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ON THE PRODUCT MARTINGALE MEASURE AND MULTIPLE STOCHASTIC INTEGRAL

HUANG ZHIYUAN (黄志远)*

Abstract

A theorem of Fubini type which reduces a multiple stochastic integral on an abstract product space to iterated stochastic integrals is proved under the condition that at least one of those Doléans measures of the component martingale measures should be admissible.

In a recent paper[1], we defined the stochastic integral of a predictable random function with respect to a martingale measure on a general topological measurable space. Since the definition is suitable for both one dimensional and multidimensional cases, one can ask the question about the relation between a product space and its component spaces. In this connection we shall prove a theorem of Fubini type which reduces a multiple stochastic integral to iterated stochastic integrals.

In this paper, we shall use all notations in [1] without explanation.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space, U_1 and U_2 be two topological spaces with their Borel σ-algebras $\mathscr{B}_1$, $\mathscr{B}_2$, algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ and their sublattices $\mathscr{C}_1$, $\mathscr{C}_2$ respectively. Let $\{\widetilde{\mathscr{F}}^1_c, c\in\mathscr{C}_1\}$ and $\{\widetilde{\mathscr{F}}^2_c, c\in\mathscr{C}_2\}$ be two independent families of sub-σ-algebras of $\mathscr{F}$ each satisfying the conditions listed in [1]. Let $U=U_1\times U_2$, $\mathscr{B}=\mathscr{B}_1\times\mathscr{B}_2$, $\mathfrak{A}=\mathfrak{A}_1\times\mathfrak{A}_2$ and $\mathscr{C}=\mathscr{C}_1\times\mathscr{C}_2$ be the product space, product σ-algebra, algebra and product lattice respectively.

Define $\mathscr{F}^1_c=\widetilde{\mathscr{F}}^1_c\vee\widetilde{\mathscr{F}}^2_{u_2}$ for $c\in\mathscr{C}_1$ and $\mathscr{F}^2_c=\widetilde{\mathscr{F}}^1_{u_1}\vee\widetilde{\mathscr{F}}^2_c$ for $c\in\mathscr{C}_2$. If μ_i is a square integrable martingale measure with respect to $\{\widetilde{\mathscr{F}}^i_c, c\in\mathscr{C}_i\}$ (denoted by $\mu_i\in M^2(U_i)$, $i=1, 2$), then it is also such a measure with respect to $\{\mathscr{F}^i_c, c\in\mathscr{C}_i\}$ $(i=1, 2)$.

We first define $\mathscr{F}_{c_1\times c_2}=\mathscr{F}^1_{c_1}\cap\mathscr{F}^2_{c_2}$ for $c_1\in\mathscr{C}_1$ and $c_2\in\mathscr{C}_2$, and then extend it to $\mathscr{C}=\mathscr{C}_1\times\mathscr{C}_2$. It is easy to verify that $\{\mathscr{F}_c, c\in\mathscr{C}\}$ satisfies the conditions (F. 1)$\sim$(F. 3) in [1]. Moreover, $\mathscr{F}_c$ and $\mathscr{F}_{c'}$ are conditionally independent for given $\mathscr{F}_{cc'}$ for all $c, c'\in\mathscr{C}$.

By choosing $t(A_1\times A_2)=t(A_1)\times t(A_2)$ for $A_1\in\mathfrak{A}_1$, $A_2\in\mathfrak{A}_2$) and

$$t\Big(\sum_{k=1}^{n}(A_1^k\times A_2^k)\Big)=\bigcap_{k=1}^{n}t(A_1^k\times A_2^k)$$

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* Department of Mathematics, Wuhan University, Wuhan, Hubei, China.

for $A_1^k\in\mathfrak{A}_1$, $A_2^k\in\mathfrak{A}_2$ $(k=1, 2, \cdots, n)$, it is easy to see that the conditions (i)$\sim$(iii) listed in [1] are satisfied.

Since μ_1 and μ_2 are supposed to be independent, we can define $\mu(A_1\times A_2)=\mu_1(A_1)\mu_2(A_2)$ for $A_1\in\mathfrak{A}_1$, $A_2\in\mathfrak{A}_2$, and extend it to $\mathfrak{A}=\mathfrak{A}_1\times\mathfrak{A}_2$ preserving its additivity. It's a square integrable martingale measure on the product space $U=U_1\times U_2$ with respect to $\{\mathscr{F}_c, c\in\mathscr{C}\}$. According to [1], we can construct a Doléans measure $\langle\mu\rangle$ on the predictable σ-algebra $\mathscr{P}$ which is generated by all the sets of the form $A\times\Lambda$, where $A\in\mathfrak{A}=\mathfrak{A}_1\times\mathfrak{A}_2$ and $\Lambda\in\mathscr{F}_{t(A)}$. Consequently, the double stochastic integral of predictable random function[1] $h\in H_\mu^2\equiv L^2(U\times\Omega, \mathscr{P}_\mu, \langle\mu\rangle)$ with respect to μ is well defined.

