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THE FIRST BOUNDARY VALUE PROBLEM FOR SOLUTIONS OF DEGENERATE QUASILINEAR PARABOLIC EQUATIONS*

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Abstract

In This paper, the author proves the existence and uniqueness of nonnegative solution for the first boundary value problem of uniform degenerated parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t}=\Sigma \frac{\partial}{\partial x_i}\left(\nu(u) A_{ij}(x, t, u) \frac{\partial u}{\partial x_j}\right)+\Sigma B_i(x, t, u) \frac{\partial u}{\partial x_i}+C(x, t, u)u \quad (x, t) \in [0, T], \\ u|_{t=0} = u_0(x), \quad x \in \Omega, \\ u|_{s\in\partial\Omega}=\psi(s, t), \quad 0\leqslant t\leqslant T \end{cases}$$

$\left(\frac{1}{\Lambda} |\alpha|^2\leqslant \Sigma A_{ij}\alpha_i\alpha_j\leqslant \Lambda|\alpha|^2, \ \forall\alpha\in\mathbf{R}^n, \ 0<\Lambda<\infty, \ \nu(u)>0, \ \forall u>0 \text{ and } \nu(u)\to 0 \text{ as } u\to 0\right)$ under some very weak restrictions, i.e. $A_{ij}(x, t, r)$, $B_i(x, t, r)$, $C(x, t, r)$, $\Sigma \frac{\partial A_{ij}}{\partial x_j}$, $\Sigma \frac{\partial B_i}{\partial x_i} \in \bar\Omega\times[0, T]\times\mathbf{R}$, $|B_i|\leqslant\Lambda$, $|C|\leqslant\Lambda$, $\left|\Sigma \frac{\partial B_i}{\partial x_i}\right|\leqslant\Lambda$, $\partial\Omega\in C^2$, $\nu(r)\in C[0, \infty)$. $\nu(0)=0$, $1\leqslant \frac{r\nu(r)}{\int_0^r \nu(s)ds}\leqslant m$, $u_0(x)\in C^\beta(\bar\Omega)$, $\psi(s, t)\in C^{\beta,\frac{\beta}{2}}(\partial\Omega\times[0, T])$, $0<\beta<1$, $u_0(s)=\psi(s, 0)$.

Introduction

In this paper we shall prove the existence and uniqueness theorems for a mixed Cauchy-Dirichret problem of a general n–dimensional degenerate quasilinear parabolic equation. It is a generalization of the porous medium equation

$$u_t=\triangle(u^m) \quad (m>1).$$

The early results in this direction for one space variable appeared in [1, 2, 3], for n-dimension in [4, 5]. The result in this paper is an extension of the results given in [4, 5], for less restricted coeffcients*.

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* The result in [4] is that the Hölder condition holds for the solution of (1.1) under the restriction $\sum_{i=1}^n |b_i(x, t, u)|/\nu(u)^2\leqslant$const. In [5], the author discusses the existence and uniqueness of generalized solutions for the first boundary value problem of the equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(\nu(u)\frac{\partial u}{\partial x_i}\right)+b_i(x, t, u)\frac{\partial u}{\partial x_i}.$$

In § 1 we state the hypytheses about the equation and boundary conditions for the existence of classical solutions, and in § 2 we introduce the generalized $\mathfrak{B}_2$ class as it was done in [6]. We give the preliminary lemmas on the generalized $\mathfrak{B}_2$ class in § 3, and we obtain the Hölder estimates for classical solutions in § 4. Finally, we give the definition of weak solutions and prove their existence and uniqueness in § 5.

§ 1

Let $\mathbf{R}^n$ be an Euclidean n-space and Ω an open and bounded domain of $\mathbf{R}^n$ and $\partial\Omega$ be the boundary of Ω. Let Q_T be the $(n+1)$-dimensional domain $\Omega\times(0, T]$ and $\Gamma=\partial\Omega\times[0, T]\cup\Omega\times\{t=0\}$. In Q_T we consider the quasilinear parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a_{ij}(x, t, u)\frac{\partial u}{\partial x_j}\right)+b_i(x, t, u)\frac{\partial u}{\partial x_i}+c(x, t, u)u, \tag{1.1}$$

where u is a scalar function of x, $t(x=(x_1, x_2, \cdots, x_n))$ and dummy summations are for $i, j=1, 2, \cdots, n$. The equation (1.1) will be degenerate when $u=0$ by the conditions below. We will find a solution of equation (1.1) on Q_T satisfying the following conditions

$$u(x, 0)=u_0(x), \quad x\in\Omega, \tag{1.2}$$

$$u(x, t)=\psi(s, t), \quad x\in\partial\Omega,\ 0\leqslant t\leqslant T, \tag{1.3}$$

where s is the parameter of $\partial\Omega$.

Suppose that Ω and the coefficients of the equation (1.1) and boundary values (1.2), (1.3) satisfy the following conditions:

(i) $a_{ij}(x, t, r)$, $b_i(x, t, r)$, $c(x, t, r)$, $\frac{\partial a_{ij}}{\partial x_j}$ and $\frac{\partial b_i}{\partial x_i}$ $(i, j=1, 2, \cdots, n)$ are continuous when $(x, t)\in\overline{Q}_T$ and $|r|<\infty$.

(ii) For any $\xi\in\mathbf{R}^n$, $|r|<\infty$,

$$\frac{1}{\Lambda}\nu(|r|)|\xi|^2\leqslant a_{ij}(x, t, r)\leqslant\Lambda\nu(|r|)|\xi|^2, \tag{1.4}$$

where Λ is a constant and $\nu(s)$ is a function which has the following properties

(a) $\nu(r)\in C[0, \infty)$,

$$\nu(0)=0 \text{ and } \nu(r)>0 \text{ if } r>0. \tag{1.5}$$

(b) Let $\varphi(r)=\int_0^r\nu(s)ds$. There exists $\delta>0$ and $m>1$ such that for $0<r\leqslant\delta$ we have

$$1\leqslant\frac{\nu(r)r}{\varphi(r)}\leqslant m. \tag{1.6}$$

Change the definition of $\nu(r)$ for $r>\delta$ to be $\nu(r)=\nu(\delta)$ $(r>\delta)$ and change the constant Λ properly, we see that (1.5) and (1.6) hold for $r\in(0, \infty)$, and (1.4) holds for any bounded r and $(x, t)\in\bar{Q}_T$.

(iii)

$$|c(x, t, r)|\leqslant\Lambda, \quad (x, t)\in\bar{Q}_T, |r|<\infty. \tag{1.7}$$

(iv) There exists $a_0>0$ and $\theta_0\in(0, 1)$ such that for any n-dimensional ball $K(\rho)$ with its center on $\partial\Omega$ and radius ρ, we have

$$\text{mes}\{K(\rho)\cap\Omega\}\leqslant(1-\theta_0)\text{mes}K(\rho) \tag{1.8}$$

if $\rho\leqslant a_0$, where mes $\{\cdot\}$ is the measure of a set in $\mathbf{R}^n$.

(v) For the existence of classical solutions, we assume that $u_0(x)$ and $\psi(s, t)$ satisfy Hölder condition and $u_0(x)>0$, $\psi(s, t)>0$, $u_0(s)=\psi(s, 0)$.

Let

$$a_\varepsilon(u)=\begin{cases}(\varepsilon^2-u^2)^m, & |u|<\varepsilon,\\ 0, & |u|\geqslant\varepsilon,\end{cases}$$

where ε is a small positive number.

Consider the approximate equation of (1.1)

$$u_t=(a^\varepsilon_{ij}(x, t, u)u_{x_j})_{x_i}+b_i(x, t, u)u_{x_i}+c(x, t, u)u, \tag{1.1$'$}$$

where

$$a^\varepsilon_{ij}(x, t, u)=a_{ij}(x, t, u)+\delta_{ij}a_\varepsilon(u), \quad \delta_{ij}=\begin{cases}1 & (i=j),\\ 0 & (i\neq j).\end{cases}$$

We assume that a classical solution of (1.1)′, (1.2) and (1.3) exists and then try to get its a-priori estimates.

Using the transformation $u=e^{\lambda t}v$, where λ is a nonnegative constant, (1.1)′ becomes

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x_i}\left(a^\varepsilon_{ij}\frac{\partial v}{\partial x_j}\right)+b_i\frac{\partial v}{\partial x_i}+(c-\lambda)v.$$

Take $\lambda=2\Lambda$, we see that v cannot have nonpositive minimum inside Q_T, hence $v>0$ and $u>0$ inside Q_T. Take $\lambda=-2\Lambda$, we see that v cannot have positive minimum inside Q_T, hence

$$v\geqslant\min\{\inf_\Omega u_0(x), \inf_{\partial\Omega\times[0,T]}\psi(s, t)\},$$

$$u\geqslant e^{-2\Lambda T}\min\{\inf_\Omega u_0(x), \inf_{\partial\Omega\times[0,T]}\psi(s, t)\}\equiv M_1.$$

Similarly

$$u\leqslant e^{-2\Lambda T}\max\{\sup_\Omega u_0(x), \sup_{\partial\Omega\times[0,T]}\psi(s, t)\}\equiv M.$$

Hence

$$0<M_1\leqslant u\leqslant M.$$

Take $\varepsilon<M_1$, (1.1)′ reduces to (1.1), hence u is the solution of (1.1), (1.2) and (1.3).

Using the transformation $u=e^{\lambda t}v$ again, we change the equation (1.1) into the form

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a_{ij}(x, t, u)\frac{\partial u}{\partial x_j}\right)+b_i(x, t, u)\frac{\partial u}{\partial x_i}+\tilde{c}(x, t, u)u \tag{1.9}$$

with the condition

$$|b_i(x, t, u)|\leqslant \Lambda, \left|\frac{\partial b_i(x, t, u)}{\partial x_i}\right|\leqslant \Lambda, \ -2\Lambda\leqslant \tilde{c}(x, t, u)\leqslant 0 \tag{1.10}$$

when $M_1\leqslant u\leqslant M$.

Let $w=\varphi(u)=\int_0^u \nu(s)ds$ and its inverse be

$$u=\Phi(w).$$

In virtue of the condition (1.6) we can prove that for $0<w_1<w_2\leqslant\varphi(\delta)$,

$$1/m\leqslant \Phi'(w_1)/\Phi'(w_2)\leqslant m(w_2/w_1)^{1-\frac{1}{m}}, \tag{1.12}$$

or

$$1/m\leqslant \nu(u_2)/\nu(u_1)\leqslant [w(u_2)/w(u_1)]^{1-\frac{1}{m}}.$$

In fact, the condition (1.6) implies

$$1/(mw)\leqslant \Phi'(w)/\Phi(w)\leqslant 1/w \text{ if } 0<w\leqslant\varphi(\delta). \tag{1.13}$$

For $0<w_1<w_2\leqslant\varphi(\delta)$, integrating (1.13) from w_1 to w_2, we have

$$(w_2/w_1)^{\frac{1}{m}}\leqslant \Phi(w_2)/\Phi(w_1)\leqslant w_2/w_1 \tag{1.14}$$

or

$$w_2/w_1\leqslant (u_2/u_1)^m\leqslant (w_2/w_1)^m.$$

Using the inequalities (1.13) and (1.14) we can obtain (1.12) without any difficulty. We may suppose that (1.12) is satisfied for $0<w_1<w_2\leqslant\varphi(M)$ if we change the constant m properly.

From (1.6) we have

$$r_2/r_1\leqslant \int_0^{r_2} \nu(s)ds\Big/\int_0^{r_1} \nu(s)ds\leqslant (r_2/r_1)^m$$

and

$$1/m\leqslant r_2\nu(r_2)\int_0^{r_1} \nu(s)ds\Big/\left[r_1\nu(r_1)\int_0^{r_2} \nu(s)ds\right]\leqslant m \quad (r_1\leqslant r_2),$$

hence we have

$$1/m\leqslant \nu(r_2)/\nu(r_1)\leqslant m(r_2/r_1)^{m-1} \quad (r_1\leqslant r_2). \tag{1.15}$$

§ 2

Let $(x^0, t^0)\in Q_T$ and $K(\rho)$ be a ball of $\mathbf{R}^n$ with its center at x^0 and radius ρ. For $0\leqslant t\leqslant T$, denote that

$$\begin{aligned} A_{k,\rho}(t)&=\{x\in K(\rho)\cap\Omega\,|\,w(x, t)>k\},\\ B_{k,\rho}(t)&=\{x\in K(\rho)\cap\Omega\,|\,w(x, t)<k\}, \end{aligned} \tag{2.1}$$

where

$$w(x, t)=\varphi(u(x, t))=\int_0^{u(x,t)} \nu(s)ds$$

and $u(x, t)$ is a solution of the equation (1.9).

