_2+\cdots+a_{t-1}+1$ for all $1<t<u$, and $a_u\neq a_{u+1}$. Then $a_{u+1}\leqslant a_1+a_2+\cdots+a_v+1$, for otherwise, α cannot have any subpartition of number $a_1+a_2+\cdots+a_u+1$. If $a_{u+1}\leqslant a_1+a_2+\cdots a_u$, then (a_{u+1}) and the a_{u+1}-subpartition of $(a_1, \cdots, a_u)$ are two distinct a_{u+1}-subpartitions of the perfect partition α, this is impossible. Therefore, $a_{u+1}=a_1+\cdots+a_u+1$. The theorem is proved.

Corollary 1. *If $\alpha=(a_1, a_2, \cdots, a_s)$ is perfect, then $(a_1, a_2, \cdots, a_t)$ are all perfect for $1\leqslant t\leqslant s$.*

Corollary 2. *The only non-trivial, perfect partition into distinct parts are* $(1, 2, 2^2, \cdots, 2^k)$, $k=2, 3, \cdots$.

Hence when and only when $n=2^{k+1}-1$ $(k=2, 3, \cdots)$, there exists non-trivial perfect partition of n into distinct parts.

Theorem 2. *A necessary and sufficient conedition under which there exists a non-trivial perfect partition of n $(n>3)$ is that $n+1$ is not a prime.*

Proof If $\alpha=(a_1, \cdots, a_s)$ is a non-trivial perfect n-partition, then for some r, $1<r<s$, $a_1+a_2+\cdots+a_r+1=a_{r+1}=\cdots=a_s$. Therefore

$$n=(s-r)a_s+a_s-1=(s-r+1)a_s-1,$$

$$n+1=(s-r+1)a_s,$$

and

$$1<a_s<n.$$

Hence $n+1$ is not a prime.

If $n+1=ak$, $1<a<n+1$. Since $n+1>4$, we may assume $a>2$. Then

$$(\underbrace{1, \cdots, 1}_{a-1}, \underbrace{a, \cdots, a}_{k-1})$$

is a non-trivial perfect partition of n.

Corollary 3. *If $n>3$ is an odd number, then there exists non-trivial perfect n-partition.*

Corollary 4. *There exists a perfect n-partition in which the greatest part is a iff a is a proper factor of $n+1$.*

Corollary 5. *When n is even, the greatest part of any perfect n-partition must be odd, and its multiplicity*[3] *is even. When n is odd, if the greatest part of a perfect*

n-partition is odd, then its multiplicity must be odd also.

Denote by $\mathrm{Per}(n)$ the number of perfect n-partitions. We deduce a recurrence formula for $\mathrm{Per}(n)$.

Theorem 3.

$$\mathrm{Per}(n)=\sum_{\substack{1<a<n+1\\ a\mid n+1}}\mathrm{Per}(a-1)=\sum_{\substack{1<a<\sqrt{n+1}\\ a\mid n+1}}\left\{\mathrm{Per}(a-1)+\mathrm{Per}\left(\frac{n+1}{a}-1\right)\right\}$$
$$+1+\mathrm{Per}(\sqrt{n+1}-1), \qquad (2)$$

where $\mathrm{Per}(0)=1$ *and* $\mathrm{Per}(\sqrt{n+1}-1)=0$ *if* $\sqrt{n+1}$ *is not an integer.*

Proof From Theorem 1 and Theorem 2, every perfect partition of n can be expressed as

$$(a_1, \cdots, a_r, \underbrace{a, \cdots, a}_{k}),$$

where $k>0$, a is a proper factor of $n+1$, and $a=a_1+a_1+\cdots+a_r+1$. Moreover, $(a_1, \cdots, a_r)$ is a perfect partition of $a_1+a_2+\cdots+a_r=a-1$. On the other hand, if a is a proper factor of $n+1$ and $(a_1, \cdots, a_r)$ is a perfect partition of $a-1$, then $(a_1, \cdots, a_r, \underbrace{a, \cdots, a}_{\left(\frac{n+1}{a}-1\right)})$ is a perfect n-partition with greatest part a. Hence the number of perfect n-partitions with greatest part a is equal to the number of perfect $(a-1)$-partitions $\mathrm{Per}(a-1)$. Since the greatest part of any perfect n-partition must be a proper factor of $n+1$, we get the recurrence formula (2).