On the other hand, we consider the predictable σ-algebra $\mathscr{P}_i$ in the space $U_i\times\Omega$ and the space $H_{\mu_i}^2\equiv L^2(U_i\times\Omega, \mathscr{P}_{\mu_i}, \langle\mu_i\rangle)$. If the iterated stochastic integrals

$$\int_{A_2} d\mu_2\int_{A_1} h(u_1, u_2, w)\,d\mu_1$$

are well defined for all $A_1\in\mathfrak{A}_1$ and $A_2\in\mathfrak{A}_2$, what are the relations between the double stochastic integral and those iterated stochastic integrals? In answering this question, we have the following theorem:

Theorem. *If $\mu=\mu_1\times\mu_2$, where $\mu_i\in M^2(U_i)$ $(i=1, 2)$ and $\langle\mu_2\rangle$ is admissibl*; if $h(u_1, u_2, w)\in H_\mu^2$ and for every $u_2\in U_2$, $h(\cdot, u_2, \cdot)\in H_{\mu_1}^2$; and if for every $A_1\in\mathfrak{A}_1$, the random function $\lambda(A_1, \cdot, \cdot)$ defined by*

$$\lambda(A_1, u_2, w)=\int_{A_1} h(u_1, u_2, w)\,d\mu_1$$

belongs to $H_{\mu_2}^2$, then we have

$$\int_{A_1\times A_2} h(u_1, u_2, w)\,d\mu=\int_{A_2} d\mu_2\int_{A_1} h(u_1, u_2, w)\,d\mu_1 \qquad (*)$$

for every $A_1\in\mathfrak{A}_1$ and $A_2\in\mathfrak{A}_2$.

We shall use two lemmas which are easy to prove.

Lemma 1. *If $\mu\in M^2$, $h\in H_\mu^2$, $A\in\mathfrak{A}$ and ξ is a $\mathscr{F}_{t(A)}$-measurable random variable such that $\xi h\in H_\mu^2$, then*

$$\int_A \xi h\,d\mu=\xi\int_A h\,d\mu.$$

Proof First we consider the simple case where $\xi=\mathbf{1}_\Lambda$, $\Lambda\in\mathscr{F}_{t(A)}$. It is easy to see that

$$\int_A \mathbf{1}_\Lambda h\,d\mu=\int_U \mathbf{1}_{A\times\Lambda}h\,d\mu=\Pi_{A\times\Lambda}\int_U h\,d\mu=\mathbf{1}_\Lambda\int_A h\,d\mu.$$

Then, by passage to limit, we can show this is true for all $\mathscr{F}_{t(A)}$-measurable random variables ξ such that $\xi h\in H_\mu^2$.

We say that a σ-finite signed measure λ on $\mathscr{P}$ is admissible if there exists a

* See the definition stated just before Lemma 2.

σ-finite measure m on $\mathscr{B}$ such that $\lambda \ll m\times P$ on $\mathscr{P}$. Denote by $\mathscr{P}_u$ the completion of σ-algebra $\mathscr{P}$ with respect to all admissible measures. Then, we have

Lemma 2. *A set N in $\mathscr{P}_u$ has measure zero for all admissible measures if and only if for each $u\in U$. the u-section of N has probability zero.*

The proof of Theorem Clearly, all functions satisfying the conditions stated in the theorem constitute a linear space $\mathscr{L}$. Suppose that $h(u_1, u_2, w)=\mathbf{1}_{B_1\times B_2\times\Lambda}$, where $B_1\in\mathfrak{A}_1$, $B_2\in\mathfrak{A}_2$ and $\Lambda\in\mathscr{F}_{t(B_1\times B_2)}$. We have

$$\int_{A_1\times A_2}\mathbf{1}_{B_1\times B_2\times\Lambda}d\mu=\mathbf{1}_\Lambda\mu(A_1B_1\times A_2B_2)$$

and

$$\int_{A_1}\mathbf{1}_{B_1\times B_2\times\Lambda}\,d\mu_1=\mathbf{1}_{B_2\times\Lambda}\mu_1(A_1B_1)$$

by Lemma 1 since $\mathbf{1}_{B_2\times\Lambda}$ is $\mathscr{F}^1_{t(A_1B_1)}$-measurable. It follows that