Lemma 1. *Suppose that the coefficients of the equation (1.9) satisfy the conditions (1.4), (1.5), (1.6) and (1.10). Let $u(x, t)$ be a classical solution of the equation (1.9)* *satisfying* $0<u(x,t)\leqslant M$ *and* $\zeta(x)$ *be a cut-off function in* $K(\rho)$, *We have*

(i) *if* $k\geqslant \max\limits_{x\in K(\rho)\cap\partial\Omega} w(x,t)$, *then*

$$\frac{\partial}{\partial t}\left\{\exp[-\gamma\Phi'(k)t]\int_{A_{k,\rho}(t)}\zeta^2\chi_k(w-k)dx\right\}+\frac{1}{2}\exp[-r\Phi'(k)t]\int_{A_{k,\rho}(t)}\zeta^2|\nabla w|^2dx$$
$$\leqslant\gamma\int_{A_{k,\rho}(t)}\exp[-\gamma\Phi'(k)t]\int_{A_{k,\rho}(t)}|\nabla\zeta|^2(w-k)^2dx. \tag{2.2}$$

(ii) if $k\leqslant \min\limits_{x\in K(\rho)\cap\partial\Omega} w(x,t)$, then

$$\frac{\partial}{\partial t}\left\{\exp[-\gamma\Phi'(k)t]\int_{B_{k,\rho}(t)}\zeta^2\tilde{\chi}_k(k-w)dx\right\}+\frac{1}{2}\exp[-\gamma\Phi'(k)t]\int_{B_{k,\rho}(t)}\zeta^2|\nabla w|^2dx$$
$$\leqslant\gamma\exp[-\gamma\Phi'(k)t]\left[\int_{B_{k,\rho}(t)}|\nabla\zeta|^2(w-k)^2dx+\text{mes }B_{k,\rho}(t)\right], \tag{2.3}$$

where $\gamma=\gamma(n,\Lambda,\text{m},M)$, ∇ *is the gradient operator with respect to* x *and*

$$\chi_k(s)=\int_0^s\Phi'(k+\tau)\tau\,d\tau,\ \tilde{\chi}_k(s)=\int_0^s\Phi'(k-\tau)\tau d\tau. \tag{2.4}$$

Proof Let $w^+=\max\{0,w\}$. Multiplying the equation (1.9) by $\zeta^2(x)(w-k)^+$ and integrating over Ω and taking notice of (1.10), we obtain

$$\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)}\zeta^2\varkappa_k(w-k)dx+\int_{A_{k,\rho}(t)}\zeta^2a_{ij}(x,t,u)\frac{\partial w}{\partial x_i}\frac{\partial u}{\partial x_j}dx$$
$$\leqslant-2\int_{A_{k,\rho}(t)}\zeta(w-k)a_{ij}\frac{\partial\zeta}{\partial x_i}\frac{\partial u}{\partial x_j}dx+\int_{A_{k,\rho}(t)}\zeta^2(w-k)b_i(x,t;u)\frac{\partial u}{\partial x_i}dx. \tag{2.5}$$

Now set

$$h_i(x,t,s)=\int_0^s b_i(x,t,\Phi(k+\tau))\Phi'(k+\tau)\tau\,d\tau \tag{2.6}$$

$$|h_i|\leqslant\Lambda\int_0^s\Phi'(k+\tau)\tau d\tau=\Lambda x_k(s). \tag{2.7}$$

In virtue of (1.4), (1.10) and (2.8), the inequality (2.5) yields

$$\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)}\zeta^2x_k(w-k)dx+\int_{A_{k,\rho}(t)}\zeta^2|\nabla w|^2dx$$
$$\leqslant2\Lambda\int_{A_{k,\rho}(t)}\zeta|\nabla\zeta||\nabla w|(w-k)dx+\int_{A_{k,\rho}(t)}\zeta^2\frac{\partial h_i(x,t,w-k)}{\partial x_i}dx$$
$$-2\int_{A_{k,\rho}(t)}\zeta^2\left[\int_0^{w-k}\frac{\partial b_i(x,t,\Phi(k+2))}{\partial x_i}\Phi'(k+\tau)\tau\,d\tau\right]dx. \tag{2.8}$$

Integrating the second term on the right side by part and using (1.10), (2.7) we find

$$\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)}\zeta^2\chi_k(w-k)dx+\int_{A_{k,\rho}(t)}\zeta^2|\nabla w|^2dx$$
$$\leqslant2\int_{A_{k,\rho}(t)}\Lambda\zeta|\nabla\zeta||\nabla w|(w-k)dx+2\Lambda\int_{A_{k,\rho}(t)}\zeta|\nabla\zeta|\chi_k(w-k)dx$$
$$+\Lambda\int_{A_{k,\rho}(t)}\zeta^2\chi_k(w-k)dx.$$

By Schwarz inequality, (1.12) and

$$\chi_k(s)/s^2=\int_0^s\Phi'(k+\tau)\tau d\tau/s^2\leqslant m\Phi'(k)\int_0^s\tau\,d\tau/s^2=m\Phi'(k)/2,$$

it follows that

$$\frac{\partial}{\partial t}\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)\,dx+\frac{1}{2}\int_{A_{k,\rho}(t)} \zeta^2|\nabla w|^2\,dx$$
$$\leqslant\gamma\left[\int_{A_{k,\rho}(t)} |\nabla\zeta|^2(w-k)^2dx+\Phi'(k)\int_{A_{k,\rho}(t)} \zeta^2\chi_k(w-k)\,dx\right],$$

which implies (2.2). The inequality (2.3) can be proved in the same way, in which we use the inequality

$$\frac{\tilde{\chi}_k(k-w)}{\Phi'/k}\Big/(k-w)^2=\int_0^{k-w}\frac{\Phi'(k-\tau)}{\Phi'(k)}\,\tau\,d\tau/(k-w)^2$$
$$\leqslant\frac{m}{(k-w)^2}\int_0^{k-w}(1-\tau/k)^{\frac{1}{m}-1}\,\tau\,d\tau\leqslant m\int_0^{k-\tau}\left(1-\frac{\tau}{k-w}\right)^{\frac{1}{m}-1}d\tau/(k-w)=m^2$$

and the term $c(x, t, u)$ yields an additional term mes $B_{k,\rho}(t)$.

We shall call the family of all the functions satisfying (2.2) and (2.3) the generalized $\mathfrak{B}_2(Q_T, M, m, \gamma)$ class (cf. [6]).

§ 3

In this section we shall discuss the properties of the generalized $\mathfrak{B}_2$ class. We apply the method used in [4] with some lemmas.

Lemma 3.1. *For any $u\in W_2^1(\Omega)$, we have*

$$\int_{A_0}|u|^2dx\leqslant C(\text{mes } A_0)^{\frac{2}{n}}\int_{A_0}|\nabla u|^2\,dx, \tag{3.1}$$

where $A_0=\{x\in\Omega\,|\,u(x)>0\}$ and $C=C(n)$.

Lemma 3.2. *For any $u\in W_m^1(K(\rho))$, $m>1$, we have*

$$(\lambda-k)\text{ mes } A_{\lambda,\rho}^{1-\frac{1}{n}}\leqslant\beta\rho^n/\text{mes}(K(\rho)\setminus A_{k,\rho})\int_{A_{k,\rho}\setminus A_{\lambda,\rho}}|\nabla u|\,dx, \tag{3.2}$$

where $\lambda>k$, $\beta=\beta(n)$ and $A_{k,\rho}=\{x\in K(\rho)\,|\,u(x)>k\}$.

These two lemmas can be found in [6].

The functions $\chi_k(s)$ and $\tilde{\chi}_k(s)$ given in (2.4) for nondegenerate equations have properties: $\chi_k(s)\sim s^2$ and $\tilde{\chi}_k(s)\sim s^2$. Now they do not have these properties here due to the degeneration. However, we have the following lemma.

Lemma 3.3. (i) *For $\mu>k\geqslant\mu/2>0$, $H\leqslant\mu-k$, $0<\beta<1$, we have*

$$H^2/(2m)\leqslant\chi_k(H)/\Phi'(\mu)\leqslant mH^2, \tag{3.3}$$
$$\chi_k(H)/\chi_k(\beta H)\leqslant1+m^2(1-\beta^2)/\beta^2. \tag{3.4}$$

(*ii*) *For $k>H>0$, $0<\beta<1$, we have*

$$H^2/(2m)\leqslant\tilde{\chi}_k(H)/\Phi'(k)\leqslant m^2H^2, \tag{3.5}$$

$$\tilde{\chi}_k(H)/\tilde{\chi}_k(\beta H)\leqslant1+\max\{4m^3(1-\beta)^{\frac{1}{m}},\ 2m^2(1-\beta^2)/\beta^2\}. \tag{3.6}$$

Proof The inequalities (3.3) are obvious if we take notice of the expression (2.4) of $\chi_k(s)$ and the condition (1.12).

Now we prove (3.4). In fact

$$\chi_k(H)/\chi_k(\beta H)-1=\int_{\beta H}^{H} s\Phi'(k+s)/\Phi'(k+\beta H)ds\Big/\int_0^{\beta H} s\Phi'(k+\beta H)ds.$$

By inequality (1.12), it follows that

$$\chi_k(H)/\chi_k(\beta H)-1\leqslant m^2\int_{\beta H}^{H} sds\Big/\int_0^{\beta H} sds=m^2(1-\beta^2)/\beta^2,$$

which is required.

The first inequality in (3.5) is easily obtained by means of (2.4) and (1.12). As for the second part of (3.5)

$$\tilde{\chi}_k(H)/\Phi'(k)=\int_0^H \Phi'(k-s)/\Phi'(k)s\,ds\leqslant m\int_0^H (k/(k-s))^{1-\frac{1}{m}}s\,ds,$$

since the integrand is monotonic with respect to k, we have

$$\tilde{\chi}_k(H)/\Phi'(k)\leqslant m\int_0^H (H/(H-s))^{1-\frac{1}{m}}s\,ds\leqslant m^2H^2.$$

Now we pass to (3.6). Like the proof of (3.4), we observe

$$\tilde{\chi}_k(H)/\tilde{\chi}_k(\beta H)-1=\int_{\beta H}^{H} s\Phi'(k-s)\Phi'(k-\beta H)ds\Big/\int_0^{\beta H} s\Phi'(k-s)/\Phi'(k-\beta H)ds$$

$$\leqslant m^2\int_{\beta H}^{H}(k-s)^{\frac{1}{m}-1}sds\Big/\int_0^{\beta H}(k-s)^{\frac{1}{m}-1}sds.$$

If $k/2\leqslant H<k$, then

$$\tilde{\chi}_k(H)/\tilde{\chi}_k(\beta H)-1\leqslant m^2\int_{\beta H}^{H}(H-s)^{\frac{1}{m}-1}sds\Big/\int_0^{\beta H}(2H)^{\frac{1}{m}-1}s\,ds\leqslant 4m^3(1-\beta)^{\frac{1}{m}},$$

and if $H<k/2$, then

$$\tilde{\chi}_k(H)/\tilde{\chi}_k(\beta H)-1\leqslant m^2\int_{\beta H}^{H}(k-H)^{\frac{1}{m}-1}sds\Big/\int_0^{\beta H}k^{\frac{1}{m}-1}sds\leqslant 2m^2(1-\beta^2)/\beta^2.$$

Since $k>H$, these inequalities imply (3.6). The proof is complete.

For any fixed $\rho\in(0,1]$, we shall consider the domain

$$Q_\rho=\{(x,t)\,|\,|x-x^0|<\rho,\ t^0-a\eta\rho^2<\alpha<t^0\},\tag{3.7}$$

where $\eta=\Phi'(\rho^\varepsilon)$, a is a constant defined in Lemma 3.4 and ε is any constant in $(0, {}^1/_2]$. Denote that

$$\varkappa_n=\text{mes}\{K(1)\}\tag{3.8}$$

$$\mu=\max_{Q_\rho\cap\Omega}\{w(x,t)\},\ \tilde{\mu}=\min_{Q_\rho\cap\Omega}\{w(x,t)\},\omega=\mu-\tilde{\mu},\tag{3.9}$$

$$\bar{\sigma}_0=2^{-\frac{1}{n}},\ \bar{\rho}_3=\bar{\sigma}_0\rho,\ \bar{\rho}_2=(1+2\bar{\sigma}_0)\rho/3,\ \bar{\rho}_1=(2+\bar{\sigma}_0)\rho/3,\tag{3.10}$$

$$\xi=\Phi'(\mu)/m^2,\ \eta_k=\Phi'(k)/m.\tag{3.11}$$

In this section we shall suppose that

$$\mu\geqslant 2\rho^\varepsilon,\tag{3.12}$$

and so by the condition (1.12)

$$\xi\leqslant\eta_k\leqslant\eta=\Phi'(\rho^\varepsilon),\ \text{if}\ \rho^\varepsilon\leqslant k\leqslant\mu.$$

In the following lemmas we shall not specify the dependence of constants on the parameters of $\mathfrak{B}_2$ class.