For any factor a $(1<a<n+1)$ of $n+1$, the term $\mathrm{Per}(a-1)$ in (2) can also be expressed by those $\mathrm{Per}(b-1)$, where b is a proper factor of a, and so on. Because for any prime q, $\mathrm{Per}(q-1)=1$, we see that $\mathrm{Per}(n)$ depends only on the decomposition form of $n+1$, and is independent on the size of n. That is:

Theorem 4. *If* $n=p_1^{k_1}p_2^{k_2}\cdots p_t^{k_t}$, $m=q_1^{k_1}q_2^{k_2}\cdots q_t^{k_t}$, *where* $p_1, \cdots, p_t$; $q_1, \cdots, q_t$ *are distinct primes respectively, then* $\mathrm{Per}(n-1)=\mathrm{Per}(m-1)$.

In the following, we discuss how to calculate the number $\mathrm{Per}(n)$ by use of the factor decomposition of $n+1$.

Therem 5. *For any prime* q, $\mathrm{Per}(q^k-1)=2^{k-1}$.

Proof We use induction on k. If for any h, $0<h<k$, $\mathrm{Per}(q^h-1)=2^{h-1}$, then

$$\mathrm{Per}(q^k-1)=1+\mathrm{Per}(q-1)+\mathrm{Per}(q^2-1)+\cdots+\mathrm{Per}(q^{k-1}-1)$$
$$=1+1+2+\cdots+2^{k-2}=2^{k-1}.$$

Let $q_1, q_2, \cdots, q_t$ be distinct primes. Since $\mathrm{Per}(q_i-1)=1$ and every proper factor of $q_1q_2\cdots q_t$ is of the form $q_{i_1}q_{i_2}\cdots q_{i_h}(h<t)$, and the number of proper factors which are products of h primes is C_t^h, we have the following theorem.

Theorem 6. *If* $q_1, \cdots, q_t$ *are distinct primes, then*

$$\mathrm{Per}(q_1q_2\cdots q_t-1)=\sum_{h=0}^{t-1}C_t^h\,\mathrm{Per}(q_1\cdots q_h-1)=\sum_{\substack{t>h_1>\cdots>h_u\\ h_u=0,1,\\ h_{u-1}=2}}C_t^{h_1}C_{h_1}^{h_2}\cdots C_{h_{u-1}}^{h_u}.$$

Corollary 6. *If $q_1, \cdots, q_t$ are distinct primes, then*

$$\operatorname{Per}(q_1q_2\cdots q_t-1)=(t+1)\operatorname{Per}(q_1q_2\cdots q_{t-1}-1)+\sum_{h=1}^{t-2}C_{t-1}^{h-1}\operatorname{Per}(q_1q_2\cdots q_h-1).$$

Proof From Theorem 6

$$\operatorname{Per}(q_1q_2\cdots q_t-1)=\sum_{h=0}^{t-1}C_t^h\operatorname{Per}(q_1q_2\cdots q_h-1),$$

$$\operatorname{Per}(q_1q_2\cdots q_{t-1}-1)=\sum_{h=0}^{t-2}C_{t-1}^h\operatorname{Per}(q_1q_2\cdots q_h-1).$$

Hence

$$\begin{aligned}&\operatorname{Per}(q_1q_2\cdots q_t-1)-\operatorname{Per}(q_1q_2\cdots q_{t-1}-1)\\&=C_t^{t-1}\operatorname{Per}(q_1q_2\cdots q_{t-1}-1)+\sum_{h=0}^{t-2}(C_t^h-C_{t-1}^h)\operatorname{Per}(q_1q_2\cdots q_h-1)\\&=t\operatorname{Per}(q_1q_2\cdots q_{t-1}-1)+\sum_{h=0}^{t-2}C_{t-1}^{h-1}\operatorname{Per}(q_1q_2\cdots q_h-1).\end{aligned}$$