$$\begin{aligned}\int_{A_2}d\mu_2\int_{A_1}\mathbf{1}_{B_1\times B_2\times\Lambda}\,d\mu_1&=\int_{A_2}\mathbf{1}_{B_2\times\Lambda}\mu_1(A_1B_1)\,d\mu_2\\&=\mathbf{1}_\Lambda\mu_1(A_1B_1)\,\mu_2(A_2B_2)=\mathbf{1}_\Lambda\mu(A_1B_1\times A_2B_2)\end{aligned}$$

since $\mathbf{1}_\Lambda\mu_1(A_1B_1)$ is $\mathscr{F}^2_{t(A_2B_2)}$-measurable (also by Lemma 1). Therefore, the formula $(*)$ is established for $h=\mathbf{1}_{B_1\times B_2\times\Lambda}$.

Now suppose a sequence $\{h_n\}$ in $\mathscr{L}$ such that $0\leqslant h_n\uparrow h\in\mathscr{L}$ for which the theorem is true. By dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_{U\times\Omega}(h_n-h)^2\,d\langle\mu\rangle=0 \tag{1}$$

and

$$\lim_{n\to\infty}\int_{U_1\times\Omega}(h_n-h)^2d\langle\mu_1\rangle=0 \tag{2}$$

for every $u_2\in U_2$. It follows that for every $A_1\in\mathfrak{A}_1$, $u_2\in U_2$, the sequence

$$\lambda_n(A_1,u_2, w)\equiv\int_{A_1}h_n\,d\mu_1$$

converges to

$$\lambda(A_1, u_2, w)\equiv\int_{A_1}h\,d\mu_1$$

in the sense of

$$\lim_{n\to\infty}E(\lambda_n(A_1)-\lambda(A_1))^2=0. \tag{3}$$

Now that the equation $(*)$ holds for every h_n, i.e.

$$\int_{A_1\times A_2}h_n\,d\mu=\int_{A_2}\lambda_n(A_1)\,d\mu_2\quad(A_1\in\mathfrak{A}_1,\ A_2\in\mathfrak{A}_2,\ n\geqslant1), \tag{4}$$

in view of (1) and continuity of the stochastic integral operator I_μ, we have

$$\lim_{n\to\infty}\int_{A_1\times A_2}h_n\,d\mu=\int_{A_1\times A_2}hd\mu\quad(\text{in } L^2). \tag{5}$$

Consequently, the right side of (4) constitute a convergence sequence in L^2. Since the stochastic integral operator I_{μ_2} is an isometry between $M^2(U_2)$ and $H^2_{\mu_2}$, it follows that $\{\lambda_n(A_1)\}$ converges to some $\tilde{\lambda}(A_1)$ in $H^2_{\mu_2}$.

On the other hand, in view of (2) and the continuity of operator I_{μ_1}, we see that $\{\lambda_n(A_1)\}$ converges to $\lambda(A_1)$ in L^2 for every $u_2\in U_2$. Hence, there exists a subsequence $\{\lambda_{n_k}(A_1)\}$ which almost surely converges to $\lambda(A_1)$ for each $u_2\in U_2$. Consider the set

$$S=\{(u_2, w): \lambda(A_1, u_2, w)\neq\tilde{\lambda}(A_1, u_2, w)\}.$$

Since $\lambda(A_1, \cdot,\cdot)$ and $\tilde{\lambda}(A_1, \cdot, \cdot)$ belong to $H^2_{\mu_2}$, it follows that $S\in\mathscr{P}_{\mu_2}$. Moreover, for every $u_2\in U_2$, the section Su_2 is a P-null set, sos $\langle\mu_2\rangle(S)=0$ by Lemma 2. In other words

$$\lambda(A_1)=\tilde{\lambda}(A_1) \text{ a.e. } \langle\mu_2\rangle.$$

Hence, by passage to limit (along the subsequence $\{n_k\}$) in (4), we have

$$\int_{A_1\times A_2} h\,d\mu=\int_{A_2}\tilde{\lambda}(A_1)\,d\mu_2=\int_{A_2}\lambda(A_1)\,d\mu_2=\int_{A_2}d\mu_2\int_{A_1}h\,d\mu_1,$$

that means h also satisfies (*). Using the monotone class theorem, we conclude that for all functions in $\mathscr{L}$ the theorem is true. This completes the proof.

References

[1] Huang, Z. Y., Stochastic integrals on general topological measurable spaces, Z. wahr. verw. Gebiete, **66** (1984), 25—40.