Lemma 3.4. *There exist constants β, a, $b\in(0,1)$ such that*

(i) *if*

$$k\geqslant\max\{\mu/2,\ \max_{Q\cap\Gamma_\rho} w(x,t)\},\quad H=\mu-k>0,$$

$$\operatorname{mes} A_{k,\bar\rho_1}(t^0-a\xi\rho^2)\leqslant 1/2\varkappa_n\bar\rho_1^n, \tag{3.13}$$

then for $t\in[t^0-a\xi\rho^2,\ t^0]$

$$\operatorname{mes}[K(\bar\rho_1)\setminus A_{k+\beta H,\bar\rho_1}(t)]\geqslant b\varkappa_n\bar\rho_1^n. \tag{3.14}$$

(ii) *if*

$$k\leqslant\min_{Q_\rho\cap\Gamma}\{w(x,t)\},\ H=k-\tilde\mu\geqslant\rho^\varepsilon,\ \operatorname{mes} B_{k,\bar\rho_1}(t^0-a\eta_k\rho^2)\leqslant 1/2\varkappa_n\bar\rho_1^n, \tag{3.15}$$

then for $t\in[t^0-a\eta_k\rho^2,\ t^0]$

$$\operatorname{mes}[K(\bar\rho_1)\setminus B_{k-\beta H,\bar\rho_1}(t)]\geqslant b\varkappa_n\bar\rho_1^n. \tag{3.16}$$

Proof For $0<\sigma<1$ let

$$\zeta(x;\rho,\rho-\sigma\rho)=\begin{cases}1, & |x-x^0|<\rho-\sigma\rho,\\ (\rho-|x-x^0|)/(\sigma\rho), & \rho-\sigma\rho\leqslant|x-x^0|\leqslant\rho,\\ 0, & |x-x^0|>\rho.\end{cases} \tag{3.17}$$

We prove the first part of the lemma. Integrating the inequality (2.2) with $\zeta(x)=\zeta(x;\bar\rho_1,\bar\rho_1-\sigma\bar\rho_1)$ with respect to t from $t^0-ai\xi\rho^2$ to t, we obtain

$$\begin{aligned}&\exp[-\gamma\Phi'(k)t]\int_{A_{k,\bar\rho_1}(t)}\zeta^2\chi_k(w-k)\,dx\\ &\leqslant\exp[-\gamma\Phi'(k)(t^0-a\xi\rho^2)]\Big[\int_{A_{k,\bar\rho_1}(t^0-a\xi\rho^2)}\zeta^2\chi_k(w-k)dx\\ &\quad+\gamma a\xi\rho^2H^2/(\sigma\bar\rho_1)^2\varkappa_n\bar\rho_1^n\Big].\end{aligned}$$

By the condition (3.13), it follows that for $t\in[t^0-a\xi\rho^2,\ t^0]$,

$$\int_{A_{k,\bar\rho_1}(t)}\zeta^2\chi_k(w-k)dx\leqslant\exp[\gamma\Phi'(k)a\xi\rho^2][\chi_k(H)\cdot 1/2\varkappa_n\bar\rho_1^n+2\gamma a\xi H^2/\sigma^2\varkappa_n\bar\rho_1^n].$$

On the other hand

$$\int_{A_{k,\bar\rho_1}(t)}\zeta^2\chi_k(w-k)dx\geqslant\int_{A_{k+\beta H,\bar\rho_1-\sigma\bar\rho_1}(t)}\chi_k(w-k)dx\geqslant\chi_k(\beta H)\operatorname{mes} A_{k+\beta H,\bar\rho_1-\sigma\bar\rho_1}(t).$$

Hence, for $t\in[t^0-a\xi\rho^2,\ t^0]$

$$\begin{aligned}\operatorname{mes} A_{k+\beta H,\bar\rho_1-\sigma\bar\rho_1}(t)\leqslant&\exp[a\gamma\Phi'(k)\xi\rho^2][\chi_k(H)/\chi_k(\beta H)\cdot 1/2\varkappa_n\bar\rho_1^n\\ &+2a\gamma\sigma^{-2}\xi H^2/\chi_k(\beta H)\varkappa_n\bar\rho_1^n].\end{aligned}$$

In virtue of (3.3) and (3.4), it follows that

$$\operatorname{mes} A_{k+\beta H,\bar\rho_1-\sigma\bar\rho_1}\leqslant\exp[a\gamma\Phi'(k)\xi\rho^2]\{(1+m^2(1-\beta^2)/\beta^2)/2+4a\gamma/(m\beta^2\sigma^2)]\varkappa_n\bar\rho_1^n.$$

We may select $\beta=\beta(m)\in(0,1)$ such that

$$(1+m^2(1-\beta^2)/\beta^2)/2\leqslant 3/4.$$

Noting that $\mu\geqslant 2\rho^\varepsilon$, $k\geqslant\rho^\varepsilon$ and $\xi=m^{-2}\Phi'(\mu)$, we may take a, b_1 and σ_0 (depend only on n, m, γ, T) so small that

$$(1+m^2(1-\beta^2)/\beta^2)\exp[a\gamma\Phi'(k)\xi\rho^2]/2<(1-b_1)(1-\sigma_0)^n,$$

then take a so small again that

$$\exp[a\gamma\Phi'(k)\xi\rho^2][(1+m^2(1-\beta^2)/\beta^2)/2+4a\gamma/(m\beta^2\sigma_0^2)]\leqslant(1-b_1)(1-\sigma_0)^n.$$

Thus, for $t\in[t^0-a\xi\rho^2, t^0]$,

$$\text{mes } A_{k+\beta H,\bar\rho_1-\sigma_0\bar\rho_1}(t)\leqslant(1-b_1)(1-\sigma_0)^n\varkappa_n\bar\rho_1^n$$

and

$$\text{mes}(K(\bar\rho_1)-A_{k+\beta H,\bar\rho_1}(t))\geqslant\text{mes}(K(\bar\rho_1-\sigma_0\bar\rho_1)-A_{k+\beta H,\bar\rho_1-\sigma_0\bar\rho_1}(t))$$

$$\geqslant(1-\sigma_0)^n\varkappa_n\bar\rho_1^n-(1-b_1)(1-\sigma_0)^n\varkappa_n\bar\rho_1^n=b_1(1-\sigma_0)^n\varkappa_n\bar\rho_1^n$$

as claimed, if we let $b=b_1(1-\sigma_0)^n$. The second part of the lemma can be proved in the same way.

Lemma 3.5. *Suppose that $Q_\rho<Q_T$. For any $\theta_1>0$ there exists $s=s(\theta_1)>0$ such that*

(i) *if*

$$k\geqslant\mu/2,\ H=\mu-k>0,\ \text{mes}A_{k,\bar\rho_1}(t^0-a\xi\rho^2)\leqslant\varkappa_n\bar\rho_1^n/2 \tag{3.18}$$

then

$$\int_{t^0-a\xi\rho^2}^{t^0}\text{mes } A_{\mu-\frac{H}{2^{s+1}},\bar\rho_1}(t)dt\leqslant\theta_1\xi\bar\rho_1^{n+2}; \tag{3.19}$$

(ii) *if*

$$\max_{t\in[t^0-a\eta\rho^2,t^0-a\xi\rho^2]}\text{mes } B_{\tilde\mu+\frac{\omega}{2},\bar\rho_1}(t)\geqslant\varkappa_n\bar\rho_1^n/2 \tag{3.20}$$

then

$$\omega\leqslant2^{s+2}\rho^\varepsilon \tag{3.21}$$

or

$$\int_{t_0-a\eta_{k_{s+2}}\rho^2}^{t^0}B_{k_{s+2},\bar\rho_1}(t)dt\leqslant\theta_1\eta_{k_{s+2}}\bar\rho_1^{n+2}, \tag{3.22}$$

where $k_s=\tilde\mu+\omega/2^s$, $\eta_{k_s}=\Phi'(k_s)/m$.

Proof We prove the first part of the lemma. By the conditions (3.18) and Lemma 3.4, it follows that

$$\text{mes}[K(\bar\rho_1)\backslash A_{k+\beta H,\bar\rho_1}(t)]\geqslant b\varkappa_n\bar\rho_1^n \text{ for } t\in[t^0-a\xi\rho^2, t^0]. \tag{3.23}$$

Take r_0 such that

$$1-\beta\geqslant2^{-r_0},$$

and denote $k_l=\mu-H/2^l$, one obtains that for $t\in[t^0-a\xi\rho^2, t^0]$,

$$\text{mes}[K(\bar\rho_1)\backslash A_{k_l,\bar\rho_1}(t)]\geqslant b\varkappa_n\bar\rho_1^n,\ \text{if } l\geqslant r_0. \tag{3.24}$$

Using Lemma 3.2, we have

$$(k_{l+1}-k_l)\text{mes}^{1-\frac{1}{n}}A_{k_{l+1},\bar\rho_1}(t)\leqslant\beta\bar\rho_1^n/\text{mes}[K(\bar\rho_1)\backslash A_{k_l,\bar\rho_1}(t)]\int_{D_l(t)}|\nabla w|dx,$$

where

$${}_kD_l(t)=A_{k_l,\bar\rho_1}(t)\backslash A_{k_{l+1},\bar\rho_1}(t).$$

In virtue of (3.24), it follows that for $t\in[t^0-a\xi\rho^2, t^0]$,

$$H/2^{l+1}\text{ mes } A_{k_{l+1},\bar\rho_1}(t)\leqslant(\beta/b\varkappa_n^{1-\frac{1}{n}})\bar\rho_1\Big(\int_{D_l(t)}|\nabla w|^2dx\Big)^{\frac12}(\text{mes}D_l(t))^{\frac12}.$$

Integrating this inequality from $t^0-a\xi\rho^2$ to t^0 and applying Schwarz inequality, one obtains

$$(H/2^{l+1})^2\left[\int_{t^0-a\xi\rho^2}^{t^0}\text{mes }A_{k_{l+1},\bar\rho_1}(t)\,dt\right]^2$$
$$\leqslant(\beta/b\varkappa_n^{1-\frac{1}{n}})^2\bar\rho_1^{\frac{2}{n}}\int_{t^0-a\xi\rho^2}^{t^0}\int_{A_{k_l,\bar\rho_1}(t)}|\nabla w|^2dx\,dt\cdot\int_{t^0-a\xi\rho^2}^{t^0}\text{mes }D_l(t)\,dt. \qquad (3.25)$$

Integrating the inequality (2.2) with $\zeta(x)=\zeta(x;\rho,\bar\rho_1)$, we find

$$\frac{1}{2}\int_{t^0-a\xi\rho^2}^{t^0}\exp[-\gamma\Phi'(k)t]\int_{A_{k_l,\bar\rho_1(t)}}|\nabla w|^2dx\,dt$$
$$\leqslant\exp[-\gamma\Phi'(k)(t^0-a\xi\rho^2)]\left\{\int_{A_{k_l,\bar\rho}(t^0-a\xi\rho^2)}\zeta^2\chi_{k_l}(w-k_l)dx+9a\gamma\xi(1-\bar\rho_0/\rho)^{-2}(H/2^l)^2\varkappa_n\rho^n\right\}$$
$$\leqslant\exp[-\gamma\Phi'(k)(t^0-a\xi\rho^2)][\chi_{k_l}(H/2^l)+9a\gamma(1-\bar\rho_0/\rho)^{-2}\xi H^2/2^{2l}]\varkappa_n\rho^n.$$

By the estimate (3.3), it follows that

$$\int_{t_0-a\xi\rho^2}^{t^0}\int_{A_{k_l,\bar\rho_1}(t)}|\nabla w|^2dx\,dt\leqslant2\exp[\gamma\Phi'(k)a\xi\rho^2][2m^3+9a\gamma(1-\bar\rho_0/\rho)^{-2}]\xi H^2/2^{2l}\varkappa_n\rho^n.$$

Substituting it and (3.23) into (3.25), we have

$$\left(\int_{t_0-a\xi\rho^2}^{t^0}\text{mes }A_{k_{l+1},\bar\rho_1}(t)\,dt\right)^2\leqslant C_1\xi\rho^{n+2}\int_{t^0-a\xi\rho^2}^{t^0}\text{mes }D_l(t)\,dt,$$

where $C_1=C_1(n,m,M,\gamma,T)$. Summing it from r_0 to s with respect to l, we find

$$(s-r_0+1)\left[\int_{t_0-a\xi\rho^2}^{t^0}\text{mes }A_{k_{s+1},\bar\rho_1}(t)\,dt\right]^2\leqslant C_1k_na\xi^2\rho^{2n+}.$$

Take s such that

$$(C_1\varkappa_na/(s-r_0+1))^{\frac{1}{2}}\leqslant\theta_1\bar\rho_0^{n+2}/\rho^{n+2},$$

we obtain (3.19).

The proof of the second part of the lemma is similar. If (3.21) fails, then $\omega\geqslant2^{s+2}\rho^\varepsilon$. We shall show that (3.23) holds at this time. By the hypothesis (3.20) and the second part of Lemma 3.4, it follows that for $t\in[t^0-a\eta\rho^2,t^0]$,

$$\text{mes}[K(\bar\rho_1)\setminus B_{\tilde\mu+\frac{\omega}{2}(1-\beta),\bar\rho_1}(t)]\geqslant b\varkappa_n\bar\rho_1^n.$$

Select r_0 such that$2^{-r_0}\geqslant(1-\beta)/2$, hence we have

$$\text{mes}[K(\bar\rho_1)\setminus B_{k_l,\bar\rho_1}(t)]\geqslant b\varkappa_n\bar\rho_1^n\text{ if }l\geqslant r_0,\ t\in[t^0-a\eta\rho^2,t^0],$$

where $k_l=\tilde\mu+\omega/2^l$. It is similar to (3.25) that for $s\geqslant l\geqslant r_0$,

$$\omega^2/2^{2l+2}\left[\int_{t^0-a\eta_{k_{s+2}}\rho^2}^{t^0}\text{mes }B_{k_l,\bar\rho_1}(t)\,dt\right]^2$$
$$\leqslant(\beta/b\varkappa^{1-\frac{1}{n}})^2\bar\rho_1^{\frac{2}{n}}\int_{t^0-a\eta_{k_{s+2}}\rho^2}^{t^0}\text{mes }\tilde D_l(t)\,dt\cdot\int_{t^0-a\eta_{k_{s+2}}\rho^2}^{t^0}\int_{B_{k_l,\bar\rho_1}(t)}|\nabla w|^2dx\,dt, \qquad (3.26)$$

where $\tilde D_l(t)=B_{k_l,\bar\rho_1}(t)\setminus B_{k_{l+1},\bar\rho_1}(t)$. We get

$$\left[\int_{t^0-a\eta_{k_{s+2}}\rho^2}^{t^0}\text{mes }B_{k_l,\bar\rho_1}(t)\,dt\right]^2\leqslant C_1\eta_{k_{s+2}}\rho^{n+2t^0}\int_{-a\eta_{k_{s+2}}\rho^2}^{t^0}\text{mes }\tilde D_l(t)\,dt.$$

The rest of the proof is analogous to the previous proof for (3.19).