Thus

$$\operatorname{Per}(q_1\cdots q_t-1)=(t+1)\operatorname{Per}(q_1\cdots q_{t-1}-1)+\sum_{h=0}^{t-2}C_{t-1}^{h-1}\operatorname{Per}(q_1\cdots q_h-1).$$

Let $n=q_1^{k_1}q_2^{k_2}\cdots q_t^{k_t}$, $k_1\geqslant k_2\geqslant\cdots\geqslant k_t>0$; $q_1, \cdots, q_t$ are distinct primes. Any proper factor of n can be expressed as $q_{i_1}^{h_1}q_{i_2}^{h_2}\cdots q_{i_t}^{h_t}$, where $i_1, i_2, \cdots, i_t$ is a permutation of 1, 2, $\cdots$, t; $h_1\geqslant h_2\geqslant\cdots\geqslant h_t\geqslant 0$, $h_i\leqslant k_i$ $(i=1, 2, \cdots, t)$. We say that this factor is of the form $(h_1, h_2, \cdots, h_t)$. Assume

$$h_1=\cdots=h_{i_1}>h_{i_1+1}=\cdots=h_{i_2}>\cdots>h_{i_{u-1}+1}=\cdots=h_{i_u}>h_{i_u+1}=\cdots=h_t,$$

$$h_{i_\rho}\leqslant k_{i_\rho},\ h_{i_{\rho+1}}\leqslant k_{i_\rho+1},\ \rho=1, 2, \cdots u.$$

Then the number of those factors of n, which are of the form $(h_1, h_2, \cdots, h_t)$, is

$$C_{j_1}^{i_1}C_{j_2-i_1}^{i_2-i_1}\cdots C_{j_u-i_{u-1}}^{i_u-i_{u-1}}.$$

We denote this number by $C_{k_1k_2\cdots k_t}^{h_1h_2\cdots h_t}$. Further, we write $(h_1, \cdots, h_t)<(k_1, \cdots, k_t)$ if $h_i\leqslant k_i$ $(i=1, 2, \cdots, t)$ and $(h_1, \cdots, h_t)\neq(k_1, \cdots, k_t)$. Then we have

Theorem 7.

$$\begin{aligned}&\operatorname{Per}(q_1^{k_1}q_2^{k_2}\cdots q^{k_t}-1)\\&=\sum_{(h_1,\cdots,h_t)<(k_1,\cdots,k_t)}C_{k_1k_2\cdots k_t}^{h_1h_2\cdots h_t}\operatorname{Per}(q_1^{h_1}\cdots q_t^{h_t}-1)\\&=\sum_{(h_{u1},\cdots h_{ut})<(k_1,\cdots,k_t)}C_{k_1,\cdots,k_t}^{h_{11}\cdots h_{1t}}C_{h_{11}\cdots h_{1t}}^{h_{21}\cdots h_{2t}}\cdots C_{h_{u-1}1\cdots h_{u-1t}}^{h_{u1}\cdots h_{ut}},\end{aligned}$$

where $\sum\limits_{(h_{u1}\cdots h_{ut})<(k_1\cdots,k_t)}$ *is taken through all* $(h_{u1}, \cdots, h_{ut})<(h_{u-1}, 1, \cdots, h_{u-1t}, t)<\cdots<(h_{11}, \cdots, h_{1t})<(k_1, \cdots, k_t)$, $h_{u1}=0$ *or* 1, $h_{u2}=\cdots=h_{ut}=0$, $h_{u-11}+h_{u-12}=2$.