Lemma 3.5. *Suppose that*

$$\omega_1=\text{osc}\{w;\Gamma_\rho\}\leqslant L\rho^{\varepsilon_1}, \qquad (3.27)$$

where $\varepsilon_1>0$ *and* $\Gamma_\rho=\Gamma\subset Q_\rho$.

(A) *If* $K(\bar\rho_1)$ *satisfies*

$$\operatorname{mes}[K(\bar{\rho}_1)\setminus(K(\bar{\rho}_1)\cap\Omega)]\geqslant b_1\bar{\rho}_1^n, \tag{3.28}$$

where b_1 is a positive constant, then for any $\theta_1>0$ there exists $s=s(\theta_1)>0$ such that we have one of the following

(i)

$$\omega=\operatorname{osc}\{w,\ Q_\rho\cap Q_T\}\leqslant 2^{s+2}\rho^\varepsilon \tag{3.29}$$

or (ii)

$$\int_{t^0-a\xi\rho^2}^{t^0}\operatorname{mes} A_{\mu-\frac{w}{2^{s+1}},\,\bar{\rho}_1}(t)\,dt\leqslant\theta_1\xi\bar{\rho}_1^{n+2} \tag{3.30}$$

or (iii)

$$\int_{t^0-a\eta_{k_{s+1}}\rho^2}^{t^0}\operatorname{mes} B_{\tilde{\mu}+\frac{w}{2^{s+1}},\,\bar{\rho}_1}(t)\,dt\leqslant\theta_1\eta_{k_{s+1}}\bar{\rho}_1^{n+2}, \tag{3.31}$$

where $k_s=\tilde{\mu}+\omega/2^s$, ε is any number in $(0,\ \varepsilon_1]$.

(B) *If*

$$t^0-a\xi\rho^2\leqslant 0,$$

then instead of (3.30) *and* (3.31), *we set respectively*

(ii) $\operatorname{mes} A_{\mu-\frac{w}{4},\bar{\rho}_1}(0)=0,$

$$\int_0^{t^0}\operatorname{mes} A_{\mu-\frac{w}{2^{s+1}},\,\bar{\rho}_1}(t)\,dt\leqslant\theta_1 t^0\bar{\rho}_1^n, \tag{3.30)$'$}$$

(iii)′ $\operatorname{mes} B_{\tilde{\mu}+\frac{w}{4},\bar{\rho}_1}(0)=0,$

$$\int_0^{t^0}\operatorname{mes} B_{\tilde{\mu}+\frac{w}{2^{s+1}},\,\bar{\rho}_1}(t)\,dt\leqslant\theta_1 t^0\bar{\rho}_1^n, \tag{3.31)$'$}$$

and we have the same conclusion.

Proof Take $r_0\geqslant 2$ such that

$$2^{-r_0}\geqslant 1-\beta,\quad 2^{r_0}\geqslant 4L.$$

If $\omega\geqslant 2^{s+2}\rho^\varepsilon$, then

$$\omega\geqslant 2^{r_0}\rho^\varepsilon\geqslant 4L\rho^\varepsilon\geqslant 4\omega_1\quad\text{if}\quad s\geqslant r_0.$$

Hence, the range of $w(x,\ t)$ on Γ_ρ superimposes at most on one of the intervals $[\tilde{\mu},\ \tilde{\mu}+\omega/4]$ and $[\mu-\omega/4,\ \mu]$. we shall show that (3.30) is true if it superimposes on $[\tilde{\mu},\ \tilde{\mu}+\omega/4]$ and that (3.31)is true in another case.

Now let the first case come up. At this time take $k=\mu-w/2^{r_0}$. It is clear that

$$k\geqslant\max\{\mu/2,\ \max_{\Gamma_\rho} w(x,\ t)\},\ \operatorname{mes}(K(\bar{\rho}_1)\setminus A_{k,\bar{\rho}_1}(t))\geqslant b_1\bar{\rho}_1^n$$

$$\text{for } t\in[t^0-a\xi\rho^2,\ t^0].$$

The proof of (3.30) is similar to that of (3.18).

If $t^0-a\xi\rho^2<0$, we have

$$\operatorname{mes} A_{\mu-\frac{\omega}{4},\bar{\rho}_1}(0)=0, \tag{3.32}$$

because the range of $w(x,\ t)$ on Γ_ρ does not superimpose on $[\mu-\omega/4,\ \mu]$ in this case. By means of the method used in Lemma 3.4 and Lemma 3.5, we can obtain (3.30)′. The rest of the lemma is similar to Lemma 3.5.

Lemma 3.6. *For any $\theta_2>0$ there exists $\theta_1>0$ such that*

(i) *If*

$$k\geqslant\max\{\mu/2, \max_{Q_\rho\cap\Gamma} w(x, t)\},\ H=\mu-k>0,\ \int_{t^0-a\xi\rho^2}^{t^0} \operatorname{mes} A_{k,\bar\rho_1}(t)dt\leqslant\theta_1\xi\bar\rho_1^{n+2}, \tag{3.33}$$

then for $t\in[t^0-a\xi\rho^2/4, t^0]$

$$\operatorname{mes} A_{k+\frac{H}{2},\bar\rho_2}(t)\leqslant\theta_2\bar\rho_2^n. \tag{3.34}$$

Moreover, if $\operatorname{mes} A_{k,\bar\rho_1}(t^0-a\xi\rho^2)=0$, *then* (3.34) *holds in* $[t^0-a\xi\rho^2, t^0]$.

(ii) *If*

$$k\leqslant\min_{Q_\rho\cap\Gamma} w(x, t),\ H=k-\tilde\mu\geqslant\rho^\varepsilon,\ \int_{t^0-a\eta_{k-\frac{H}{2}}\rho^2}^{t^0} \operatorname{mes} B_{k,\bar\rho_1}(t)dt\leqslant \theta_1\eta_{k-\frac{H}{2}}\bar\rho_1^{n+2}, \tag{3.35}$$

then for $t\in[t^0-a\eta_{k-\frac{H}{2}}\rho^2/4, t^0]$,

$$\operatorname{mes} B_{k-\frac{H}{2},\bar\rho_2}(t)\leqslant\theta_2\bar\rho_2^n. \tag{3.36}$$

Moreover, if $\operatorname{mes} B_{k,\bar\rho_1}(t^0-a\eta_{k-\frac{H}{2}}\rho^2)=0$, *then* (3.36) *holds in* $[t^0-a\eta_{k-\frac{H}{2}}\rho^2, t^0]$.

Proof We prove the second part of the lemma as an example. Integrating (2.3) with $\zeta(x)=\zeta(x;\bar\rho_1,\bar\rho_2)$ with respect to t from τ to t for $t^0\geqslant t>\tau\geqslant t^0-a\rho^2\eta_{k-\frac{H}{2}}$, we find

$$\exp[-\gamma\Phi'(k)t]\tilde\chi_k(H/2)\operatorname{mes} B_{k-\frac{H}{2},\bar\rho_2}(t)$$
$$\leqslant\exp[-\gamma\Phi'(k)\tau]\Big[\tilde\chi_k(H)\operatorname{mes} B_{k,\bar\rho_1}(\tau)+\gamma[H^2/(\bar\rho_1-\bar\rho_2)^2+1]\int_\tau^t \operatorname{mes} B_{k,\bar\rho_1}(t)dt\Big]. \tag{3.37}$$

Since

$$\int_{t^0-a\eta_{k-\frac{H}{2}}\rho^2}^{t^0-\frac14 a\eta_{k-\frac{H}{2}}\rho^2} \operatorname{mes} B_{k,\bar\rho_1}(t)\,dt\leqslant\theta_1\eta_{k-\frac{H}{2}}\bar\rho_1^{n+2},$$

there exists $\tau\in[t^0-a\eta_{k-\frac{H}{2}}\rho^2,\ t^0-a\eta_{k-\frac{H}{2}}\rho^2/4]$ such that

$$\operatorname{mes} B_{k,\bar\rho_1}(\tau)\leqslant4/(3a)\theta_1\bar\rho_1^n.$$

Substituting it into (3.37), we obtain that for $t\in[t^0-a/4\eta_{k-\frac{H}{2}}\rho^2, t^0]$

$$\operatorname{mes} B_{k-\frac{H}{2},\bar\rho_2}(t)\leqslant\exp[\gamma\Phi'(k)a\rho^2\eta_{k-\frac{H}{2}}]$$
$$\cdot\{\tilde\chi_k(H)/\chi_k(H/2)4/(3a)+\gamma\eta_{k-\frac{H}{2}}/\tilde\chi_k(H/2)(9H^2(1-\bar\rho_0)^{-2}+\rho^2/\rho)\}\theta_1\bar\rho_1^n.$$

By Lemma 3.3, one has

$$\tilde\chi_k(H)/\tilde\chi_k(H/2)\leqslant1+4m^3,\ \tilde\chi_k(H/2)/\eta_{k-\frac{H}{2}}\geqslant1/(8m)H^2.$$

Therefore there exists θ_1 so small such that (3.36) holds.

If $\operatorname{mes} B_{k,\bar\rho_1}(t^0-a\eta_{k-\frac{H}{2}}\rho^2)=0$, it suffices to take $\tau=t^0-a\eta_{k-\frac{H}{2}}\rho^2$ in the above-mentioned argument.

Lemma 3.7. *There exists $\theta_2>0$ such that*

(i) *If*

$$k \geqslant \max\{\mu/2, \max_{\Gamma_\rho} w(x, t)\}, \; H=\mu-k>0, \; \max_{t\in[t^0-a\xi\rho^2, t^0]} \operatorname{mes} A_{k,\bar\rho_2}(t) \leqslant \theta_2 \bar\rho_2^n, \tag{3.38}$$

then for $t \in [t^0 - a\xi\rho^2/4, t^0]$,

$$\operatorname{mes} A_{k+\frac{H}{2}, \bar\rho_3}(t) = 0. \tag{3.39}$$

Moreover, if $\operatorname{mes} A_{k,\bar\rho_2}(t^0 - a\xi\rho^2) = 0$, *then* (3.39) *holds for* $t \in [t^0 - a\xi\rho^2, t^0]$.

(ii) *If*

$$k \leqslant \min_{\Gamma_\rho} w(x, t), \; H = k - \tilde\mu \geqslant \rho^\varepsilon, \; \max_{t\in[t^0 - a\eta_k\rho^2, t^0]} \operatorname{mes} B_{k,\bar\rho_2}(t) \leqslant \theta_2 \bar\rho_2^n, \tag{3.40}$$

then for $t \in [t^0 - a\eta_k\rho^2/4, t^0]$,

$$\operatorname{mes} B_{k-\frac{H}{2}, \bar\rho_3}(t) = 0. \tag{3.41}$$

Moreover, if $\operatorname{mes} B_{k,\bar\rho_2}(t^0 - a\eta_k\rho^2) = 0$, *then* (3.41) *holds for* $t \in [t^0 - a\eta_k\rho^2, t^0]$.

Proof We still prove only the second part of the lemma. Let

$$\begin{aligned} &k_h = k - H/2 + H/2^{h+1}, \; t_h = t^0 - 1/4a\eta_k\rho^2 - 3/2^{h+2}a\eta_k\rho^2, \\ &\rho_h = \bar\rho_3 + (\bar\rho_2 - \bar\rho_3)/2^h, \; \mu_h = \max_{t\in[t_h, t^0]} \operatorname{mes} B_{k_h,\rho_h}(t), \\ &\zeta_h(x) = \zeta(x; \rho_h, \rho_{h+1}), \; I_h(t) = \exp[-\gamma\Phi'(k_h)(t - t^0 + a\eta_k\rho^2)] \\ &\quad \times \int_{B_{k_h,\rho_h}(t)} \tilde\chi_{k_h}(k_h - w)\,dx. \end{aligned} \tag{3.42}$$

Since $k_h \geqslant k/2$, by Lemma 3.3 it follows that

$$\tilde\chi_{k_h}(k_h - w) \leqslant m^2\Phi'(k_h)(k_h - w)^2 \leqslant 2m^4\eta_k(k_h - w)^2.$$

Hence when $t^0 - a\eta_k\rho^2 \leqslant t \leqslant t^0$, we have

$$I_h(t) \leqslant 2m^4\eta_k \int_{B_{k_h,\rho_h}(t)} (k_h - w)^2 \zeta_h^2\,dx. \tag{3.43}$$

In virtue of Lemma 3.1, it follows that

$$I_h(t) \leqslant C\eta_k\mu_h^{\frac{2}{n}}\Big[\int_{B_{k_h,\rho_h}(t)} \zeta_h^2 |\nabla w|^2 dx + H^2/(\rho_h - \rho_{h+1})^2 \mu_h\Big], \tag{3.44}$$

where $C = C(n, m)$. The inequality (2.3) implies that

$$\begin{aligned} &I_h'(t) + 1/2\exp[-\gamma\Phi'(k_h)(t - t^0 + a\eta_k\rho^2)]\int_{B_{k_h,\rho_h}(t)} |\nabla w|^2 \zeta_h^2\,dx \\ &\leqslant \gamma[H^2/(\rho_h - \rho_{h+1})^2 + 1]\mu_h. \end{aligned} \tag{3.45}$$

For any fixed $t \in [t_{h+1}, t^0]$, there are three cases:

(a) If $I_h'(t) \geqslant 0$, we obtain from (3.45) that

$$\begin{aligned} \int_{B_{k_h,\rho_h}(t)} |\nabla w|^2 \zeta_n^2 dx &\leqslant 2\gamma \exp[\gamma\Phi'(h_n)a\eta_k\rho^2][H^2/(\rho_h - \rho_{h+1})^2 + 1]\mu_h \\ &\leqslant 2\gamma\exp(\gamma C_1)[H^2/(\rho_h - \rho_{h+1})^2 + 1]\mu_h, \end{aligned}$$

where $C_1 = C_1(n, m, \gamma, T)$ when $\rho \leqslant 1$, $\varepsilon \leqslant \varepsilon_0 \leqslant 1/2$. Substiting it into (3.44) we find

$$I_h(t) \leqslant C\eta_h\mu_n^{\frac{2}{n}+1}[(2\Lambda\gamma\exp(\gamma C_1) + 1)H^2/(\rho_h - \rho_{h+1})^2 + 2\Lambda\gamma\exp(\gamma C_1), \tag{3.46}$$

(b) If $I_h'(t) < 0$ and there exists $\tau \in [t_h, t]$ such that $I_h'(\tau) = 0$, then we may select τ such that $I_h'(s) < 0$ for $s \in (\tau, t]$ and so $I_h(t) \leqslant I_h(\tau)$. $I_h(\tau)$ has the estimate (3.46) and so does $I_h(t)$.