From these formulas we can see that the number $\operatorname{Per}(n)$ depends on the number of proper factors of $n+1$ and the numbers of proper factors of the proper factors of $n+1$, etc. We say that m is a subfactor of n if for some numbers $m_1, \cdots, m_\rho$ $(\rho\geqslant 0)$, $m<m_1<\cdots<m_\rho<n$,

$$m\,|\,m_1\,|\,m_2\,|\cdots|\,m_\rho\,|\,n.$$

If there are two different series $m_1, \cdots, m_\rho$ $(\rho>0)$ and $m'_1, \cdots, m'_{\rho'}$, $(\rho'>0)$ such that

$$m|m_1|\cdots|m_\rho|n, \quad m<m_1<\cdots<m_\rho<n,$$
$$m|m'_1|\cdots|m'_{\rho'}|n, \quad m<m'_1<\cdots<m'_{\rho'}<n,$$

then m should be counted twice as we enumerate the number of proper subfactors of n. We can calculate $\mathrm{Per}(n)$ by calculating the number of non-unity proper subfactors of $n+1$.

Theorem 8. $\mathrm{Per}(n)$ *is equal to the number of non-unity proper subfactors of* $n+1$ *plus* 1.

Proof From Theorem 3

$$\mathrm{Per}(n)=\sum_{\substack{1<a<n+1\\ a|n+1}}\mathrm{Per}(a-1)+1,$$
$$\mathrm{Per}(a-1)=\sum_{\substack{1<b<a\\ b|a}}\mathrm{Per}(b-1)+1.$$

....

If we regard the term 1 in the expression of $\mathrm{Per}(a-1)$ as the non-unity proper factor a of $n+1$, regard the term 1 in the expression of $\mathrm{Per}(b-1)$ as the non-unity proper factor b of a, etc, then we can get the result from Theorem 3 by induction on n.

Using these methods we calculate the $\mathrm{Per}(n)$ up to 100 and for some other n with $n+1$ expressible as products of a small number of primes. The values of these $\mathrm{Per}(n)$ are listed in the following tables.

Table of values of $\mathrm{Per}(n)$

n	$\mathrm{Per}(n)$	n	$\mathrm{Per}(n)$	n	$\mathrm{Per}(n)$	n	$\mathrm{Per}(n)$	n	$\mathrm{Per}(n)$
1	1	21	3	41	13	61	3	81	3
2	1	22	1	42	1	62	3	82	1
3	2	23	20	43	8	63	32	83	44
4	1	24	2	44	8	64	3	84	3
5	3	25	3	45	3	65	3	85	3
6	1	26	4	46	1	66	1	86	3
7	4	27	8	47	48	67	8	87	20
8	2	28	1	48	2	68	3	88	1
9	3	29	13	49	8	69	13	89	44
10	1	30	1	50	3	70	1	90	3
11	8	31	16	51	8	71	76	91	8
12	1	32	3	52	1	72	1	92	3
13	3	33	3	53	20	73	3	93	3
14	3	34	3	54	3	74	8	94	3
15	8	35	26	55	20	75	8	95	112
16	1	36	1	56	3	76	3	96	1
17	8	37	3	57	3	77	13	97	8
18	1	38	3	58	1	78	1	98	8
19	8	39	20	59	44	79	48	99	26
20	3	40	1	60	1	80	8	100	1

Table of values of Per$(q_1^{k_1}\cdots q_t^{k_t}-1)$

$n+1$	Per(n)	$n+1$	Per(n)	$n+1$	Per(n)	$n+1$	Per(n)
q	1	q^4	8	q^5	15	q^6	32
		$q_1^3q_2$	20	$q_1^4q_2$	48	$q_1^5q_2$	112
q^2	2	$q_1^2q_2^2$	26	$q_1^3q_2^2$	76	$q_1^4q_2^2$	208
q_1q_2	3	$q_1^2q_2q_3$	44	$q_1^3q_2q_3$	132	$q_1^4q_2q_3$	368
		$q_1q_2q_3q_4$	75	$q_1^2q_2^2q_3$	176	$q_1^3q_2^3$	252
q^3	4			$q_1^2q_2q_3q_4$	308	$q_1^3q_2^2q_3$	604
$q_1^2q_2$	8			$q_1q_2q_3q_4q_5$	541	$q_1^3q_2q_3q_4$	1076
$q_1q_2q_3$	13					$q_1^2q_2^2q_3^2$	818
						$q_1^2q_2^2q_3q_4$	1460
						$q_1^2q_2q_3q_4q_5$	2612
						$q_1q_2q_3q_4q_5q_6$	4683

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