(c) If $I_h' < 0$ for any $\tau \in [t_h, t]$, it follows from (3.45) that

$$1/(2\Lambda)\int_{t_h}^{t}\exp[-r\Phi'(k_h)\ (t-t^0+a\eta_k\rho^2)]\int_{B_{k_h},\rho_h(t)}|\nabla w|^2\xi_h^2\,dx\,dt$$
$$\leqslant I_h(t_h)+\gamma(t-t_h)[H^2/(\rho_h-\rho_{h+1})^2+1]\mu_h$$

or

$$\int_{t_h}^{t}\int_{B_{k_h},\rho_h(t)}(t)\,|\nabla w|^2\zeta_h^2\,dx\,dt\leqslant 2\Lambda\exp(\gamma C_1)\{I_h(t_h)+\gamma(t-t_h)[H^2/(\rho_h-\rho_{h+1})^2+1]\mu_h\}.$$

Integrating (3.44) from t_h to t and using the previous inequality one finds

$$\int_{t_h}^{t}I_h(\tau)\,d\tau\leqslant C\eta_k\mu_h^{\frac{2}{n}}[2\Lambda\exp(\gamma C_1)I_h(t_h)$$
$$+(2\Lambda\gamma\exp(\gamma C_1)+1)H^2(t+t_h)\mu_h/(\rho_h-\rho_{h+1})^2+2\Lambda\gamma\exp(\gamma C_1)(t-t_h)\mu_h].$$

In viratue of the decrease of $I_h(\tau)$ in $[t_h,t]$ and the inequality (3.43), it follows that for $t\in[t_{h+1},\ t^\circ]$,

$$I_h(t)\leqslant C\eta_k\mu_h^{\frac{2}{n}+1}[4m^4\Lambda\exp(\gamma C_1)\eta_k H^2/(t_{h+1}-t_h)$$
$$+(2\Lambda\gamma\exp(\gamma C_1)+1)H^2/(\rho_h-\rho_{h+1})^2+2\Lambda\gamma\exp(\gamma C_1).\qquad(3.47)$$

Thus, no matter which case it is, one has (3.47) for $t\in[t_{h+1},\ t^\circ]$.

On the other hand, applying Lemma 3.3 and the inequality (1.12), we have

$$I_h(t)\geqslant\exp(-\gamma C_1)\tilde{\chi}_{k_h}(k_h-k_{h+1})\ \mathrm{mes}\ B_{k_{h+1}},\ \rho_{h+1}(t)$$
$$\geqslant\exp(-\gamma C_1)\cdot\frac{1}{4m}\eta_k(k_h-k_{h+1})^2\ \mathrm{mes}B_{k_{h+1},\rho_{h+1}}(t).$$

Combining this with (3.47), we find that

$$\mu_{h+1}\leqslant 4mC\exp(\gamma C_1)\mu^{\frac{2}{n}+1}2^{2(h+1)}[4m^4\Lambda\exp(\gamma C_1)\eta_k/(t_{h+1}-t_h)$$
$$+(2\Lambda\gamma\exp(\gamma C_1)+1)/(\rho_h-\rho_{h+1})^2+2\Lambda\gamma\exp(\gamma C_1)/H^2].$$

By the definition (3.42) of t_h and ρ_h, it follows that

$$\mu_{h+1}\leqslant C_2 2^{4h}\mu_h^{\frac{2}{n}+1}/\rho^2,$$

where $C_2=C_2\ (n,\ m,\ \gamma,\ M,\ T)$. Setting $y_h=\mu_h/\rho^n$, we obtain

$$y_{h+1}\leqslant C_2 2^{4h}\,y_h^{\frac{2}{n}+1}.\qquad(3.48)$$

The hypothesis (3.40) implies that

$$y_0\leqslant\theta_2.$$

We shall prove by induction that

$$y_h\leqslant\theta_2 2^{-2nh}\quad(h=0,\ 1,\ 2,\ \cdots).\qquad(3.49)$$

In fact

$$y_{h+1}\leqslant C_2 2^{4h}\,y^{\frac{2}{n}+1}\leqslant C_2 2^{4h}(\theta_2 2^{-2nh})^{\frac{2}{n}+1}\leqslant C_2\theta_2^{\frac{2}{n}+1}2^{-2nh}.$$

If we take θ_2 satisfying

$$\theta_2^{\frac{2}{n}}\leqslant 2^{-2n}/C_2,$$

the induction argument will be valid. Thus (3.49) holds for any natural number h. Setting $h\to\infty$ in (3.49), we obtain (3.41).

if mes $B_{k,\rho}(t^\circ-a\eta_k\rho^2)=0$, it suffices to take $t_h=t^\circ-a\eta_k\rho^2$ $(h=0,\ 1,\ 2,\ \cdots)$ in the previous argument.

Lemma 3.8. *Suppose that $Q_\rho < Q_T$. Then there exists $s>0$ such that*
(i) *if*

$$k \geq \mu/2,\ H=\mu-k>0,\ \text{mes}\ A_{k,\bar\rho_1}(t^\circ - a\xi\rho^2) \leq 1/2\varkappa_n\bar\rho_1^n, \tag{3.50}$$

then for $t \in \left[t^\circ - \frac{1}{16} a\xi\rho^2,\ t^\circ\right]$,

$$\text{mes}\ A_{\mu-\frac{H}{2^{s+3}},\bar\rho_3}(t)=0. \tag{3.51}$$

(ii) *if*

$$\max_{t\in[t^\circ - a\eta\rho^2,\, t^\circ - a\xi\rho^2]} \text{mes}\ B_{\tilde\mu+\frac{\omega}{2},\bar\rho_1}(t) \leq 1/2\varkappa_n\bar\rho_1^n, \tag{3.52}$$

then

$$\omega \leq 2^{s+2}\rho^\varepsilon \tag{3.53}$$

or

$$\text{osc}\ \{w;\ \tilde Q_{\frac{\rho}{4}}\} \leq (1-1/2^{s+3})\,\text{osc}\{w;\ Q_\rho\}, \tag{3.54}$$

where

$$\tilde Q_{\frac{\rho}{4}} = \{(x,t)\,|\,x\in K(\rho/4),\ t^\circ - a\xi(\rho/4)^2 < t < t^\circ\}. \tag{3.55}$$

Proof We first determine the constant θ_2 by Lemma 3.7 and then θ_1 by Lemma 3.6 and finally s by Lemma 3.5. One can derive (3.51) from these lemmas without difficulty. As for the second part of the lemma, in the same way one can obtain

$$\omega \leq 2^{s+2}\rho^\varepsilon$$

or

$$\text{mes}\ B_{\tilde\mu+\frac{\omega}{2^{s+3}},\bar\rho_3}(t)=0 \ \text{for}\ t\in[t^\circ - a\eta_{k_{s+2}}(\rho/4)^2,\ t^\circ],$$

where $\eta_{k_s}=1/m\Phi'(k_s)$, $k_s=\tilde\mu+\omega/2^s$. For the second possibility, it is clear that

$$\text{osc}\{w;\ \tilde Q_{\frac{\rho}{4}}\} \leq \mu - (\tilde\mu + \omega/2^{s+3}) \leq (1-1/\,2^{s+3})\omega,$$

which is required.

Lemma 3.8′. *Suppose that $\partial\Omega$ satisfies the condition (1.9) and $w(x,t)$ belongs to $C^{\varepsilon_1,\varepsilon_1/2}(\Gamma)$. If $\tilde Q_{\frac{\rho}{4}} \cap \Gamma \neq \varnothing$, then for any $0<\varepsilon\leq\varepsilon_1$ there exists const ant s such that*

$$\text{osc}\ \{w;\ \tilde Q_{\frac{\rho}{4}} \cap Q_T\} \leq 2^{s+2}\rho^\varepsilon \tag{3.56}$$

or

$$\text{osc}\{w;\ \tilde Q_{\frac{\rho}{4}} \cap Q_T\} \leq (1-1/2^{s+3})\,\text{osc}\{w;\ Q_\rho \cap Q_T\}. \tag{3.57}$$

Proof $\tilde Q_{\frac{\rho}{4}} \cap \Gamma \neq \varnothing$ and the condition (1.9) imply that

$$\text{mes}\ [K(\bar\rho_1)\setminus(K(\bar\rho_1)\cap\Omega)] \geq b_1\bar\rho_1^n \ \text{or}\ t^\circ - a\xi\rho^2 \leq 0.$$

Applying Lemma 3.5′, 3.6 and 3.7 we can obtain what we want.

§ 4

In order to find Hölder estimate for $w(x,t)$, we still need the following lemma.

Lemma 4.1. *For $\rho_0 \leq 1$, suppose that $Q_{\rho_0} < Q_T$. Let $u(x,t)$ be a classical solution of the equation (1.9) in Q_{ρ_0} satisfying $0<u(x,t)\leq M$. If*

$$\omega_0 \geq \hat C\rho_0^\varepsilon,\quad \mu_0 \geq 2\hat C\rho_0^\varepsilon, \tag{4.1}$$

We can obtain Hölder interior estimate for $w(x, t)$ now.

Theorem 4.1. *Suppose that the coefficients of the equation* (1.9) *satisfy the conditions* (1.4), (1.5), (1.6) and (1.10). *Let* $u(x, t)$ *be a classical solution of the equation* (1.9) *satisfring* $0<u(x, t)\leqslant M$. *For any* $(x^\circ, t^\circ)\in Q_T$, *denote that* $\rho_0=\min\{1, [mM^{1-\frac{1}{m}}\Phi'(M)]^{-1}, d(x^\circ), [mt^\circ/(a\Phi'(1))]^{\frac{1}{2}}\}$, *where* $d(x^\circ)s=\mathrm{dist}\{x^\circ, \partial\Omega\}$.

Then, for any $(x, t)\in Q^*_{\rho_0}$, *we have*

$$|w(x, t)-w(x^\circ, t^\circ)|\leqslant C\rho_0^{-2}[|x-x_0|^\alpha+|t-t^\circ|^{\alpha/2}] \quad (\alpha>0), \tag{4.10}$$

where C, α *depend only on* n, m, δ, Λ, M *and* T, *and*

$$Q^*_{\rho_0}=\{(x, t)\,|\,|x-x^\circ|<\rho_0,\ t^\circ-a/m\Phi'(1)\rho_0^2<t\leqslant t^\circ\}.$$

Proof Let

$$\hat{C}=\max\{(2^{s+8}m^3)^m,\ 4\varphi(M)/\rho_0\}, \tag{4.11}$$

$$\tau=64m^3\hat{C}^{1-\frac{1}{m}}, \tag{4.12}$$

where the constant s is defined by Lemma 3.8 and take ε so small that

$$0<\varepsilon<1/2,\ (1/\tau)^\varepsilon\geqslant 1-2^{-[1+\frac{\tau}{16}(s+3)]}. \tag{4.13}$$

Denote that

$$\rho_l=\rho_0/\tau^l \quad (l=0, 1, 2, \cdots),$$

$$Q_l=Q_{\rho_l}=\{(x, t)\,|\,|x-x^\circ|<\rho_l,\ t^\circ-a\Phi'(\rho_l^\varepsilon)\rho_l^2<t<t^\circ\},$$

$$\mu_l=\max_{Q_l}\{w(x, t)\},\quad \tilde{\mu}_l=\min_{Q_l}\{w(x, t)\}, \tag{4.14}$$

$$\omega_l=\mu_l-\tilde{\mu}_l,\quad l^*=\min_{l\geqslant 0}\{l\,|\,\mu_l\geqslant 2\hat{C}\rho_l^\varepsilon\}.$$

By (4.11) we have

$$\omega_0\leqslant\hat{C}\rho_0^\varepsilon. \tag{4.15}$$

If $l^*=0$, one immediately obtains (4.10) by Lemma 4.1.

Now suppose that $l^*>0$. We shall argue by induction

$$\omega_l\leqslant\hat{C}\rho_l^\varepsilon \text{ if } l\leqslant l^*. \tag{4.16}$$

Assume that (4.16) is true for $l(<l^*)$. If $\mu_l\leqslant 2\rho_l^\varepsilon$, it is easy to show that $\omega_{l+1}\leqslant\mu_l\leqslant\hat{C}\rho_{l+1}^\varepsilon$. So let $\mu_l>2\rho_l^\varepsilon$, we can apply the results in § 3. Consider the following two cases:

(i) If

$$\max_{t\in[t^\circ-a\eta\rho^2,\ t^\circ-a\xi\rho_l^2]} \operatorname{mes} B_{\tilde{\mu}_l+\frac{\omega_l}{2},\rho_{1l}}(t)\leqslant 1/2\varkappa_n\rho_{1l}^n, \tag{4.17}$$

where $\eta=\Phi'(\rho_l^\varepsilon)$, $\xi=1/m^2\Phi'(\mu_l)$, $\rho_{1l}=(2+\bar{\sigma}_0)/3\rho_l$, $\bar{\sigma}_0=2^{-\frac{1}{n}}$, by Lemma 3.8, it follows that

$$\omega_l\leqslant 2^{s+2}\rho_l^\varepsilon\leqslant 2^{s+2}\tau^\varepsilon\rho_{l+1}^\varepsilon\leqslant 2^{-6}/m^3\hat{C}^{\frac{1}{m}}(64m^3\hat{C}^{1-\frac{1}{m}})^\varepsilon\rho_{l+1}^\varepsilon\leqslant\hat{C}\rho_{l+1}^\varepsilon \tag{4.18}$$

or

$$\operatorname{osc}\{w:\ \tilde{Q}_{\frac{\rho_0}{4}}\}\leqslant(1-1/2^{s+3})\operatorname{osc}\{w,\ Q_{\rho_l}\}\leqslant(1/\tau)^\varepsilon\hat{C}\rho_l^\varepsilon\leqslant\hat{C}\rho_{l+1}^\varepsilon. \tag{4.19}$$

By the selection (4.12) of τ and the condition $l<l^*$ which means $\mu_l<2\hat{C}\rho_l^\varepsilon$, we can show that $Q_{l+1}\subset\tilde{Q}_{\rho_l/4}$. In fact, in virtue of (1.12), we have

$$a\xi(\rho_l/4)^2/(a\Phi'(\rho_{l+1}^\varepsilon)\rho_{l+1}^2)\geqslant\tau^2/(16m^3)(2\hat{C}\tau^3)^{\frac{1}{m}-1}$$
$$\geqslant\tau^{1+\frac{1}{m}}/(32m^3\hat{C}^{1-\frac{1}{m}})\geqslant 2, \tag{4.20}$$

which implies $Q_{l+1}\subset\tilde{Q}_{\rho_l/4}$. Therefore, (4.16) holds by induction in this case.

(ii) If (4.17) fails, then there exists $\bar{t}\in[t^\circ-a\eta\rho_l^2,\ t^\circ-a\xi\rho_l^2]$ such that

$$\operatorname{mes} A_{\mu_l-\frac{\omega_l}{2},\rho_{1l}}(\bar{t})\leqslant 1/2\varkappa_n\rho_{1l}^n. \tag{4.21}$$

Divide the interval $[\bar{t},\ t^\circ]$ into N equal parts such that

$$1/2a\xi\rho_l^2\leqslant(t^\circ-\bar{t})/N\leqslant a\xi\rho_l^2, \tag{4.22}$$

where

$$N\leqslant 2a\Phi'(\rho_l^\varepsilon)\rho_l^2/(a\xi\rho_l^2)\leqslant 2m^3(2\hat{C}\rho_l^\varepsilon/\rho_l^\varepsilon)^{1-\frac{1}{m}}\leqslant 4m^3\hat{C}^{1-\frac{1}{m}}=\frac{\tau}{16}.$$

Let $t_p=\bar{t}+\frac{p-1}{N}(t^\circ-\bar{t})$ $(p=1,\ 2,\ \cdots,\ N)$ and $t_{N+1}=t^\circ$. By (4.21), we apply Lemma 3.8 to the interval $[t_1,\ t_2]$ and then obtain

$$\operatorname{mes} A_{\mu-\frac{\omega}{2^{s+4}},\rho_3}(t_2)=0,$$

where $\rho_3=\bar{\sigma}_0\rho$, $\bar{\sigma}_0=2^{-\frac{1}{n}}$ and we omit the subscript l.

Thus

$$\operatorname{mes} A_{\mu-\frac{\omega}{2^{s+4}},\rho_1}(t_2)\leqslant\operatorname{mes} A_{\mu-\frac{\omega}{2^{s+6}},\rho_3}(t_2)+\varkappa_n(\rho_1^n-\rho_3^n)\leqslant 1/2\varkappa_n\rho_1^n.$$

Arguing by induction, we obtain

$$\operatorname{mes} A_{\mu-\frac{\omega}{2^{(s+3)(N+1)}},\rho_3}(t_N)\leqslant 1/2\varkappa_n\rho_1^n$$

and

$$\operatorname{mes} A_{\mu-\frac{\omega}{2^{(s+1)N+1}}\rho_3}(t)=0\ \text{for}\ t\in\left[t^\circ-\frac{a}{32}\xi\rho^2,\ t^\circ\right]. \tag{4.23}$$

Combing with (4.20), we have

$$\operatorname{osc}\{w;\ Q_{l+1}\}\leqslant(1-1/2^{(s+3)N+1})\operatorname{osc}\{w;\ Q_l\}\leqslant\tau^{-s}\hat{C}\rho_l^\varepsilon\leqslant\hat{C}\rho_{l+1}^\varepsilon. \tag{4.24}$$

So far we have shown that (4.16) holds for $l<l^*$. We have $w_{l^*}\leqslant\hat{C}\rho_{l^*}^\varepsilon$ in particular and $\mu_{l^*}\geqslant 2\hat{C}\rho_{l^*}^\varepsilon$ by the definition of l^*. Applying Lemma 4.1 we obtain (4.10) for $(x,\ t)\in Q_{l^*}$.

When $(x,\ t)\in Q_l\backslash Q_{l+1}$ $(l<l^*)$, applying (4.16) and (1.12), we have

$$(|x-x^\circ|^2+|t-t^\circ|)^{\frac{1}{2}}\geqslant\min\{\rho_{l+1},\ [a\Phi'(\rho_{l+1}^\varepsilon)\rho_{l+1}^2]^{\frac{1}{2}}\}$$
$$\geqslant\min\{1,\ [a/m\Phi'(M)]^{\frac{1}{2}}\}\rho_{l+1}\geqslant\min\{1,\ [a/m\Phi'(M)]^{\frac{1}{2}}\}\frac{1}{\tau}(\omega_l/\hat{C})^{\frac{1}{\varepsilon}}$$

or

$$|w(x,\ t)-w(x^\circ,\ t^\circ)|<[\tau^2m/(a\Phi'(M))]^{\frac{\varepsilon}{2}}\hat{C}(|x-x^\circ|^2+|t-t^\circ|)^{\frac{\varepsilon}{2}}.$$

Hence we obtain (4.10) for $(x,t)\in Q_l\backslash Q_{l+1}$ $(l\leqslant l^*)$.

Theorem 4.2. *Suppose that* (1.4), (1.5), (1.6), (1.7) *and* (1.8) *are satisfied. Let* $u(x,\ t)$ *be a classical solution of the equation* (1.1) *satisfying* $0<u\leqslant M$ *and belong to* $C^{\varepsilon_1,\varepsilon_1/2}(\Gamma)$. *Then, for any* $(x^\circ,\ t^\circ)\in\bar{Q}_T$ *and* $(x,\ t)\in Q_{\rho_0}^*\cap Q_T\Big(\rho_0=a_0,\ Q_{\rho_0}^*=\Big\{(x,\ t)\,|\ |x-x^\circ|<\rho_0,\ t-\frac{a}{m}\Phi'(1)\rho_0^2<t\leqslant t^\circ\Big\}\Big)$, *we have*

$$|w(x, t)-w(x^\circ, t^\circ)|\leqslant C[|x-x^\circ|^\alpha+|t-t^\circ|^{\frac{\alpha}{2}}],$$

where α, C depend only on n, m, T, Λ, ε_1, δ, M, $\|u\|_{C^{\varepsilon_1,\varepsilon_1/2}}(\Gamma)$ and the constants θ_0, a_0 of the condition (1.8).

Proof Let $\rho_0=a_0$ and $\hat{C}$, τ, ε are defined in (4.11), (4.12), (4.13) respectively and, moreover, $\varepsilon\leqslant\varepsilon_1$. Assume that $(x^\circ, t^\circ)\in Q_T$. If $Q_{\rho_0}\subset Q_T$, then (4.25) is just the result of Theorem 4.1. Now let $Q_{\rho_0}\cap\Gamma\neq\phi$. Then there exists $l_0\geqslant 1$ such that

$$Q_{l_0-1}\cap\Gamma\neq\phi,\quad Q_{l_0}\cap\Gamma=\phi.$$

By Lemma 3.8′, it follows that

$$\operatorname{osc}\{w, Q_l\cap Q_T\}\leqslant\hat{C}\rho_l^\varepsilon \text{ if } l\leqslant l_0-1$$

and hence

$$\operatorname{osc}\{w, Q_{l_0}\cap Q_T\}\leqslant\hat{C}\rho_{l_0-1}^\varepsilon\leqslant\hat{C}\tau^\varepsilon\rho_{l_0}^\varepsilon=\tilde{C}\rho_{l_0}^\varepsilon. \tag{4.26}$$

Using $\tilde{C}$, ρ_{l_0} instead of $\hat{C}$, ρ_0 respectively in the proof of Theorem 4.1, we can obtain (4.25). The validity of (4.25) for $(y^\circ, t^\circ)\in\bar{Q}_T$, is proved by a limit process. The proof is thus complete.

In virtue of the inequalities (1.12), (1.13) and (1.14), it is easy to find

$$|\Phi(w_1)-\Phi(w_2)|\leqslant C|w_1-w_2|^{\frac{1}{m}}, \tag{4.27}$$

where $C=C(m, M)$. We can immediately establish Hölder estimate for $u(x, t)$ from that for $w(x, t)$.

§ 5

We give the definition of weak solution as follows.

Definition. *A function $u(x, t)$ defined on $\bar{Q}_T$ is said to be a weak solution of* (1.1), (1.2) *and* (1.3) *if u is real, nonnegative and $u(x, t)\in C(\bar{Q}_T)$, satisfying* (1.2) *and* (1.3) *in the usual sense and satisfying* (1.1) *in the generalized sense as follows.*

For all $\varphi\in C^{2,1}(Q_T)\cap C^1(\bar{Q}_T)$ with $\varphi|_{t=T}=\varphi|_{\partial\Omega\times[0,T]}=0$, the following identity holds

$$\begin{aligned}&\int_\Omega u\varphi\,dx|_{t=0}+\int_{Q_T}\{u\varphi_t+A_{ij}(x, t, u)\varphi_{x_ix_j}+[A_i(x, t, u)-B_i(x, t, u)]\varphi_{x_i}\\&\quad+[c(x, t, u)u-B(x, t, u)]\varphi\}dx\,dt-\int_{\partial\Omega\times[0,T]}A_{ij}(s, t, \psi(s, t))\\&\quad\times\cos(N, x_i)\cos(N, x_j)\frac{\partial\varphi}{\partial N}(s, t)\,ds\,dt=0,\end{aligned} \tag{5.1}$$

where N is the outer normal of $\partial\Omega$ and

$$\begin{aligned}A_{ij}(x, t, r)&=\int_0^r a_{ij}(x, t, s)ds,\quad A_j(x, t, r)=\int_0^r\frac{\partial a_{ij}}{\partial x_i}(x, t, s)ds,\\B_i(x, t, r)&=\int_0^r b_i(x, t, s)ds,\quad B(x, t, r)=\int_0^r\frac{\partial b_i}{\partial x_i}(x, t, s)ds.\end{aligned} \tag{5.2}$$

Theorem 5.1. *Suppose that* (i), (ii), (iii), (iv) *of* § 1 *are satisfied, and*

$$\partial\Omega \in C^{1+\beta_0}(\beta_0>0),\ u_0(x)\geqslant 0,\ \psi(s,t)\geqslant 0,\ u_0(s)=\psi(s,0),\ u_0(x)\in C^{\beta}(\bar{\Omega}),$$

$$\psi(s,t)\in C^{\beta,\frac{\beta}{2}}(\partial\Omega\times[0,T])\quad (0<\beta<1).$$

Then there exists weak, solution u of (1.1), (1.2) and (1.3) and u satisfies Holder condition on $\bar{Q}_T$.

Proof For $k=1, 2, \cdots$, let

$$\bar{u}_{0k}(x)=\begin{cases} u_0(x), & \text{when } u_0(x)>1/k,\\ 1/k, & \text{when } u_0(x)\leqslant 1/k,\end{cases}\quad \bar{\psi}_k(s,t)=\begin{cases}\psi(s,t), & \text{when } (s,t)>1/k,\\ 1/k, & \text{when}(s,t)\leqslant 1/k.\end{cases} \tag{5.3}$$

Let $u_{0k}(x)$, $\psi_k(s,t)$ be the convolution of $\bar{u}_{0k}$, $\bar{\psi}_k$ with molifier such that $u_{0k}(s)=\psi_k(s,0)$,

$$|u_{0k}(x)-\bar{u}_{0k}(x)|\leqslant \nu(1/k)/k,\ |\psi_k(s,t)-\bar{\psi}_k(s,t)|\leqslant \nu(1/k)/k. \tag{5.4}$$

We have

$$u_{0k},\ \psi_k \in C^{1+\beta_0},\ u_{0k}(x)\geqslant 1/k,\ \psi_k(s,t)\geqslant 1/k. \tag{5.5}$$

Let $\bar{a}_{ij\varepsilon}(x,t,r)$, $b_{i\varepsilon}(x,t,k)$, $c_\varepsilon(x,t,r)$ be the convolution of $\bar{a}_{ij}(x,t,r)=a_{ij}(x,t,r)/\nu(r)$, $b_i(x,t,r)$, $c(x,t,r)$ with molifier.

Let

$$r\nu(r)\Big/\int_0^r \nu(s)ds=\alpha(r),$$

we have

$$\nu(r)=\int_0^1 \nu(s)ds\,\frac{\alpha(r)}{r}\ \exp\Big[\int_0^1 \alpha(s)\frac{ds}{s}\Big].$$

It follows from (1.6) that

$$1\leqslant\alpha(r)\leqslant m,\quad \alpha(r)\in C(0,\infty).$$

Let $\alpha_\varepsilon(r)$ be the smooth approximation of $\alpha(r)$ such that

$$|\alpha_\varepsilon(r)-\alpha(r)|\leqslant\varepsilon/[(\log r)^2+1]. \tag{5.6}$$

(5.6) can be achieved as follows.

Let $\alpha(\log r)=\beta(s)$, then $\beta(s)\in C(-\infty,\infty)$. $\forall\varepsilon>0$, we can find $\delta_{k,\varepsilon}(k=1,2,\cdots)$ such that

$$|\beta(s')-\beta(s)|\leqslant\varepsilon/(k^2+1),\ \forall|s|\leqslant k,\ |s'-s|\leqslant\delta_{k,\varepsilon}.$$

Let $\sigma(s)$ be a smooth decreasing function of $s\in[0,\infty)$ such that

$$\sigma(k-1)=\delta_{k,\varepsilon}\quad (k=1,2,\cdots).$$

Let $\beta_\varepsilon(s)$ be the convolution of $\beta(s)$ with molifier of variable radius $\sigma(s)$, then $\beta_\varepsilon(s)\in C^\infty(-\infty,\infty)$ and

$$\beta_\varepsilon(s)=\int_{|\tau|\leqslant\sigma(s)}\beta(s-\tau)\exp\Big[\frac{\tau^2}{\sigma(s)^2}-1\Big]^{-1}d\tau/K(s),$$

where

$$K(s)=\int_{|\tau|\leqslant\sigma(s)}\exp\Big[\frac{\tau^2}{\sigma(s)^2}-1\Big]^{-1}d\tau.$$

When $k-1\leqslant|s|\leqslant k$, we have

$$|\beta_\varepsilon(s)-\beta(s)|=\left|\int_{|\tau|\leq\sigma(s)}[\beta(s-\tau)-\beta(s)]\exp\left[\frac{\tau^2}{\sigma(s)^2}-1\right]^{-1}d\tau/K(s)\right|$$
$$\leq\max_{|s-s'|\leq\sigma(s)}|\beta(s')-\beta(s)|\leq\max_{|s|\leq k,|s'-s|\leq\delta_{k,\varepsilon}}|\beta(s')-\beta(s)|\leq\varepsilon/(k^2+1)$$
$$\leq\varepsilon/(s^2+1).$$

(5.6) follows by taking $\alpha_\varepsilon(r)=\beta_\varepsilon(e^r)$.

Let

$$\nu_\varepsilon(r)=\int_0^1\nu(s)ds\frac{\alpha_\varepsilon(r)}{r}\exp\left[\int_1^r\alpha_\varepsilon(s)\frac{ds}{s}\right].$$

It is easily seen that

$$\nu_\varepsilon(r)\in C[0,\infty),\ \nu_\varepsilon(0)=0,\ 1\leq r\nu_\varepsilon(r)\Big/\int_0^r\nu_\varepsilon(s)ds\leq m,$$

$$e^{-\frac{\pi}{2}}/m\leq\nu_\varepsilon(r)/\nu(r)\leq me^{\frac{\pi}{2}}. \tag{5.7}$$

Let

$$a_{ij\varepsilon}(x,t,r)=\bar{a}_{ij\varepsilon}(x,t,r)\nu_\varepsilon(r).$$

Solve the first boundary value problem

$$\begin{cases}u_t=(a_{ij\varepsilon}u_{x_j})_{x_i}+b_{i\varepsilon}u_{x_i}+c_\varepsilon u, & (5.8)\\ u|_{t=0}=u_{0k}(x), & (5.9)\\ u|_{\partial\Omega\times[0,T]}=\psi_k(s,t), & (5.10)\end{cases}$$

By maximum principle we have

$$e^{-2AT}/k\leq u\leq M,$$

where $M=\max\{\sup_\Omega u_0(x),\ \sup_{\partial\Omega\times[0,T]}\psi(s,t)\}$. Hence problem (5.8)—(5.10) is non-degenerate, its solution $u_{k\varepsilon}$ exists, and $u_{k\varepsilon}\in C^{2,1}(Q_T)\cap C(\bar{Q}_T)$. By Theorems 4.1 and 4.2, we have $u_{k\varepsilon}\in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$, where α and $\|u_{k\varepsilon}\|_{C^{\alpha,\frac{\alpha}{2}}}(Q)$ are independent of k and ε. We also have $u_{k\varepsilon}\in C^{1+\alpha,1}(\bar{Q}_T)$, where $\|u\|_{C^{1+\alpha,1}}$ is independent of ε. $u_{k\varepsilon}$ is compact in $C(\bar{Q}_T)$, hence we can select a partial sequence $\{u_{k\varepsilon}\}$ such that $u_{k\varepsilon}\to u_k$ $(i\to\infty)$ $\forall(x,t)\in\bar{Q}_T$. u_k satisfies (5.9) and (5.10) in the usual sense, and satisfies (6.1) by $u_{0k}(x)$, $\phi_k(s,t)$ instead of $u_0(x)$, $\psi(s,t)$. Moreover $e^{-2AT}/k\leq u_k\leq M$, α and $\|u_k\|_{C_{\alpha,\frac{\alpha}{2}}}(\bar{Q}_T)$ are independent of k. $\{u_k\}$ is compact in $C(\bar{Q}_T)$. Hence we can select a partial sequence of $\{u_k\}$ (still denote it by $\{u_k\}$) such that $u_k\to u$, $u(x,t)\in\bar{Q}_T$. u satisfies (1.2), (1.3), (5.1) and $u\in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$. This completes the proof of the theorem.

Turn to consider the uniqueness problem of the weak solution. If there exists another weak solution $\tilde{u}(x,t)\in C(\bar{Q}_T)$ satisfying (1.2), (1.3) and (5.1). From (5.1) we have

$$\int_{Q_T}(u_k-\tilde{u})[\varphi_t+a_{ij}^{(k)}(x,t)\varphi_{x_ix_j}+b_i^{(k)}(x,t)\varphi_{x_i}+c^{(k)}(x,t)\varphi]dx\,dt$$
$$+\int_\Omega[u_{0k}(x)-u_0(x)]\varphi(x,0)dx-\int_{\partial\Omega\times[0,T]}(\psi_k-\psi)a_{ij}^{(k)}(s,t)\frac{\partial\varphi}{\partial N}\cos(N,x_i)$$
$$\times\cos(N,x_j)ds\,dt=0 \tag{5.11}$$

$$\forall\varphi\in C^{2,1}(Q_T)\cap C^1(\bar{Q}_T)\ \text{with}\ \varphi|_{t=T}=\varphi|_{\partial\Omega\times[0,T]}=0,$$

where

$$a_{ij}^{(k)}(x,t)=[A_{ij}(x,t,u_k)-A_{ij}(x,t,\tilde{u})]/(u_k-\tilde{u})$$
$$=\int_0^1 a_{ij}(x,t,\theta u_k+(1-\theta)\tilde{u})d\theta, \tag{5.12}$$

$$b_i^{(k)}=-\int_0^1\Big(\frac{\partial a_{ij}}{\partial x_j}+b_i\Big)(x,t,\theta u_k+(1-\theta)\tilde{u})d\theta,\quad c^{(k)}=\cdots. \tag{5.13}$$

Let $\tilde{u}_\varepsilon$ be the smooth appproximation of $\tilde{u}$ such that $\tilde{u}\leqslant\tilde{u}_\varepsilon\leqslant\tilde{u}+\varepsilon$. Let

$$a_{ij}^{(k,\varepsilon)}=\int_0^1 a_{ij\varepsilon}(x,t,\theta u_k+(1-\theta)\tilde{u}_\varepsilon)d\theta,\ b_i^{k,\varepsilon)}=\cdots,\ c^{(k,\varepsilon)}=\cdots.$$

Solve the first boundary value problem

$$\varphi_t+a_{ij}^{(k,\varepsilon)}\varphi_{x_ix_j}+b_i^{(k,\varepsilon)}\varphi_{x_i}+c^{(k,\varepsilon)}\varphi=U(x,t), \tag{5.14}$$
$$\varphi|_{t=T}=\varphi|_{\partial\Omega\times[0,T]}=0,$$

where $U(x,t)$ is any smooth function in $\bar{Q}_T$. Since $a_{ij}^{(k,\varepsilon)}$, $b_i^{(k,\varepsilon)}$, $c^{(k,\varepsilon)}\in C^{2,1}(Q_T)\cap C^{1,\frac{\alpha}{2}}(\bar{Q}_T)$ if $\partial\Omega\in C^{1+\beta_0}$, in this case we also have $\varphi\in C^{2,1}(Q_T)\cap C^{1,\frac{\alpha}{2}}(\bar{Q}_T)$.

Taking φ to be the solution $\varphi^{(k,\varepsilon)}$ of (5.14) into (5.11) and applying (5.14) we have

$$\int_{Q_T}[u(x,t)-\tilde{u}(x,t)]U(x,t)dx\,dt=\int_{Q_T}(u-u_k)U dx\,dt+\int_{Q_T}(u_k-\tilde{u})$$
$$\times[a_{ij}^{(k,\varepsilon)}-a_{ij}^{(k)}]\varphi_{x_ix_j}+[b_i^{(k,\varepsilon)}-b_i^{(k)}]\varphi_{x_i}+[c^{(k,\varepsilon)}-c^{(k)}]\varphi\}dx\,dt$$
$$-\int_\Omega(u_{0k}-u_0)\varphi(x,0)dx+\int_{\partial\Omega\times[0,T]}(\psi_k-\psi)a_{ij}^{(k)}(s,t)\frac{\partial\varphi}{\partial N}\cos(N,x_i)$$
$$\times\cos(N,x_j)ds\,dt. \tag{5.15}$$

The terms on the right-hand side of (5.15) can be estimated by the following three lemmas.

Lemma 5.1.

$$|\varphi^{(k,\varepsilon)}|\leqslant e^{2AT}\sup_{Q_T}U(x,t)|. \tag{5.16}$$

Proof (5.16) follows easily by applying maximum principle.

Lemma 5.2. *If $\partial\Omega\in C^2$, we have*

$$\left|\frac{\partial\varphi^{(k,\varepsilon)}}{\partial N}\right|\nu(1/k)\leqslant k. \tag{5.17}$$

Proof Let $\varphi^{(k,\varepsilon)}=e^{-2At}\psi$. (5.14) becomes

$$\mathscr{L}(\psi)=\psi_t+a_{ij}^{(k,\varepsilon)}\psi_{x_ix_j}+b_i^{(k,\varepsilon}\psi_{x_i}+(c^{(k,\varepsilon)}-2A)\psi=Ue^{2At}. \tag{5.18}$$

Since $\partial\Omega\in C^2$, $\forall(x^\circ,t^\circ)\in\partial\Omega\times[0,T]$, we can take $\bar{x}^\circ$ such that

$$\{x||x-\bar{x}^\circ|\leqslant R\}\cap\bar{\Omega}=\{x^\circ\},\ R=|x^\circ-\bar{x}^\circ|.$$

Let

$$\Psi=1-R^q[|x-\bar{x}^\circ|^2+(t-t^\circ)^2]^{-\frac{q}{2}},$$

where q is a constant to be determined. We have

$$\mathscr{L}(\Psi)=-qR^q[|x-\bar{x}^\circ|^2+(t-t^\circ)^2]^{-(\frac{q}{2}+1)}\Big[(q-2)a_{ij}^{(k,\varepsilon)}\frac{(x_i-\bar{x}_i^0)(x_j-\bar{x}_j^0)}{|x-\bar{x}^\circ|^2+(t-t^\circ)^2}$$
$$-a_{ii}^{(k,\varepsilon)}-(t-t^\circ)-b_i^{(k,\varepsilon)}(x_i-\bar{x}_i^0)\Big]+[c^{(k,\varepsilon)}-2\Lambda]\Psi. \tag{5.19}$$

Take $q=k/\nu(1/k)$. Applying (5.7) and (1.15) we have

$$qa_{ij}^{(k,\varepsilon)}\frac{(x_i-\bar{x}_i^0)x_j-\bar{x}_j^0)}{|x-\bar{x}^\circ|^2+(t-t^\circ)^2}\geqslant\frac{q}{2\Lambda}\int_0^1\nu_\varepsilon(\theta u_k+(1-\theta)\tilde{u}_\varepsilon)d\theta\frac{R^2}{(R+\operatorname{diam}\Omega)^2+T^2}$$
$$\geqslant CK\int_0^1\nu(\theta u_k+(1-\theta)\tilde{u}_\varepsilon)/\nu(1/k)\,d\theta\geqslant\frac{CK}{2m}\,\nu(e^{-2\Lambda T}/(2k))/\nu(1/k)$$
$$\geqslant\frac{CK}{2m^2}(e^{-2\Lambda T}/2)^{m-1}.$$

Taking K large enough, from (5.19) and the above inequality, we have

$$\mathscr{L}(\Psi)\leqslant-C_1q\Big[\frac{R^2}{|x-\bar{x}^\circ|^2+(t-t^\circ)^2}\Big]^{\frac{q}{2}}-\Lambda\Psi/2=-C_1q(1-\Psi)-\Lambda\Psi/2$$
$$\leqslant-\min(C,\,q,\,\Lambda/2)=\leqslant-\Lambda/2,$$

hence

$$\pm\psi+K_1\Psi\quad\Big(K_1=\frac{3}{\Lambda}\sup_{Q_T}|U(x,\,t)|\Big)$$

cannot take nonpositive minimum in Q_T, and the minimum of $\pm\psi+K_1\Psi$, in $\bar{Q}_T$ is zero, and it only can take at the point $(x^\circ,\,t^\circ)$. Hence

$$\Big[\pm\frac{\partial\psi}{\partial N}+K_1\frac{\partial\Psi}{\partial N}\Big]_{(x^\circ,t^\circ)}\leqslant 0,$$

where N is the outer normal of $\partial\Omega$. Hence we have

$$\pm e^{2\Lambda t}\frac{\partial\varphi(k,\,\varepsilon)}{\partial N}\Big|_{(x^\circ,t^\circ)}\leqslant-K_1\frac{\partial\Psi}{\partial N}\Big|_{(x^\circ,t^\circ)}\leqslant Kq.$$

This proves the lemma.

Lemma 5.3. *When $\partial\Omega\in C^2$ and ε small enough, we have*

$$\int_{Q_T}[\Sigma\varphi_{x_ix_j}^{(k,\varepsilon)2}+\Sigma\varphi_{x_i}^{(k,\varepsilon)2}]dx\,dt\leqslant K(k), \tag{5.20}$$

i. e. the integral is bounded by a constant independent of ε.

Proof From (5.14) we have

$$|\varphi_t+a_{ij}^{(k)}\varphi_{x_ix_j}|\leqslant\omega(\varepsilon)\Sigma|\varphi_{x_ix_j}|+K(\Sigma|\varphi'_{x_i}|+1),$$

where $\omega(\varepsilon)$ is a decreasing-to-zero function of its argument. This follows from that $\tilde{u}_\varepsilon$, $\tilde{a}_{ij\varepsilon}(x,\,t,\,r)$ and $\nu_\varepsilon(r)$ are continuous functions of ε. For fixed $(x^\circ,\,t^\circ)\in\bar{Q}_T$, the above relation can also be reduced to

$$|\varphi_t+a_{ij}^{(k)}(x^\circ,\,t^\circ)\varphi_{x_ix_j}|\leqslant[\bar{\omega}(x-x^\circ|+|t-t^\circ|)+\omega(\varepsilon)]\Sigma|\varphi_{x_ix_j}|$$
$$+K(\Sigma|\varphi_{x_i}|+1), \tag{5.21}$$

where $\bar{\omega}$ posseses the same property as ω. This follows from $a_{ij}\in C(\bar{Q}_T)$, $\tilde{u}\in C(\bar{Q}_T)$ and $u_k\in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ uniformly with respect to k.

By rotating coordinate axis we can assume that $a_{ij}^{(k)}(x^\circ,\,t^\circ)=0$ when $i\neq j$.

Let $\zeta(x)$, $\eta(t)$ be the cut-off functions such that

$$\zeta(x)=\begin{cases}1, & |x-x^\circ|\leqslant\delta/2,\\ 0, & |x-x^\circ|\geqslant\delta,\end{cases}\quad |\nabla\zeta|\leqslant 2/\delta,\ \eta(t)=\begin{cases}1, & |t-t^\circ|\leqslant\delta/2,\\ 0, & |t-t^\circ|\geqslant\delta,\end{cases}\quad |\eta'(t)|\leqslant 2/\delta.$$

From (5.21) we have

$$\int_{Q_T}[\varphi_t^2+\sum_{i,j}a_{ii}^{(k)}(x^\circ,t^\circ)a_{jj}^{(k)}(x^\circ,t^\circ)\varphi_{x_ix_j}]\zeta^2\eta\,dx\,dt$$

$$-\int_\Omega\sum_i a_{ii}^{(k)}(x^\circ,t^\circ)\varphi_{x_i}^2\zeta^2\eta]_{t=0}^T\,dx+\Sigma a_{ii}^{(k)}(x^\circ,t^\circ)a_{jj}^{(k)}(x^\circ,t^\circ)$$

$$\times\int_{\partial\Omega\times[0,T]}\varphi_{x_i}[\varphi_{x_ix_j}\cos(N,x_j)-\varphi_{x_jx_j}\cos(N,x_i)]\zeta^2\eta\,ds\,dt$$

$$+\int_{Q_T}[\Sigma a_{ii}^{(k)}(x^\circ,t^\circ)-(4\varphi_t\varphi_{x_i}\zeta\zeta_{x_i}\eta+\varphi_{x_i}^2\zeta^2\eta')+2\Sigma a_{ii}^{(k)}(x^\circ,t^\circ)a_{jj}^{(k)}(x^\circ,t^\circ)$$

$$\times(\varphi_{x_i}\varphi_{x_jx_j}\zeta\zeta_{x_i}\eta-\varphi_{x_i}\varphi_{x_ix_j}\zeta\zeta_{x_i}\eta)]\,dx\,dt\leqslant 2n^2[\bar\omega(2\delta)+\omega(\varepsilon)]\int_{Q_T}\Sigma\varphi_{x_ix_j}^2\zeta^2\eta\,dx\,dt$$

$$+K\int_{Q_T}(\Sigma\varphi_{x_i}^2+1)\zeta^2\eta\,dx\,dt. \tag{5.22}$$

Because of $\varphi(x,T)=0$, we have

$$-\int_\Omega\varphi_{x_i}^2\zeta^2\eta\,|_{t=0}^T\,dx\geqslant 0,$$

hence the second term on the left of (5.22) is nonnegative. Since

$$u_k(x^\circ,t^\circ)\geqslant e^{-2AT}/k,$$

$$a_{ii}^{(k)}(x^\circ,t^\circ)\geqslant\int_0^1\nu(\theta u_k(x^\circ,t^\circ)+(1-\theta)\tilde u_k(x^\circ,t^\circ))\,d\theta\geqslant\nu(e^{-2AT}/(2k))/(2m),$$

hence by (5.22) we have

$$\int_{Q_T}(\varphi_t^2+\Sigma\varphi_{x_ix_j}^2)\zeta^2\eta\,dx\,dt\leqslant K(k)\int_{Q_T}[\varphi_t^2+\Sigma a_{ii}^{(k)}(x^\circ,t^\circ)a_{jj}^{(k)}(x^\circ,t^\circ)\varphi_{x_ix_j}^2]\zeta^2\eta\,dx\,dt$$

$$\leqslant K(k)[\bar\omega(2\delta)+\omega(\varepsilon)+\varepsilon_1]\int_{Q_T}(\varphi_t^2+\Sigma_{x_ix_j}^2)\zeta^2\eta\,dx\,dt+K(k,\delta,\varepsilon_1)$$

$$\int_{Q_T}\cap\{(x,t)\,|\,|x-x^\circ|<\delta,\ |t-t^\circ|<\delta\}(\Sigma\varphi_{x_i}^2+1)\,dx\,dt+K(k)\Sigma a_{ii}^{(k)}(x^\circ,t^\circ)$$

$$\times a_{jj}^{(k)}(x^\circ,t^\circ)\int_{\partial\Omega x[0,T]}\varphi_{x_i}[\varphi_{x_ix_j}\cos(N,x_i)-\varphi_{x_ix_j}\cos(N,x_j)]\zeta^2\eta\,ds\,dt. \tag{5.23}$$

Let the local coordinate of $\partial\Omega$ be $(s_1, s_2, \cdots, s_{n-1})$ and let

$$\sigma_{ijl}=\cos(N,x_i)\cos(N,x_j)[\cos(N,x_i)\cos(s_l,x_j)-\cos(N,x_j)\cos(s_l,x_i)]$$

$$\tau_{ij}=\Big[\frac{\partial\cos(N,x_j)}{\partial x_j}-\frac{\partial\cos(N,x_i)}{\partial x_j}\Big]\cos(N,x_i),$$

we have

$$\int_{\partial\Omega}\varphi_{x_i}[\varphi_{x_jx_j}\cos(N,x_i)-\varphi_{x_ix_j}\cos(N,x_j)]\zeta^2\,ds=\int_{\partial\Omega}[\sum_l\varphi_N\varphi_{Ns_l}\sigma_{ijl}+\varphi_N^2\tau_{ij}]\zeta^2\,ds$$

$$=\int_{\partial\Omega}\varphi_N^2\Big[\Big(\tau_{ij}-1/2\,\Sigma\frac{\partial\sigma_{ijl}}{\partial s_l}\Big)\zeta^2-\sigma_{ijl}\zeta\zeta_{s_l}\Big]ds\leqslant\frac{k}{\delta}\int_{\partial\Omega}\varphi_N^2\zeta\,ds=\frac{k}{\delta}\int_\Omega\frac{\partial}{\partial N}(\varphi_N^2\zeta)\,dx$$

$$=\frac{k}{\delta}\int_\Omega(\varphi_N^2\zeta_N+2\varphi_N\varphi_{NN}\zeta)\,dx\leqslant\varepsilon_2\int_\Omega\Sigma\varphi_{x_ix_j}^2\zeta^2\,dx+K(\varepsilon_2,\delta)\int_{\partial\Omega\cap\{x\,|\,|x-x^\circ|<\delta\}}\Sigma\varphi_{x_i}^2\,dx.$$

Substituting this inequality into (5.23) and taking

$$K(k)[\bar{\omega}(2\delta)+\omega(\varepsilon_0)+\varepsilon_1+\varepsilon_2]\leqslant 1/2,$$

we have

$$\int_{Q_T\cap\{(x,t)\,|\,|x-x^0|<\frac{\delta}{2},|t-t^0|<\frac{\delta}{2}\}}(\varphi_t^2+\Sigma\varphi_{x_ix_j}^2)\,dx\,dt$$

$$\leqslant K(k)\int_{Q_T\cap\{(x,t)\,|\,|x-x^0|<\delta,|t-t^0|<\delta\}}(\Sigma\varphi_{x_i}^2+1)\,dx\,dt$$

if $\varepsilon\leqslant\varepsilon_0$. Take suitable (x^0, t^0) net such that the union of $\{(x, t)\,|\,|x-x^0|\leqslant\delta/2, |t-t^0|\leqslant\delta/2\}$ covers Q_T, then the above expression yields

$$\int_{Q_T}(\varphi_t^2+\Sigma\varphi_{x_ix_j}^2)\,dx\,dt\leqslant K(k)\int_{Q_T}(\Sigma\varphi_{x_i}^2+1)\,dx\,dt \tag{5.24}$$

when $\varepsilon\leqslant\varepsilon_0$. But we have

$$\int_{Q_T}\Sigma\varphi_{x_i}^2\,dx\,dt=-\int_{Q_T}\varphi\Delta\varphi\,dx\,dt\leqslant\varepsilon_3\int_{Q_T}(\Sigma\varphi_{x_ix_j}^2)\,dx\,dt+K(\varepsilon_3). \tag{5.25}$$

We obtain (5.20) when $\varepsilon\leqslant\varepsilon_0$ by (5.24) and taking $\varepsilon_3=(2K(k))^{-1}$ in (5.25). The lemma is proved.

Theorem 5.2. *Suppose that $\partial\Omega\in C^2$ and that the hypothesis of Theorem 5.1 is satisfied, then the weak solution of (1.1), (1.2) and (1.3) is unique.*

Proof Using (5.3), (5.4), (5.15) and Lemma 2, we have

$$\left|\int_{Q_T}(u-\tilde{u})U\,dx\,dt\right|\leqslant\sup_{Q_T}|U|\int_{Q_T}|u-u_k|\,dx\,dt+\sup_{\Omega}|\varphi(x,0)|\int_{\Omega}|u_{0k}-u_0|\,dx$$
$$+K\Big\{\sup_{Q_T}[\Sigma|a_{ij}^{(k,\varepsilon)}-a_{ij}^{(k)}|+\Sigma|b_i^{(k,\varepsilon)}-b_i^{(k)}|+|c^{(k,\varepsilon)}-c^{(k)}|]$$
$$\times\int_{Q_T}(\Sigma\varphi_{x_ix_j}^2+\Sigma\varphi_{x_i}^2+\varphi^2)\,dx\,dt\Big\}^{\frac{1}{2}}$$
$$+\int_{\partial\Omega\times\{\psi<\frac{1}{k}\}}\left[\frac{1}{k}+\frac{\nu\left(\frac{1}{k}\right)}{k}\right]_{\Lambda}\int_0^1\nu\left(\frac{1}{k}\theta+\psi(1-\theta)\right)d\theta\,\frac{k}{\nu\left(\frac{1}{k}\right)}\,ds$$
$$+\int_{\partial\Omega\times\{\psi\geqslant\frac{1}{k}\}}\frac{\nu\left(\frac{1}{k}\right)}{k}\,\frac{k}{\nu\left(\frac{1}{k}\right)}\,ds.$$

When $\psi\leqslant 1/k$ we have

$$\int_0^1\nu\left(\frac{1}{k}\theta+\psi(1-\theta)\right)d\theta/\nu(1/k)\leqslant m$$

by (1.15). Substituting this expression into the above inequality and letting $\varepsilon\to 0$ first, then $k\to\infty$, we have

$$\int_{Q_T}(u-\tilde{u})U\,dx\,dt=0.$$

Since U is arbitrary, we have

$$u(x,t)=\tilde{u}(x,t),\ \forall(x,t)\in\bar{Q}_T.$$

This proves the theorem.

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